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Some results on generalized (k,μ)-space forms

Shanmukha B[1], *Venkatesha*[1] *and Vishunuvardhana S. V.*[2]

[1]Department of Mathematics, Kuvempu University, Shankaraghatta, Shimoga, Karnataka, India
[2] Department of Mathematics, Gitam School of technology, Gitam (Deemed to be University), Bangalore, Karnataka,India

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Abstract: In this paper we have studied Ricci symmetric and Ricci pseudosymmetric generalized (k,μ)-space forms and generalized (k,μ)-space forms with quasi umblical hypersurface and τ-flat curvature tensor.

Keywords: Generalized (k,μ)-space form, Ricci symmetric, Ricci pseudosymmetric, quasi umblical hypersurface, τ-curvature tensor.

1 Introduction

An almost contact metric manifold (M,g) is a Riemannian manifold with a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η on M satisfying [5,6]

$$\phi^2 = -I + \eta\circ\xi,\ \eta(\xi) = 1,\ \eta\circ\phi = 0,\ \phi\xi = 0, \tag{1}$$
$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2}$$
$$g(\phi X,Y) = -g(X,\phi Y),\ g(\phi X,X) = 0,\ g(X,\xi) = \eta(X), \tag{3}$$

for all vector fields X, Y on M. We know that a real space form is a Riemannian manifold having constant sectional curvature and a complex space form is a Kaehlerian manifold (M,J,g) with constant holomorphic sectional curvature c.

Generalized Sasakian space forms were studied extensively in [1,2,3,15,18,19,25]. An almost contact metric manifold (M,ϕ,ξ,η,g) is a generalized (k,μ) space form if there exists differential functions $f_1, f_2,\cdots, f_6$ on $M^{2n+1}(f_1,\cdots,f_6)$, whose curvature tensor R is given by [8,9]

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{4}$$

where R_1,R_2,R_3,R_4,R_5 and R_6 are given by;

$$\begin{aligned}
R_1(X,Y)Z &= \{g(Y,Z)X - g(X,Z)Y\},\\
R_2(X,Y)Z &= \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\},\\
R_3(X,Y)Z &= \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\
R_4(X,Y)Z &= \{g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y\},\\
R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,\\
R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,
\end{aligned}$$

* Corresponding author e-mail: vensmath@gmail.com

Here h is defined by $2h = L_\xi \phi$ is a symmetric tensor and satisfies the following conditions

$$h\xi = 0, \quad h\phi = -\phi h, \quad tr(h) = 0, \quad \eta \circ h = 0, \tag{5}$$

and where L is the usual Lie derivative. In particular if $f_4 = f_5 = f_6 = 0$, then generalized (k,μ)-space form $M^{2n+1}(f_1,\cdots,f_6)$ reduces to generalized Sasakian space forms. Also in [17] it was proved that (k,μ)-space forms are natural examples of generalized (k,μ)-space forms for constant functions $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$, $f_3 = \frac{c+3}{4} - k$, $f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1-\mu$.

In a generalized (k,μ) space forms, the following relations hold [9];

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) + ((2n-1)f_4 - f_6)g(hX,Y), \tag{6}$$

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)\eta(Y)\} + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}, \tag{7}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \tag{8}$$

Definition 1. *A Riemannian manifold M is said to be*

(I) *Einstein manifold if $S(X,Y) = \lambda_1 g(X,Y)$,*
(II) *η-Einstein manifold if $S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y)$,*
(III) *Special type of η-Einstein manifold if $S(X,Y) = \lambda_1 \eta(X)\eta(Y)$,*

where S is the Ricci tensor and λ_1 and λ_2 are constants.

In a generalized quasi-Einstein manifold the Ricci tensor S is given by [13]

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \tag{9}$$

Here α,β,γ are non zero scalars and A,B are non-zero 1-forms which are defined by

$$g(X,U) = A(X) \ and \ g(X,V) = B(X),$$

where U and V are two orthogonal vectors. If $\gamma = 0$, then the manifold reduces to a quasi Einstein manifold.

