New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.296

Asymptotics of eigenvalues for Sturm-Liouville problem including quadratic eigenvalue in the boundary condition

Elif Baskaya

Department of Mathematics, Karadeniz Technical University, Trabzon, Turkey

Received: 15 December 2017, Accepted: 25 December 2017
Published online: 30 July 2018.

Abstract: In this paper, we obtain asymptotic estimates of eigenvalues for regular Sturm-Liouville problems having the quadratic eigenvalue parameter in the boundary condition. Also the potential of the problem is integrable.

Keywords: Sturm-Liouville problems, integrable potential, eigenvalues, asymptotics.

1 Introduction

In this paper, we consider the boundary value problem

$$y''(t)+\{\lambda-q(t)\}y(t)=0,\ \ t\in[a,b], \tag{1}$$

$$y'(a)-\left[c\lambda^2+d\lambda+e\right]y(a)=0,\ \ c,d,e\in\mathbb{R},\ c\neq 0, \tag{2}$$

$$y(b)\cos\beta-y'(b)\sin\beta=0,\ \ \beta\in[0,\pi). \tag{3}$$

where λ is a real parameter; $q(t)$ is a real-valued function. Also we assume that $q(t)$ is integrable on $[a,b]$. This problem differs from the usual regular Sturm-Liouville problem in the sense that eigenvalue parameter λ is contained in the boundary condition at a. Problems of this type arise from the method of separation of variables applied to mathematical models for certain physical problems including heat conduction and wave propagation, etc. [8]. It is shown by Walter [15] that this problem is self-adjoint problem. The purpose of this paper is to obtain asymptotic approximations for the eigenvalues of (1)-(3).

Approximations of this type have been derived before. We mention in particular [7,8] and [2]. Fulton's approach in [7] is based on an iteration of the usual Volterra integral equation, producing an asymptotic expansion of the solution in higher powers of $1/\lambda^{1/2}$ as $\lambda\to\infty$ and in [8] is based on the analysis of [14] for regular Sturm-Liouville problems on a finite closed interval and involves some operator-theoretical results of [15]. The approach used in [2] is based on an iterative procedure solving the associated Riccati equation and producing an asymptotic expansion of the solution in the higher powers of $1/\lambda^{1/2}$ as $\lambda\to\infty$ for smooth $q(t)$. There is also a vast amount of literature dealing with asymptotic estimates of eigenvalues for standart Sturm-Liouville problems with regular endpoints [3,4,5,6,9,10,11,13,14]. Here we follow the similar approach in [4,10,12]. We assume without loss of generality, that $q(t)$ has mean value zero. That is $\int_a^b q(t)\,dt=0$.

* Corresponding author e-mail: elifbekar@ktu.edu.tr

2 Conclusion

Theorem 1. *The eigenvalues λ_n of (1)-(3) satisfy as $n \to \infty$,*

(i) *if* $\beta = 0$,

$$\lambda_n^{1/2} = \frac{(n+2)\pi}{(b-a)} - \frac{1}{2(n+2)\pi}\int_a^b \left[\cos\frac{2(n+2)\pi(x-a)}{b-a}\right] q(x)\,dx + O\left(n^{-2}\eta(n)\right) + O\left(n^{-1}\eta^2(n)\right),$$

(ii) *if* $\beta \neq 0$,

$$\lambda_n^{1/2} = \frac{(2n+3)\pi}{2(b-a)} - \frac{1}{(2n+3)\pi}\left\{2\cot\beta + \int_a^b \left[\cos\frac{(2n+3)\pi(x-a)}{b-a}\right] q(x)\,dx\right\} + O\left(n^{-2}\eta(n)\right) + O\left(n^{-1}\eta^2(n)\right).$$

3 The Method

We associate with (1) the Riccati equation

$$v'(t,\lambda) = -\lambda + q - v^2.$$

We define

$$S(t,\lambda) = Re[v(t,\lambda)], \tag{4}$$
$$T(t,\lambda) = Im[v(t,\lambda)]. \tag{5}$$

