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Generalized order and related growth measure of composite entire function of several complex variables on the basis of central index

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Abstract: In this paper, we define generalized order of an entire function of several complex variables in terms of central index and use it to estimate the growth properties of composite entire function of several complex variables with respect to one of the factor of the composition function.

Keywords: Entire function, maximum modulus, maximum term, central index, generalized order (generalized lower order).

1 Introduction, Definitions and Notations

We denote complex n-space by $\mathbb{C}^n$ and indicate its elements (points):

$$(z_1,z_2,...,z_n),(|z_1|,|z_2|,...,|z_n|),(r_1,r_2,...,r_n),(k_1,k_2,...,k_n)$$

by their corresponding symbols $z,|z|,r,k$ etc. Throughout $\Omega=\Omega_n$ stands for a nonempty open complete n-circular region in C^n(see §3.3 of [3]) with center at $(0,0,...,0)$, the zero element of $\mathbb{C}^n$. We write

$$|\Omega|=\{r:r=|z| \text{ for some } z\in\Omega\}$$

and

$$\Omega^+=\left\{r:r\in|\Omega|\,,\text{no}\,r_j=0,\ 1\le j\le n\right\}$$

and regard these as subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. For any $r,s\in\mathbb{R}^n$, we say that

(i) $r\le s$ or $s\ge r$, if and only if $r_j\le s_j$ for $1\le j\le n$,
(ii) $r<s$ or $s>r$, if and only if $r\le s$ but r is not equal to s
and
(iii) $r<<s$ or $s>>r$, if and only if $r_j<s_j$ for $1\le j\le n$.

A function $f(z)$, $z\in\mathbb{C}^n$ is said to be analytic at a point $\xi\in\mathbb{C}^n$ if it can be expanded in some neighborhood of ξ as an absolutely convergent power series. If we assume $\xi=(0,0,...,0)$, then $f(z)$ has representation(see [6] and [8]).

$$f(z)=\sum_{k=(0,0,....,0)}^{\infty}a_{k_1,k_2,...,k_n}z_1^{k_1}z_2^{k_2}...z_n^{k_n}=\sum_{|k|=0}^{\infty}a_kz^k,$$

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where $k=(k_1,k_2,.....,k_n)$ belongs to $\mathscr{N}=\{k:k\in\mathbb{C}^n$, each k_j is rational integer$\}$ and $|k|=k_1+k_2+.....+k_n$.

For $r>(0,0,...,0)$, the maximum term $\mu(r)=\mu(r,f)$, the maximum modulus $M(r)=M(r,f)$ and the central index $\nu(r)=\nu(r,f)=(\nu_1(r,f),\nu_2(r,f),...,\nu_n(r,f))$ of entire function $f(z)$ are given by (see [6] and [7]).

$$\mu(r)=\mu(r,f)=\max_{k\in\mathscr{N}}\{|a_k|r^k\}$$

$$M(r)=M(r,f)=\max_{|z|=r}|f(z)|$$

and

$$\nu_j(r)=\nu_j(r,f)=\left\{\begin{array}{c}\max\left[k_j:|a_k|r^k=\mu(r)\right],\text{ if }\mu(r)>0\\ 0,\text{ if }\mu(r)=0,\text{ for }1\le j\le n.\end{array}\right\}$$

Also, the central index $\nu(r,f)$ for which maximum term is achieved

$$|\nu(r,f)|=\nu_1(r,f)+\nu_2(r,f)+...+\nu_n(r,f).$$

Definition 1. ([3], *p*.339) *The order ρ_f and lower order λ_f of an entire function $f(z)=f(z_1,z_2,.....,z_n)$ are defined as follows*

$$\rho_f=\limsup_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[2]}M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)}$$

and

$$\lambda_f=\liminf_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[2]}M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)}.$$

where

$$\log^{[k]}x=\log\left(\log^{[k-1]}x\right)\text{ for }k=1,2,3,...\text{ and }\log^{[0]}x=x.$$

Following Datta and Mallik (see [2]) definitions of hyper order (hyper lower order), generalized order (generalized lower order) of entire functions of two complex variables, we may give the same for the entire functions of n-complex variables.