2 Main results

Theorem 1. *Let M be quasi-umbilical hypersurface of a generalized (k,μ)-space form is a generalized quasi-Einstein hypersurface if and only if $f_6 = (2n-1)f_4$.*

Proof. We know that hypersurface of $(M^{2n+1},\tilde{g})$ is (M^{2n},g). If A is the $(1,1)$-tensor corresponding to the normal valued second fundamental tensor H, then we have [11]

$$g(A_\rho(X),Y) = \tilde{g}(H(X,Y),\rho), \tag{10}$$

where ρ is a unit normal vector field and X,Y are tangent vector fields. Let H_ρ be the symmetric $(0,2)$ tensor corresponding to A_ρ in the hypersurface, defined by

$$g(A_\rho(X),Y) = H_\rho(X,Y). \tag{11}$$

A hypersurface of a Riemannian manifold (M^{2n+1},g) is called quasi-umbilical if its second fundamental tensor has the form [11]

$$H_\rho(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y), \tag{12}$$

where ω is a 1-form and α,β are scalars. Suppose $\alpha=0$ (resp. $\beta=0$ or $\alpha=\beta=0$) holds, then it is called cylindrical (resp. umbilical or geodesic). Now from (10), (11) and (12) we obtain

$$g(H(X,Y),\rho)=\alpha g(X,Y)\tilde{g}(\rho,\rho)+\beta\omega(X)\omega(Y)\tilde{g}(\rho,\rho),$$

which implies that

$$H(X,Y)=\alpha g(X,Y)_{\rho}+\beta\omega(X)\omega(Y)_{\rho}. \tag{13}$$

The Gauss equation tangent to the hypersurface is given by [11]

$$R(X,Y,Z,W)=\tilde{R}(X,Y,Z,W)-g(H(X,W),H(Y,Z))+g(H(X,Z),H(Y,W)), \tag{14}$$

where $\tilde{R}(X,Y,Z,W)=\tilde{g}(R(X,Y)Z,W)$ and $R(X,Y,Z,W)=g(R(X,Y)Z,W)$.
Let us consider quasi umbilical hypersurface of generalized (k,μ)-space forms. Then from (13) and (14) we have

$$\begin{aligned}R(X,Y,Z,W)=\tilde{R}(X,Y,Z,W)&-g([\alpha g(X,W)_{\rho}+\beta\omega(X)\omega(W)_{\rho}],[\alpha g(Y,Z)_{\rho}+\beta\omega(Y)\omega(Z)_{\rho}])\\&+g([\alpha g(Y,W)_{\rho}+\beta\omega(Y)\omega(W)_{\rho}],[\alpha g(X,Z)_{\rho}+\beta\omega(X)\omega(Z)_{\rho}]).\end{aligned} \tag{15}$$

Using (4) in (15) and then contracting over X and W, we get

$$\begin{aligned}S(Y,Z)=&(2nf_1+3f_2-f_3+2n\alpha^2+\alpha^2)g(Y,Z)-(3f_2+(2n-1)f_3)\eta(Y)\eta(Z)\\&+((2n-1)f_4-f_6)g(hY,Z)+(2n-1)\alpha\beta\omega(Y)\omega(Z).\end{aligned} \tag{16}$$

This complete the proof of the theorem

Theorem 2. *A generalized (k,μ)-space form $M^{2n+1}(f_1,\cdots,f_6)$ satisfying $(S(X,\xi)\cdot R)(Y,Z)W=0$ is an η-Einstein manifold.*