It is shown in [3] that any real-valued solution of (1) is in the form

$$y(t,\lambda) = R(t,\lambda)\cos\theta(t,\lambda) \tag{6}$$

with

$$S(t,\lambda) = \frac{R'(t,\lambda)}{R(t,\lambda)}, \tag{7}$$

$$T(t,\lambda) = \theta'(t,\lambda). \tag{8}$$

Our approach to calculating λ_n is to approximate those λ which are such that

$$\theta(b,\lambda) - \theta(a,\lambda) = \int_a^b T(x,\lambda)\,dx. \tag{9}$$

We suppose that there exist functions $A(t)$ and $\eta(\lambda)$ so that

$$\left|\int_t^b e^{2i\lambda^{1/2}x} q(x)\,dx\right| \leq A(t)\,\eta(\lambda), \quad t \in [a,b] \tag{10}$$

where

(i) $A(t) := \int_t^b |q(x)|\,dx$ is a decreasing function of t,
(ii) $A(.) \in L[a,b]$,
(iii) $\eta(\lambda) \to 0$ as $\lambda^{1/2} \to \infty$.

For $q \in L[a,b]$ the existence of the A and η functions may be established for λ positive as follows. We note that, avoiding the trivial case $\int_t^b |q(x)|\,dx = 0$. $\left|\int_t^b e^{2i\lambda^{1/2}x} q(x)\,dx\right| \le \int_t^b |q(x)|\,dx < \infty$ so, if we define

$$F(t,\lambda) := \begin{cases} \left|\int_t^b e^{2i\lambda^{1/2}x} q(x)\,dx\right| / \int_t^b |q(x)|\,dx, & \text{if } \int_t^b |q(x)|\,dx \neq 0, \\ 0, & \text{if } \int_t^b |q(x)|\,dx = 0, \end{cases} \tag{11}$$

then $0 \le F(t,\lambda) \le 1$ and we set $\eta(\lambda) := \sup_{a \le t \le b} F(t,\lambda)$. $\eta(\lambda)$ is well defined by (11) and $\eta(\lambda) \to 0$ as $\lambda \to \infty$ [12].

Our method of approximating a solution of $v'(t,\lambda) = -\lambda + q - v^2$ on $[a,b]$ is similar to [12], so we set

$$v(t,\lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(t,\lambda). \tag{12}$$

When we put this serie into the Riccati equation and solve differential equations, we hold

$$v_1(t,\lambda) = -e^{-2i\lambda^{1/2}t} \int_t^b e^{2i\lambda^{1/2}x} q(x)\,dx,$$

$$v_2(t,\lambda) = e^{-2i\lambda^{1/2}t} \int_t^b e^{2i\lambda^{1/2}x} v_1^2(x,\lambda)\,dx, \tag{13}$$

$$v_n(t,\lambda) = e^{-2i\lambda^{1/2}t} \int_t^b e^{2i\lambda^{1/2}x} \left[v_{n-1}^2(x,\lambda) + 2v_{n-1}(x,\lambda) \sum_{m=1}^{n-2} v_m(x,\lambda) \right] dx, \quad n \ge 3.$$

Also we found $\theta(b,\lambda) - \theta(a,\lambda) = \int_a^b T(x,\lambda)\,dx$, so with (8) and (12) we have

$$\theta(b,\lambda) - \theta(a,\lambda) = \int_a^b \left[\lambda^{1/2} + Im \sum_{n=1}^{\infty} v_n(x,\lambda) \right] dx,$$

then

$$\theta(b,\lambda) - \theta(a,\lambda) = \lambda^{1/2}(b-a) + \sum_{n=1}^{\infty} Im \int_a^b v_n(x,\lambda)\,dx. \tag{14}$$

Theorem 2. [1] *If $v(t,\lambda)$ as in (12), as $\lambda \to \infty$*

$$v(t,\lambda) = i\lambda^{1/2} + v_1(t,\lambda) + O\left(\eta^2(\lambda)\right)$$

where

$$\begin{aligned} v_1(t,\lambda) &= -e^{-2i\lambda^{1/2}t} \int_t^b e^{2i\lambda^{1/2}x} q(x)\,dx \\ &= -\left[\cos 2\lambda^{1/2}t - i\sin 2\lambda^{1/2}t\right] \times \int_t^b \left[\cos 2\lambda^{1/2}x + i\sin 2\lambda^{1/2}x\right] q(x)\,dx \end{aligned}$$

and $\eta(\lambda)$ is defined (11).