Definition 2. *The hyper order $\overline{\rho}_f$ and the hyper lower order $\overline{\lambda}_f$ of an entire function f are defined as follows:*

$$\overline{\rho}_f=\limsup_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[3]}M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)}$$

and

$$\overline{\lambda}_f=\liminf_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[3]}M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)}.$$

Definition 3. *Let l be an integer ≥ 1. The generalized order $\rho_f^{[l]}$ and the generalized lower order $\lambda_f^{[l]}$ of an entire function f are defined as follows:*

$$\rho_f^{[l]}=\limsup_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[l+1]}M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)}$$

and

$$\lambda_f^{[l]}=\liminf_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[l+1]}M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)}.$$

When $l=1$, Definition 3 coincides with Definition 1 and when $l=2$, Definition 3 coincides with Definition 2.
In 1988, He and Xiao [5] define the order of an entire function in terms of its central index as follows:

Definition 4. *The order ρ_f of an entire function $f(z)$ is defined by*

$$\rho_f = \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log r}.$$

Similarly, the lower order λ_f of of an entire function $f(z)$ is defined as

$$\lambda_f = \liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log r}.$$

Later in 1999, Chen and Yang [1] define the hyper order of an entire function in terms of the central index in the following manner.

Definition 5. *The hyper order $\overline{\rho}_f$ of an entire function $f(z)$ is defined by*

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log r}.$$

Similarly, the hyper lower order $\overline{\lambda}_f$ of of an entire function f is defined as

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log r}.$$

So it is interesting to investigate that whether or not the generalized order of an entire function of several complex variables can be define in terms of its central index.

In this paper, we establish that the generalized order (generalized lower order) of an entire function of several complex variables can be defined in terms of its central index. Also we study some comparative growth measure of composite entire function of several complex variables with respect to left (right) factor of the composite entire function based on their central index.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[6] Let $p, r \in |\Omega|$ and let $\mu(p)$ and $\mu(r)$ be both positive. Then the line integral,*

$$I = \int_p^r \sum_{j=1}^n \frac{\nu_j(x)}{x_j} dx_j$$

taken over any connected polygon in $|\Omega|$ with sides parallel to the axes and from p to r,

(i) *exists,*
(ii) *is independent of the polygon and*
(iii) *is such that* $\log \mu(r) = \log \mu(p) + I$.

Lemma 2. *[6] Let $r \in |\Omega|$. Let $p \in |C^n|$ and be such that $p >> (1,1,...,1)$, while $pr = (p_1r_1, p_2r_2, \ldots, p_nr_n)$ still $\in |\Omega|$. Let*

$$N_j = \max_{r\le t\le pr} \nu_j(t) \text{ for } 1 \le j \le n.$$

Then

(i) $\mu(r) \leq M(r) \leq \mu(r) \prod_{j=1}^{n} \left[N_j + \frac{p_j}{p_j - 1}\right]$,

(ii) $\mu(r) = M(r)$, *if and only if the series* $\sum_{|k|=0}^{\infty} a_k r^k$ *has at most one non vanishing term*,

(iii) *the last relation in (i) is an equality if and only if* $\mu(r) = 0$.

Lemma 3. *Let* $f(z)$ *be an entire function of n-complex variables with generalized order* $\rho_f^{[l]}$, *where l be a positive integer* ≥ 1. *Then*

$$\rho_f^{[l]} = \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[l]} |\nu(r_1, r_2, \ldots, r_n, f)|}{\log(r_1 r_2 \ldots r_n)}.$$

Proof. Set

$$f(z) = \sum_{k=(0,0,\ldots,0)}^{\infty} a_{k_1, k_2, \ldots, k_n} z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k z^k.$$

By Lemma 1, we see the maximum term $\mu(r)$ of f satisfies

$$\log \mu(r) = \log \mu(p) + \int_p^r \sum_{j=1}^{n} \frac{\nu_j(x)}{x_j} dx_j \tag{1}$$

Krishna ([6], Corollary 2.9) proved that $\nu_j(r)$ is increasing and right continuous in j-th variable for $1 \leq j \leq n$. Therefore, for any $p, r \in |\Omega|$ such that $\mu(r) > 0$ and $p >> (1, 1, \ldots, 1)$, we get for $1 \leq j \leq n$,

$$\nu_j(r) \leq \frac{1}{\log p_j} \int_p^r \nu_j(r_1, \ldots, r_{j-1}, \ldots, r_n) \frac{dx_j}{x_j}. \tag{2}$$