Proof. Consider a generalized (k,μ)-space form $(n>1)$ satisfying $(S(X,\xi)\cdot R)(Y,Z)W=0$, then we have

$$\begin{aligned}0=(S(X,\xi)\cdot R)(Y,Z)W&=(X\wedge\xi)R(Y,Z)W\\&=((X\wedge_s\xi)\cdot R)(Y,Z)W+R((X\wedge_s\xi)Y,Z)W\\&+R(Y,(X\wedge_s\xi)Z)W+R(Y,Z)(X\wedge_s\xi)W,\end{aligned} \tag{17}$$

where $(X\wedge_s Y)$ is an endomorphism and is defined by

$$(X\wedge_s Y)Z=S(Y,Z)X-S(X,Z)Y. \tag{18}$$

Using (7) and (18) in (17) and taking inner product with ξ to the resulting equation, then we have

$$\begin{aligned}&2n(f_1-f_3)\{\eta(R(Y,Z)W)\eta(X)+\eta(Y)\eta(R(X,Z)W)+\eta(Z)\eta(R(Y,X)W)+\eta(W)\eta(R(Y,Z)X)\}\\&-\{S(X,R(Y,Z)W)+S(X,Y)\eta(R(\xi,Z)W)+S(X,Z)\eta(R(Y,\xi)W)+S(X,W)\eta(R(Y,Z)\xi)\}=0.\end{aligned}$$

Setting $Y=W=\xi$ in the above equation, we get

$$\begin{aligned}&(f_1-f_3)\{2n(f_1-f_3)\eta(X)\eta(Z)-S(X,Z)\}+2n(f_1-f_3)^2\{g(X,Z)-\eta(Z)\eta(X)\\&+(f_4-f_6)2n(f_1-f_3)g(hZ,X)\}=0.\end{aligned} \tag{19}$$

By virtue of (6) in (19), we have

$$S(X,Z) = Ag(X,Z) + B\eta(X)\eta(Z),$$

where

$$A = -[\frac{((2n-1)f_4 - f_6)2n(f_1 - f_3) - (2nf_1 + 3f_2 - f_3)}{2n(f_1 - f_3)((2n-1)f_4 - f_6) + (f_4 - f_6)2n}]$$

and

$$B = -[\frac{-4n(f_1 - f_3) + (3f_2 + (2n-1)f_3)}{2n(f_1 - f_3)((2n-1)f_4 - f_6) + (f_4 - f_6)2n}].$$

Hence the proof.

Definition 2. *A generalized (k,μ)-space forms is said to be Ricci symmetric if it satisfies*

$$(\nabla_W S)(X,Y) = 0$$

for all X,Y are orthogonal to ξ.

Theorem 3. *Let M be a 3-dimensional generalized (k,μ)-space form has constant ϕ-sectional curvature with constants $(f_1 - f_3)$ and $(f_4 - f_6)$, then the following are equivalent:*

(1) *Ricci symmetric with $(f_4 - f_6) \neq 0$,*
(2) *Tensor h is parallel.*

Proof. Differentiating (6) with respect to W, we get

$$\begin{aligned}(\nabla_W S)(X,Y) = &\{2nd(f_1 - f_3)(W) + 3df_2(W) + (2n-1)df_3(W)\}g(X,Y) - \{3df_2(W) + (2n-1)df_3(W)\}\eta(X)\eta(Y) \\ &+ \{2(n-1)df_4(W) + d(f_4 - f_6)(W)\}g(hX,Y) - (3f_2 + (2n-1)f_3)\{(\nabla_W\eta)(X)\eta(Y) \\ &+ (\nabla_W\eta)(Y)\eta(X)\} + \{2(n-1)f_4 + (f_4 - f_6)\}g((\nabla_W h)X,Y). \end{aligned} \tag{20}$$

If X and Y are orthogonal to ξ, then we have from (20)

$$\begin{aligned}(\nabla_W S)(X,Y) = &\{2nd(f_1 - f_3)(W) + 3df_2(W) + (2n-1)df_3(W)\}g(X,Y) + \{2(n-1)df_4(W) \\ &+ d(f_4 - f_6)(W)\}g(hX,Y) + \{2(n-1)f_4 + (f_4 - f_6)\}g((\nabla_W h)X,Y). \end{aligned} \tag{21}$$