After some calculations by using the last theorem, with (4) we gain

$$S(t,\lambda) = -\left(\cos 2\lambda^{1/2}t\right) \int_t^b \left[\cos 2\lambda^{1/2}x\right] q(x)\,dx - \left(\sin 2\lambda^{1/2}t\right) \int_t^b \left[\sin 2\lambda^{1/2}x\right] q(x)\,dx + O\left(\eta^2(\lambda)\right).$$

Let define the following notations:

$$\sin\xi_t := \int_t^b \left(\cos 2\lambda^{1/2}x\right) q(x)\,dx,$$
$$\cos\xi_t := \int_t^b \left(\sin 2\lambda^{1/2}x\right) q(x)\,dx,$$

thus we can write $S(t,\lambda)$ as

$$S(t,\lambda) = -\sin\left(2\lambda^{1/2}t+\xi_t\right) + O\left(\eta^2(\lambda)\right).$$

Similarly, with (5) we find $T(t,\lambda)$ as

$$T(t,\lambda) = \lambda^{1/2} - \cos\left(2\lambda^{1/2}t+\xi_t\right) + O\left(\eta^2(\lambda)\right). \tag{15}$$

Also, by using integration by part to (13), we determine

$$\int_a^b v_1(x,\lambda)\,dx = \frac{i}{2\lambda^{1/2}} \int_a^b e^{2i\lambda^{1/2}(x-a)} q(x)\,dx$$

and again with integration by part

$$\int_a^b v_1(x,\lambda)\,dx = \frac{i}{2}\lambda^{-1/2}[\int_a^b iq(x)\left[\sin 2\lambda^{1/2}x\right]\left[\cos 2\lambda^{1/2}a\right]dx - \int_a^b iq(x)\left[\cos 2\lambda^{1/2}x\right]\left[\sin 2\lambda^{1/2}a\right]dx$$
$$+\frac{i}{2}\lambda^{-1/2}[\int_a^b q(x)\left[\cos 2\lambda^{1/2}x\right]\left[\cos 2\lambda^{1/2}a\right]dx + \int_a^b q(x)\left[\sin 2\lambda^{1/2}x\right]\left[\sin 2\lambda^{1/2}a\right]dx,$$

so

$$Im\int_a^b v_1(x,\lambda)\,dx = \frac{1}{2}\lambda^{-1/2}\left[\cos 2\lambda^{1/2}a\right][\int_a^b q(x)\left[\cos 2\lambda^{1/2}x\right]dx + \left[\sin 2\lambda^{1/2}a\right]\int_a^b q(x)\left[\sin 2\lambda^{1/2}x\right]dx$$

We also have from equation (13),

$$\int_a^b v_2(x,\lambda)\,dx = \frac{i}{2\lambda^{1/2}}\int_a^b \left[1-e^{2i\lambda^{1/2}(x-a)}\right] v_1^2(x,\lambda)\,dx$$

and for $n \geq 3$

$$\int_a^b v_n(x,\lambda)\,dx = \frac{i}{2\lambda^{1/2}} \times \int_a^b [\,1-e^{2i\lambda^{1/2}(x-a)}\,][\,v_{n-1}^2(x,\lambda)+2v_{n-1}(x,\lambda)\sum_{m=1}^{n-2} v_m(x,\lambda)\,]\,dx.$$