From (1) and (2), we get

$$\log \mu(r) \geq \log \mu(p) + \sum_{j=1}^{n} \nu_j(r) \log p_j \tag{3}$$

By Lemma 2, we have

$$\mu(r, f) \leq M(r, f) \tag{4}$$

It follows from (3) and (4) that

$$\sum_{j=1}^{n} \nu_j(r) \log p_j \leq \log M(r, f) + C_1 \tag{5}$$

As $p >> (1, 1, \ldots, 1)$ i.e., $p = (p_1, p_2, \ldots, p_n) >> (1, 1, \ldots, 1)$, choosing $p_j = 2$ for $1 \leq j \leq n$, we get

$$\begin{aligned} &\sum_{j=1}^{n} \nu_j(r) \log 2 \leq \log M(r, f) + C_1 \\ &\Rightarrow log^{[l]} |\nu(r, f)| + log^{[l+2]} 2 \leq log^{[l+2]} M(r, f) + C_2 \\ &\Rightarrow \log^{[l]} |\nu(r, f)| + \log^{[l+1]} 2 \leq \log^{[l+1]} M(r, f) + C_2 \end{aligned} \tag{6}$$

where $C_j (> 0) (j = 1, 2)$ is a suitable constant.

By (6) and Definition 3, we have

$$\limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[l]} |\nu(r_1, r_2, \ldots, r_n, f)|}{\log(r_1 r_2 \ldots r_n)} \leq \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[l+1]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)} = \rho_f^{[l]}. \tag{7}$$

On the other hand, by choosing $p_j = 2$ for $1 \le j \le n$ i.e., $p = (2,2,...,2)$ in (i) of Lemma 2, we have

$$M(r,f) \le \mu(r,f) \prod_{j=1}^{n} [N_j + 2],$$

where $N_j = \max_{r \le t \le pr} \nu_j(t)$, for $1 \le j \le n$.

$$\Rightarrow M(r,f) \le \left|a_{\nu(r,f)}\right| r^{\nu(r,f)} \prod_{j=1}^{n} [N_j + 2] \tag{8}$$

Since $\{|a_k|\}$ is bounded, from (8) we get

$$\begin{aligned}
\log M(r,f) &\le \sum_{j=1}^{n} \nu_j(r) \log r_j + \sum_{j=1}^{n} \log N_j + C_3 \\
&\le \sum_{j=1}^{n} |\nu(r,f)| \log r_j + \sum_{j=1}^{n} \log N_j + C_3 \\
&\le |\nu(r,f)| \log(r_1 r_2 ... r_n) + \log(N_1 N_2 ... N_n) + C_3 \\
&\Rightarrow \log^{[l+1]} M(r,f) \le \log^{[l]} |\nu(r,f)| + \log^{[l+1]}(r_1 r_2 ... r_n) + \log^{[l+1]}(N_1 N_2 ... N_n) + C_4 \qquad (9)
\end{aligned}$$

where $C_j (> 0)(j = 3,4)$ are suitable constants. By (9) and Definition 3, we get

$$\rho_f^{[l]} = \limsup_{r_1,r_2,...,r_n \to \infty} \frac{\log^{[l+1]} M(r_1,r_2,...,r_n,f)}{\log(r_1 r_2 ... r_n)} \le \limsup_{r_1,r_2,...,r_n \to \infty} \frac{\log^{[l]} |\nu(r_1,r_2,...,r_n,f)|}{\log(r_1 r_2 ... r_n)}. \tag{10}$$

By (7) and (10), Lemma 3 follows.

In the line of Lemma 3, we can prove the following lemma:

Lemma 4. *Let $f(z)$ be an entire function of n-complex variables with generalized lower order $\lambda_f^{[l]}$, where l is a positive integer ≥ 1. Then*

$$\lambda_f^{[l]} = \liminf_{r_1,r_2,...,r_n \to \infty} \frac{\log^{[l]} |\nu(r_1,r_2,...,r_n,f)|}{\log(r_1 r_2 ... r_n)}.$$

The proof is omitted.

3 Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f and g be two entire functions of n-complex variables. Also, let $0 < \lambda_{f \circ g}^{[l]} \le \rho_{f \circ g}^{[l]} < \infty$ and $0 < \lambda_g^{[l]} \le \rho_g^{[l]} < \infty$. Then*

$$\begin{aligned}
\frac{\lambda_{f\circ g}^{[l]}}{\rho_g^{[l]}} &\le \liminf_{r_1,r_2,...,r_n \to \infty} \frac{\log^{[l]} |\nu(r_1,r_2,...,r_n,f \circ g)|}{\log^{[l]} |\nu(r_1,r_2,...,r_n,g)|} \le \min\{\frac{\lambda_{f\circ g}^{[l]}}{\lambda_g^{[l]}}, \frac{\rho_{f\circ g}^{[l]}}{\rho_g^{[l]}}\} \\
&\le \max\{\frac{\lambda_{f\circ g}^{[l]}}{\lambda_g^{[l]}}, \frac{\rho_{f\circ g}^{[l]}}{\rho_g^{[l]}}\} \le \limsup_{r_1,r_2,...,r_n \to \infty} \frac{\log^{[l]} |\nu(r_1,r_2,...,r_n,f \circ g)|}{\log^{[l]} |\nu(r_1,r_2,...,r_n,g)|} \le \frac{\rho_{f\circ g}^{[l]}}{\lambda_g^{[l]}}.
\end{aligned}$$