Suppose $(f_1 - f_3)$ and $(f_4 - f_6)$ are non-zero constants, then (21) becomes

$$(\nabla_W S)(X,Y) = \{3df_2(W) + (2n-1)df_3(W)\}g(X,Y) + \{2(n-1)df_4(W)\}g(hX,Y) + \{(2n-1)f_4 - f_6)\}g((\nabla_W h)X,Y). \tag{22}$$

For $n = 1$ (i.e. for three-dimensional case) in (22), we have

$$(\nabla_W S)(X,Y) = d(3f_2 + f_3)(W)g(X,Y) + (f_4 - f_6)g((\nabla_W h)X,Y). \tag{23}$$

Let the ϕ-sectional curvature $(3f_2 + f_1)$ (see [9]) of generalized (k,μ)-space form be constant. Then (23) gives

$$(\nabla_W S)(X,Y) = (f_4 - f_6)g((\nabla_W h)X,Y). \tag{24}$$

Hence the proof, moreover.

Theorem 4. *Let $(2n+1)$-dimensional (k,μ)-space form with $X,Y \in \xi^{\perp}$, then the following are equivalent:*

(I) *Ricci symmetric with $\{(2n-1)f_4 - f_6\} \neq 0$,*
(II) *Tensor h is parallel.*

Proof. Consider the functions $f_1, \ldots f_6$ in (4) are constants, then generalized (k,μ)-space form reduces to the (k,μ)-space form. Now suppose that the functions $f_1 \ldots f_6$ are constants. Then by taking covariant differentiation of (6) with respect to W, we get

$$(\nabla_W S)(X,Y) = -(3f_2+(2n-1)f_3)\{(\nabla_W \eta)(X)\eta(Y)+\eta(X)(\nabla_W \eta)(Y)\}+\{(2n-1)f_4-f_6\}g((\nabla_W h)X,Y). \quad (25)$$

If X,Y are orthogonal to ξ, then (25) yields

$$(\nabla_W S)(X,Y) = \{(2n-1)f_4-f_6\}g((\nabla_W h)X,Y).$$

This completes the proof of the theorem.

When generalizing the spaces of constant curvature, a locally symmetric spaces were introduced by Cartan [7]. All locally symmetric space satisfies $R\cdot R=0$, where the first R represents the curvature operator which acts as a derivation and the second R represents the Riemannian curvature tensor. Manifolds satisfying the condition $R\cdot R=0$ are called semisymmetric manifolds and were classified by Szabo [24]. The condition of semisymmetry was weakened by Deszcz as pseudosymmetry which are characterized by the condition $R\cdot R = LQ(g,R)$, here by L is a real function and $Q(g,R)$ is the Tachibana tensor.

Definition 3. *A Riemannian manifold M is said to be pseudosymmetric, in the sense of Deszcz [14] if*

$$(R(X,Y)\cdot R)(Z,U)V = L_R\{((X\wedge Y)\cdot R)(Z,U)V)\}, \quad (26)$$

holds. Where L_R is some smooth function on $U_R=\{x\in M|R-\frac{r}{n(n-1)}G\neq 0$ at $x\}$, G is the $(0,4)$-tensor defined by $G(X_1,X_2,X_3,X_4)=g((X_1\wedge X_2)X_3,X_4)$ and $(X_1\wedge X_2)X_3$ is the endomorphism and it is defined as,

$$(X_1\wedge X_2)X_3 = g(X_2,X_3)X_1-g(X_1,X_3)X_2. \quad (27)$$

Definition 4. *In a Riemannian manifold M, if $R\cdot S$ and $Q(g,S)$ are linearly dependent then M is called a Ricci pseudo symmetric manifold and this condition is given by*

$$R\cdot S = f_s Q(g,S), \quad (28)$$

where $U_S=\{x\in M : S-\frac{k}{n}\neq 0$ at $x\}$ and f_S is a function defined on U_S.