Thus, with the last equations

$$\int_a^b \sum_{n=1}^{\infty} Im\{v_n(x,\lambda)\}\,dx = \sum_{n=1}^{\infty} Im\left\{\int_a^b v_n(x,\lambda)\,dx\right\} = \frac{1}{2}\lambda^{-1/2}\sin\left(2\lambda^{1/2}t+\xi_t\right) + O\left(\lambda^{-1}\eta(\lambda)\right). \tag{16}$$

4 Proof of the theorem

(i) If $\beta = 0$, our problem is

$$y''(t)+\{\lambda-q(t)\}y(t)=0,\ \ t\in[a,b],$$
$$y'(a)-\left[c\lambda^2+d\lambda+e\right]y(a)=0,\ \ c,d,e\in\mathbb{R},\ c\neq 0,$$
$$y(b)=0.$$

The real solution of $y''(t)+[\lambda-q(t)]y(t)=0$ is $y(t,\lambda)=R(t,\lambda)\cos\theta(t,\lambda)$ from (6). We use this equation for boundary $t=a$, we find

$$R(a,\lambda)\left\{\cos\theta(a,\lambda)\left[\frac{R'(a,\lambda)}{R(a,\lambda)}-\left(c\lambda^2+d\lambda+e\right)\right]-\theta'(a,\lambda)\sin\theta(a,\lambda)\right\}=0.$$

If we choose α_1 as

$$\sin\alpha_1:=\frac{R'(a,\lambda)}{R(a,\lambda)}-\left(c\lambda^2+d\lambda+e\right),$$
$$\cos\alpha_1:=-\theta'(a,\lambda),$$

we have $R(a,\lambda)\sin[\alpha_1+\theta(a,\lambda)]=0$ so $\sin(\alpha_1+\theta(a,\lambda))=0$, or $\theta(a,\lambda)=-\alpha_1$. Using by equations (7) and (8) as $S(t,\lambda)=\frac{R'(t,\lambda)}{R(t,\lambda)}$, $T(t,\lambda)=\theta'(t,\lambda)$ and their asymptotic expansions (9)-(15), we calculate

$$\frac{\cos\alpha_1}{\sin\alpha_1}=\frac{-\lambda^{1/2}+\cos\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\eta^2(\lambda)\right)}{-\sin\left(2\lambda^{1/2}a+\xi_a\right)-c\lambda^2-d\lambda-e+O\left(\eta^2(\lambda)\right)},$$

hence

$$\frac{\cos\alpha_1}{\sin\alpha_1}=\frac{-\lambda^{1/2}+\cos\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\eta^2(\lambda)\right)}{-c\lambda^2\left[1+\frac{d}{c}\lambda^{-1}+\frac{e}{c}\lambda^{-2}+\frac{1}{c}\lambda^{-2}\sin\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\lambda^{-2}\eta^2(\lambda)\right)\right]}.$$

Then

$$\cot\alpha_1=\left[\frac{1}{c}\lambda^{-3/2}-\frac{1}{c}\lambda^{-2}\cos\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\lambda^{-2}\eta^2(\lambda)\right)\right]$$
$$\times\left[1-\frac{d}{c}\lambda^{-1}-\frac{e}{c}\lambda^{-2}-\frac{1}{c}\lambda^{-2}\sin\left(2\lambda^{1/2}a+\xi_a\right)+\frac{d^2}{c^2}\lambda^{-2}+O\left(\lambda^{-2}\eta^2(\lambda)\right)\right],$$

so

$$\cot\alpha_1=\frac{1}{c}\lambda^{-3/2}-\frac{1}{c}\lambda^{-2}\cos\left(2\lambda^{1/2}a+\xi_a\right)-\frac{d}{c^2}\lambda^{-5/2}+O\left(\lambda^{-2}\eta^2(\lambda)\right).$$

In the last equation, by using Taylor expansion of *arccotx* at $x=0$, we obtain

$$\alpha_1=\frac{\pi}{2}-\frac{1}{c}\lambda^{-3/2}+\frac{1}{c}\lambda^{-2}\cos\left(2\lambda^{1/2}a+\xi_a\right)+\frac{d}{c^2}\lambda^{-5/2}+O\left(\lambda^{-2}\eta^2(\lambda)\right). \tag{17}$$