Proof. Using respectively Lemma 3 and Lemma 4 for the entire function g, we have for arbitrary positive ε and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log^{[l]} |\nu(r_1, r_2, ..., r_n, g)| \leq (\rho_g^{[l]} + \varepsilon) \log(r_1 r_2 ... r_n) \tag{11}$$

and

$$\log^{[l]} |\nu(r_1, r_2, ..., r_n, g)| \geq (\lambda_g^{[l]} - \varepsilon) \log(r_1 r_2 ... r_n). \tag{12}$$

Also, for a sequence of values of each of $r_1, r_2, ..., r_n$ tending to infinity

$$\log^{[l]} |\nu(r_1, r_2, ..., r_n, g)| \leq (\lambda_g^{[l]} + \varepsilon) \log(r_1 r_2 ... r_n) \tag{13}$$

and

$$\log^{[l]} |\nu(r_1, r_2, ..., r_n, g)| \geq (\rho_g^{[l]} - \varepsilon) \log(r_1 r_2 ... r_n). \tag{14}$$

Using respectively Lemma 3 and Lemma 4 for the composite entire function $f \circ g$, we have for arbitrary positive ε and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log^{[l]} |\nu(r_1, r_2, ..., r_n, f \circ g)| \leq (\rho_{f \circ g}^{[l]} + \varepsilon) \log(r_1 r_2 ... r_n) \tag{15}$$

and

$$\log^{[l]} |\nu(r_1, r_2, ..., r_n, f \circ g)| \geq (\lambda_{f \circ g}^{[l]} - \varepsilon) \log(r_1 r_2 ... r_n). \tag{16}$$

Again, for a sequence of values of each of $r_1, r_2, ..., r_n$ tending to infinity

$$\log^{[l]} |\nu(r_1, r_2, ..., r_n, f \circ g)| \leq (\lambda_{f \circ g}^{[l]} + \varepsilon) \log(r_1 r_2 ... r_n) \tag{17}$$

and

$$\log^{[l]} |\nu(r_1, r_2, ..., r_n, f \circ g)| \geq (\rho_{f \circ g}^{[l]} - \varepsilon) \log(r_1 r_2 ... r_n). \tag{18}$$

Now from (11) and (16), it follows for all sufficiently large values of $r_1, r_2 ..., r_n$ that

$$\frac{\log^{[l]} |\nu(r_1, r_2, ..., r_n, f \circ g)|}{\log^{[l]} |\nu(r_1, r_2, ..., r_n, g)|} \geq \frac{\lambda_{f \circ g}^{[l]} - \varepsilon}{\rho_g^{[l]} + \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain

$$\liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[l]} |\nu(r_1, r_2, ..., r_n, f \circ g)|}{\log^{[l]} |\nu(r_1, r_2, ..., r_n, g)|} \geq \frac{\lambda_{f \circ g}^{[l]}}{\rho_g^{[l]}}. \tag{19}$$

Again, combining (12) and (17), we get for a sequence of values of each of $r_1, r_2, ..., r_n$ tending to infinity

$$\frac{\log^{[l]} |\nu(r_1, r_2, ..., r_n, f \circ g)|}{\log^{[l]} |\nu(r_1, r_2, ..., r_n, g)|} \leq \frac{\lambda_{f \circ g}^{[l]} + \varepsilon}{\lambda_g^{[l]} - \varepsilon}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$\liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[l]} |\nu(r_1, r_2, ..., r_n, f \circ g)|}{\log^{[l]} |\nu(r_1, r_2, ..., r_n, g)|} \leq \frac{\lambda_{f \circ g}^{[l]}}{\lambda_g^{[l]}}. \tag{20}$$

Similarly, from (14) and (15) it follows for a sequence of values of each of $r_1, r_2, \ldots, r_n$ tending to infinity

$$\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \le \frac{\rho^{[l]}_{f\circ g}+\varepsilon}{\rho^{[l]}_g-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \le \frac{\rho^{[l]}_{f\circ g}}{\rho^{[l]}_g}. \tag{21}$$