Theorem 5. *Let $(2n+1)$-dimensional Ricci pseudosymmetric generalized (k,μ)-space form is Ricci semisymmetric, provided $f_s\neq f_1-f_3$ and $f_4=f_6$.*

Proof. Now equation (28) can be written as

$$(R(X,Y)S)(Z,W) = -f_s\{S((X\wedge_g Y)Z,W)+S(Z,(X\wedge_g Y)W)\}, \quad (29)$$

where the endomorphism $(X\wedge_g Y)Z = g(Y,Z)X-g(X,Z)Y$. Now (29) yields

$$-S(R(X,Y)Z,W)-S(Z,R(X,Y)W) = -f_s\{S(Y,W)g(X,Z)-S(X,W)g(Y,Z)+S(Z,Y)g(X,W)-S(Z,X)g(Y,W)\}. \quad (30)$$

Putting $X=Z=\xi$ in (30), we get

$$-S(R(\xi,Y)\xi,W)-S(\xi,R(\xi,Y)W) = -f_s\{S(Y,W)g(\xi,\xi)-S(\xi,W)g(W,\xi)+S(\xi,Y)g(\xi,W)-S(\xi,\xi)g(Y,W)\}. \quad (31)$$

Using (1), (6) and (7) in the above equation, we have

$$((f_1-f_3)-f_s)[S(Y,W)-2n(f_1-f_3)g(Y,W)]+(f_4-f_6)[S(hY,W)-2n(f_1-f_3)g(hY,W)]=0. \tag{32}$$

Hence proof. Further, from (32) We can state the following statement.

Theorem 6. *A $(2n+1)$-dimensional Ricci pseudosymmetric generalized (k,μ)-space form is Ricci pseudosymmetric generalized Sasakian space form if and only if $(f_4-f_6)=0$.*

In [22] Tripathi et.al., introduced τ-curvature tensor and which is the generalization of conformal, concircular, projective etc. This curvature tensor was studied on K-contact, Sasakian and (k,μ)-contact metric manifolds by the authors in [21, 22]. The τ-curvature tensor is defined by [22]

$$\begin{aligned}\tau(X,Y)Z=&\,a_0R(X,Y)Z+a_1S(Y,Z)X+a_2S(X,Z)Y+a_3S(X,Y)Z+a_4g(Y,Z)QX\\&+a_5g(X,Z)QY+a_6g(X,Y)QZ+a_7r\{g(Y,Z)X-g(X,Z)Y\}.\end{aligned} \tag{33}$$

Definition 5. *A $(2n+1)$-dimensional generalized (k,μ)-space form is said to be τ-flat if it satisfies*

$$\tau(X,Y)Z=0. \tag{34}$$

Theorem 7. *A τ-flat $(2n+1)$-dimensional generalized (k,μ)-space form is an η-Einstein manifold.*

Proof. Consider a τ-flat $(2n+1)$-dimensional generalized (k,μ)-space form. Then from (34) we have

$$\begin{aligned}a_0\,g(R(X,Y)Z,W)=&-a_1S(Y,Z)g(X,W)-a_2S(X,Z)g(Y,W)-a_3S(X,Y)g(Z,W)-a_4g(Y,Z)S(X,W)\\&-a_5g(X,Z)S(Y,W)-a_6g(X,Y)S(Z,W)-a_7r\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}.\end{aligned} \tag{35}$$

Putting $X=W=\xi$ in the above equation and using (6) and (1), it follows that

$$S(Y,Z)=Ag(Y,Z)+B\eta(Y)\eta(Z),$$

where

$$A=-\{\frac{(2n-1)f_4-f_6)(a_42n(f_1-f_3)+a_7r)-a_0(f_4-f_6)(2nf_1+3f_2-f_3)}{a_0(f_4-f_6)-a_1((2n-1)f_4-f_6)}\}$$

and

$$B=-\{\frac{((2n-1)f_4-f_6)(-a_0+2n(a_2+a_3+a_5+a_6))(f_1-f_3)+a_7r)+(3f_2+(2n-1)f_3}{a_0(f_4-f_6)-a_1((2n-1)f_4-f_6)}\}.$$

This complete the proof of the theorem.

Definition 6. *A Riemaniann manifold (M,g) satisfying the condition [21,22]*

$$\phi^2(\tau(\phi X,\phi Y)\phi Z)=0, \tag{36}$$

is called ϕ-τ flat.