Similarly, when we use the form $y(t,\lambda)=R(t,\lambda)\cos\theta(t,\lambda)$ for boundary $t=b$, we find $\cos\theta(b,\lambda)=0$ so

$$\theta(b,\lambda)=\frac{\pi}{2}+(n+1)\pi. \tag{18}$$

Let use these findings (16), (17) and (18) in $\theta(b,\lambda)-\theta(a,\lambda)=\lambda^{1/2}(b-a)+\sum_{n=1}^{\infty} Im\int_a^b v_n(x,\lambda)\,dx$, we see that

$$\frac{\pi}{2}+(n+1)\pi+\frac{\pi}{2}-\frac{1}{c}\lambda^{-3/2}+\frac{1}{c}\lambda^{-2}\cos\left(2\lambda^{1/2}a+\xi_a\right)+\frac{d}{c^2}\lambda^{-5/2}+O\left(\lambda^{-2}\eta^2(\lambda)\right)$$
$$=\lambda^{1/2}(b-a)+\frac{1}{2}\lambda^{-1/2}\sin\left(2\lambda^{1/2}t+\xi_t\right)+O\left(\lambda^{-1}\eta(\lambda)\right)+O\left(\lambda^{-1/2}\eta^2(\lambda)\right).$$

We prove the theorem by using definitions of $\sin\xi_t,\cos\xi_t$ and $\eta(\lambda)$; also series error computation in the last equation.

(ii) If $\beta\neq 0$, our problem is

$$y''(t)+\{\lambda-q(t)\}y(t)=0,\quad t\in[a,b],$$
$$y'(a)-\left[c\lambda^2+d\lambda+e\right]y(a)=0,\quad c,d,e\in\mathbb{R},\ c\neq 0,$$
$$y(b)\cos\beta-y'(b)\sin\beta=0,\quad \beta\in(0,\pi).$$

For the first boundary condition, we found $\theta(a,\lambda)=-\alpha_1$ and α_1 is in (17).

For boundary $t=b$, by using $y(b,\lambda)=R(b,\lambda)\cos\theta(b,\lambda)$, we find

$$R(b,\lambda)\left\{\cos\theta(b,\lambda)\left[\cos\beta-\frac{R'(b,\lambda)}{R(b,\lambda)}\sin\beta\right]+\theta'(b,\lambda)\sin\beta\sin\theta(b,\lambda)\right\}=0.$$

If we choose α_2 as

$$\sin\alpha_2:=\cos\beta-\frac{R'(b,\lambda)}{R(b,\lambda)}\sin\beta,$$
$$\cos\alpha_2:=-\theta'(b,\lambda)\sin\beta,$$

we have $R(b,\lambda)\sin[\alpha_2-\theta(b,\lambda)]=0$ so $\sin[\alpha_2-\theta(b,\lambda)]=0$, or $\theta(b,\lambda)=(n+1)\pi+\alpha_2$. Using by $S(t,\lambda)=\frac{R'(t,\lambda)}{R(t,\lambda)}$, $T(t,\lambda)=\theta'(t,\lambda)$ and their asymptotic expansions, one writes

$$\frac{\sin\alpha_2}{\cos\alpha_2}=\frac{\cos\beta+O\left(\eta^2(\lambda)\right)}{-\lambda^{1/2}\sin\beta+O\left(\eta^2(\lambda)\right)}=\frac{\cos\beta+O\left(\eta^2(\lambda)\right)}{-\lambda^{1/2}\sin\beta\left[1-O\left(\lambda^{-1/2}\eta^2(\lambda)\right)\right]}$$
$$=\left[-\lambda^{-1/2}\cot\beta+O\left(\lambda^{-1/2}\eta^2(\lambda)\right)\right]\times\left[1+O\left(\lambda^{-1/2}\eta^2(\lambda)\right)\right],$$

then

$$\tan\alpha_2=-\lambda^{-1/2}\cot\beta+O\left(\lambda^{-1/2}\eta^2(\lambda)\right).$$