Now combining (19), (20) and (21), we get

$$\frac{\lambda^{[l]}_{f\circ g}}{\rho^{[l]}_g} \le \liminf_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \le \min\{\frac{\lambda^{[l]}_{f\circ g}}{\lambda^{[l]}_g},\frac{\rho^{[l]}_{f\circ g}}{\rho^{[l]}_g}\}. \tag{22}$$

Now, from (13) and (16) we obtain for a sequence of values of each of $r_1, r_2, \ldots, r_n$ tending to infinity that

$$\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \ge \frac{\lambda^{[l]}_{f\circ g}-\varepsilon}{\lambda^{[l]}_g+\varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \ge \frac{\lambda^{[l]}_{f\circ g}}{\lambda^{[l]}_g}. \tag{23}$$

Again from (12) and (15), it follows for all sufficiently large values of $r_1, r_2, \ldots, r_n$

$$\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \le \frac{\rho^{[l]}_{f\circ g}+\varepsilon}{\lambda^{[l]}_g-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \le \frac{\rho^{[l]}_{f\circ g}}{\lambda^{[l]}_g}. \tag{24}$$

Similarly, combining (11) and (18), we get for a sequence of values of each of $r_1, r_2, \ldots, r_n$tending to infinity

$$\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \ge \frac{\rho^{[l]}_{f\circ g}-\varepsilon}{\rho^{[l]}_g+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|} \ge \frac{\rho^{[l]}_{f\circ g}}{\rho^{[l]}_g}. \tag{25}$$

Therefore, combining (23), (24) and (25) we get

$$\max\{\frac{\lambda^{[l]}_{f\circ g}}{\lambda^{[l]}_{g}},\frac{\rho^{[l]}_{f\circ g}}{\rho^{[l]}_{g}}\}\leq \limsup_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,g)|}\leq\frac{\rho^{[l]}_{f\circ g}}{\lambda^{[l]}_{g}}. \tag{26}$$

Thus the theorem follows from (22) and (26).

Remark. If we take $0<\lambda^{[l]}_f\leq\rho^{[l]}_f<\infty$ instead of $0<\lambda^{[l]}_g\leq\rho^{[l]}_g<\infty$ and the other conditions remain the same, then also Theorem 1 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 2. *Let f and g be two entire functions of n-complex variables. Also let $0<\lambda^{[l]}_{f\circ g}\leq\rho^{[l]}_{f\circ g}<\infty$ and $0<\lambda^{[l]}_f\leq\rho^{[l]}_f<\infty$. Then*

$$\begin{aligned}\frac{\lambda^{[l]}_{f\circ g}}{\rho^{[l]}_{f}}&\leq\liminf_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f)|}\leq\min\{\frac{\lambda^{[l]}_{f\circ g}}{\lambda^{[l]}_{f}},\frac{\rho^{[l]}_{f\circ g}}{\rho^{[l]}_{f}}\}\\&\leq\max\{\frac{\lambda^{[l]}_{f\circ g}}{\lambda^{[l]}_{f}},\frac{\rho^{[l]}_{f\circ g}}{\rho^{[l]}_{f}}\}\leq\limsup_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f)|}\leq\frac{\rho^{[l]}_{f\circ g}}{\lambda^{[l]}_{f}}.\end{aligned}$$

The proof is omitted.

Example 1. Taking $f=\exp z$, $g=z$ and $n=1$ one can easily verify that the sign " $\leq$ " in Theorem 2 cannot be replaced by " $<$ " only.

Corollary 1. *Let f and g be two entire functions of n-complex variables such that $0<\lambda^{[l]}_{f\circ g}\leq\rho^{[l]}_{f\circ g}<\infty$. Also let $0<\lambda^{[l]}_f=\rho^{[l]}_f<\infty$. Then*

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f)|}=\frac{\lambda^{[l]}_{f\circ g}}{\lambda^{[l]}_{f}},$$

and

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f\circ g)|}{\log^{[l]}|\nu(r_1,r_2,\ldots,r_n,f)|}=\frac{\rho^{[l]}_{f\circ g}}{\rho^{[l]}_{f}}.$$

4 Conclusion

The main aim of the paper is to investigate some growth properties of entire function of several complex variables on the basis of central index. There are several other growth properties of entire function of several complex variables, namely type(weak type), exponent of convergence, L-order(lower order), L*-order(lower order), and properties related to these can be investigate using the central index and we feel that our theory will provide a helping tool for the investigation.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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