Suppose that the generalized (k,μ)-space form $M^{2n+1}(f_1,\cdots,f_6)$ is ϕ-τ-flat. Then

$$g(\tau(\phi X,\phi Y)\phi Z,\phi W)=0, \tag{37}$$

for all vector fields X,Y,Z and W.

Theorem 8. *A ϕ-τ flat generalized (k,μ)-space form $M^{2n+1}(f_1,.....f_6)$ is an η-Einstein manifold.*

Proof. We know that $M^{2n+1}(f_1,\cdots,f_6)$ is ϕ-τ flat, (33) can be written as

$$\begin{aligned} a_0\ g(R(\phi X,\phi Y)\phi Z,\phi W) = &-a_1 S(\phi Y,\phi Z)g(\phi X,\phi W) - a_2 S(\phi X,\phi Y)g(\phi Z,\phi W)\\ &- a_3 S(\phi X,\phi Y)g(\phi Z,\phi W) - a_4 g(\phi Y,\phi Z)S(\phi X,\phi W)\\ &- a_5 g(\phi X,\phi Z)S(\phi Y,\phi W) - a_6 g(\phi X,\phi Y)S(\phi Z,\phi W)\\ &- a_7 r[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{38}$$

Let $\{e_1,...,e_{n-1},\xi\}$ be a local orthonormal basis of vector fields in $M^{2n+1}(f_1,\cdots,f_6)$ and using the fact that $\{\phi e_1,...,\phi e_{n-1},\xi\}$ is also a local orthonormal basis, if we put $Y=Z=e_i$ in (38) and sum up with respect to i, then we get

$$\begin{aligned} a_0\sum_{i=1}^{2n} g(R(\phi X,\phi e_i)\phi e_i,\phi W) = &-a_1\sum_{i=1}^{2n} S(\phi e_i,\phi e_i)g(\phi X,\phi W) - a_2\sum_{i=1}^{2n} S(\phi X,\phi e_i)g(\phi e_i,\phi W)\\ &- a_3\sum_{i=1}^{2n} S(\phi X,\phi e_i)g(\phi e_i,\phi W) - a_4\sum_{i=1}^{2n} g(\phi e_i,\phi e_i)S(\phi X,\phi W)\\ &- a_5\sum_{i=1}^{2n} g(\phi X,\phi e_i)S(\phi e_i,\phi W) - a_6\sum_{i=1}^{2n} g(\phi X,\phi e_i)S(\phi e_i,\phi W)\\ &- a_7 r\sum_{i=1}^{2n}[g(\phi e_i,\phi e_i)g(\phi X,\phi W) - g(\phi X,\phi e_i)g(\phi e_i,\phi W)]. \end{aligned} \tag{39}$$

It can be easily verified that,

$$\sum_{i=1}^{2n} g(R(X,\phi e_i)\phi e_i,W) = S(\phi X,\phi W) + g(\phi X,\phi W), \tag{40}$$

$$\sum_{i=1}^{2n} g(e_i,e_i) = \sum_{i=1}^{2n} g(\phi e_i,\phi e_i) = 2n, \tag{41}$$

$$\sum_{i=1}^{2n} S(\phi e_i,\phi e_i) = r - 2n(f_1 - f_3), \tag{42}$$

$$\sum_{i=1}^{2n} g(X,\phi e_i)g(\phi e_i,W) = g(\phi X,\phi W). \tag{43}$$

Using (3), (8) and (40)-(43) in (39) we get

$$S(X,W) = Ag(X,W) + B\eta(X)\eta(W).$$

where

$$A = -\{\frac{a_0 + a_4(r - 2n(f_1 - f_3)) + a_7 r(2n-1)}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6}\}$$

and

$$B = \{\frac{a_0 + a_4(r - 2n(f_1 - f_3)) + a_7 r(2n-1) + 2n(f_1 - f_3)(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6}\}.$$

Hence the result.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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