In the last equation, by using Taylor expansion of $\arctan x$ at $x=0$, we obtain

$$\alpha_2=-\lambda^{-1/2}\cot\beta+O\left(\lambda^{-1/2}\eta^2(\lambda)\right)$$

so

$$\theta(b,\lambda)=(n+1)\pi-\lambda^{-1/2}\cot\beta+O\left(\lambda^{-1/2}\eta^2(\lambda)\right). \tag{19}$$

Let use these findings in $\theta(b,\lambda)-\theta(a,\lambda)=\lambda^{1/2}(b-a)+\sum_{n=1}^{\infty} Im\int_a^b v_n(x,\lambda)\,dx$, we estimate that

$$
\begin{aligned}
&(n+1)\pi - \lambda^{-1/2}\cot\beta + \frac{\pi}{2} - \frac{1}{c}\lambda^{-3/2} + \frac{1}{c}\lambda^{-2}\cos\left(2\lambda^{1/2}a + \xi_a\right) + O\left(\lambda^{-1/2}\eta^2(\lambda)\right) \\
&= \lambda^{1/2}(b-a) + \frac{1}{2}\lambda^{-1/2}\sin\left(2\lambda^{1/2}t + \xi_t\right) + O\left(\lambda^{-1}\eta(\lambda)\right) + O\left(\lambda^{-1/2}\eta^2(\lambda)\right).
\end{aligned}
$$

We prove the theorem by using definitions of $\sin\xi_t, \cos\xi_t$ and $\eta(\lambda)$; also series error computation in the last equation.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Başkaya, E., Regular Sturm-Liouville problems with eigenvalue parameter in the boundary conditions, PhD. Thesis, Karadeniz Technical University, The Graduate School of Natural and Applied Sciences Biology Graduate Program, Trabzon, 2013.

[2] Coskun, H.; Bayram, N., Asymptotics of eigenvalues for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition, J. Math. Anal. Appl. 306 (2005), 548-566.

[3] Coskun, H.; Harris, B. J., Estimates for the periodic and semi-periodic eigenvalues of Hill's equation, Proc. Roy. Soc. Edinburgh (A) 130 (2000), 991-998.

[4] Coskun, H, On the spectrum of a second-order periodic differential equation, Rocky Mountain J. Math. 33 (2003), 1261-1277.

[5] Eastham, M. S., The Spectral Theory of Periodic Differential Equations, Scottish Academic Press, Edinburgh 1973.

[6] Fix, G., Asymptotic eigenvalues of Sturm-Liouville systems, J. Math. Anal. Appl. 19 (1967), 519-525.

[7] Fulton, C. T., An integral equation iterative scheme for asymptotic expansions of spectral quantities of regular Sturm-Liouville problems, J. Integral Equations 4 (1982), 163-172.

[8] Fulton, C. T., Two point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edinburgh (A) 77 (1977), 293-308.

[9] Fulton, C. T.; Pruess, S. A., Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems, J. Math. Anal. Appl. 182 (1994), 297-340.

[10] B. J. Harris, A series solution for certain Riccati equations with applications to Sturm-Liouville problems, J. Math. Anal. Appl. 137 (1989), 462-470.

[11] B. J. Harris, Asymptotics of eigenvalues for regular Sturm-Liouville problems, J. Math. Anal. Appl. 183 (1994), 25-36.

[12] B. J. Harris, The form of the spectral functions associated with Sturm-Liouville problems with continuous spectrum, Mathematika 44 (1997), 149-161.

[13] Hochstadt, H., Asymptotic estimates for the Sturm-Liouville spectrum, Comm. Pure Appl. Math. 14 (1961), 749-764.

[14] Titchmarsh, E. C., Eigenfunction Expansions Associated with Second Order Differential Equations I, 2 nd edn., Oxford Univ. Press, Oxford 1962.

[15] Walter, J., Regular eigenvalue problems with eigenvalue parameter in the boundary condition, Math. Z. 153 (1973), 301-312.