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Jacobi polynomial solutions of Volterra integro-differential equations with weakly singular kernel

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Abstract: An advanced matrix method is formulated for the solution of Volterra integro-differential equations with weakly singular kernel by using orthogonal Jacobi polynomials. Employing this practical method, it is possible to solve various types of these equations routinely in a systematic fashion. An error estimation procedure is prescribed to estimate the error of the basic Jacobi method and then the error correction term is added to the basic method to obtain more accurate results. Four test experiments are provided to confirm the validity and systematic approach of the advanced method. These experiments also certified that this advanced method surpasses the basic Jacobi method, as well as several alternative approaches.

Keywords: Jacobi polynomials, Volterra integro-differential equation, weakly singular kernel, residual error function.

1 Introduction

1.1 Volterra integro-differential equations with weakly singular kernel

An mth order linear Volterra integro-differential equation with weakly singular kernel (WSVIDE) having variable coefficients can be expressed as [1,2]

$$\sum_{i=0}^{J} P_i(x)\,u^{(i)}(x) = \sigma(x) + \lambda_1 \int_0^x \frac{u^{(k)}(t)}{(x-t)^{\xi}}dt + \lambda_2 \int_0^x K(x,t)\,u^{(r)}(t)\,dt, \quad 0 \le x,t \le \beta \tag{1}$$

subject to the mixed conditions

$$\sum_{k=0}^{m-1} \left[a_{jk}u^{(k)}(0) + b_{jk}u^{(k)}(\beta)\right] = \varphi_j, \quad m = max(J,r,k), \ j = 0,\ 1,\ 2,\ \ldots,\ m-1 \tag{2}$$

where $P_i(x)$ and $\sigma(x)$ are known functions defined on the interval $0 \le x \le \beta$; a_{jk}, b_{jk}, φ_j, λ_1 and λ_2 are real or complex constants, $0 < \xi < 1$ and, $u(x)$ is the unknown function to be determined.

Eq. (1) contains both integral and differential operators and may take several forms as follows:

(i) $\lambda_1 = 0$ and $r = 0$;

$$\sum_{i=0}^{J} P_i(x)\,u^{(i)}(x) = \sigma(x) + \lambda_2 \int_0^x K(x,t)\,u(t)\,dt \tag{3}$$

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Eq. (3) is defined as Volterra integro-differential equation (VIDE). Many researchers have been interested in solving VIDEs by employing miscellaneous numerical methods [3,4].

(ii) $\lambda_1 = 0$, $r = 0$ and $P_0(x) = 1$ and all other $P_i(x)$ are zero;

$$u(x) = \sigma(x) + \lambda_2 \int_0^x K(x,t)\, u(t)\, dt, \quad 0 < x < \beta \tag{4}$$

Eq. (4) is called Volterra integral equation (VIE). Finding out numerical solutions for VIE has been a research area for mathematicians [5,6,7].

VIEs and VIDEs are faced in various problems in mechanics, physics and biology. Some of the common examples are the heat conduction problem, unsteady Poiseuille flow in a pipe, electroelasticity, population growth model, and diffusion problems [7].

(iii) $\lambda_2 = 0$, $k = 0$ and $P_i(x) = 0$ such that $0 \leq i \leq J$;

$$\sigma(x) = -\lambda_1 \int_0^x \frac{u(x)}{(x-t)^{\xi}} dt \tag{5}$$

Eq. (5) is defined as the Abel integral equation of the first kind [8]. Its exact solution for $\xi = 0.5$ is stated as

$$u(x) = \frac{-1}{\lambda_1 \pi} \int_0^x \frac{1}{\sqrt{x-t}} \frac{d\sigma(t)}{dt} \tag{6}$$

assuming $\sigma(0) = 0$ [9].

(iv) $\lambda_2 = 0$, $k = 0$, $P_0(x) = 1$ and $P_i(x) = 0$ such that $1 \leq i \leq J$;

$$u(x) = \sigma(x) + \lambda_1 \int_0^x \frac{u(x)}{(x-t)^{\xi}} dt, \quad 0 < \xi < 1 \tag{7}$$

Eq. (7) is called the Abel integral equation of the second kind [9,10]. Vanani and Soleymani [11] gave a numerical solution for Eq. (7) using Tau method.

Both types of the Abel integral equations constitute the model for many physical problems and are solved numerically by many scientists even in the last decade [8,9,10,11,12].

Since Eq. (1) covers Eqs. (3,7), its numerical solutions have become an important subject for researchers. Numerical solution algorithms have been given by collocation methods based on Bernstein polynomials [1] and Bessel polynomials [2] for $\xi = 0.5$. Tang [13] presented a numerical solution for the first order WSVIDEs by using spline collocation method.

The present study comes up with a matrix method on the basis of orthogonal Jacobi polynomials to numerically solve Eq. (1). Error estimation is exploited to enhance the accuracy of results. The advantage of the presented method is that it is practical and can be beneficial in solving many different problems.

1.2 Orthogonal Jacobi polynomials

The polynomial systems have always been extensively studied; however, many researchers focused especially on orthogonal polynomials and applied them to miscellaneous problems. Jacobi, Hermite and Laguerre systems are altogether named the classical orthogonal polynomials [14].
Jacobi polynomials can be characterized using the following relationship [14]

$$P_n^{(\zeta,\eta)}(x) = (-2)^n (n!)^{-1} (1-x)^{-\zeta} (1+x)^{-\eta} \frac{d^n}{dx^n} \left[(1-x)^{n+\zeta} (1+x)^{n+\zeta} \right] \tag{8}$$

In this equation, the parameters ζ and η are taken as $\zeta > -1$, $\eta > -1$ for integrability purposes. The following are the most essential cases.

(i) The Legendre polynomials $(\zeta = \eta = 0)$
(ii) The Chebyshev polynomials $(\zeta = \eta = -1/2)$
(iii) The Gegenbauer polynomials $(\zeta = \eta)$.

The Jacobi polynomials $P_n^{(\zeta,\eta)}(x)$ are characterized [15,16] with reference to weight function $w^{\zeta,\eta}(x) = (1-x)^{\zeta}(1+x)^{\eta}$ $(\zeta > -1, \eta > -1)$ on $(-1,1)$. On the other hand, these polynomials comply the relation [6]:

$$P_n^{(\zeta,\eta)}(x) = \sum_{k=0}^{n} B_n^{(\zeta,\eta,n)} (x-1)^k; \;\; \zeta,\eta > -1 \tag{9}$$

where

$$B_n^{(\zeta,\eta,n)} = 2^{-k} \binom{n+\zeta+\eta+k}{k} \binom{n+\zeta}{n-k}; \quad k = 0,\ 1,\ 2,\ \ldots,\ n \tag{10}$$

In the last decade, Jacobi polynomials have been used to design new numerical algorithms for particular issues. Eslahchi et al. [17] derived approximate solutions and presented explicit formulae for certain nonlinear ordinary differential equations. Bojdi et al. [18] came up with a spectral method for numerically solving boundary value problems involving differential-difference equations. Fractional-order differential equations were solved using these polynomials via the Tau method in [19]. Cai [20] and Behzadi [21] employed the Jacobi method for analyzing Fredholm integral and Fredholm-Volterra integro-differential equations, respectively. In the last few years, Bhrawy and co-workers [22] used Jacobi polynomials to solve numerically various problems by spectral method, tau method and collocation methods. The authors of this paper [23] applied the present method recently to Fredholm integro-differential-difference equations and obtained commendable results.

We propose the basic Jacobi solution in the form of truncated series of Jacobi polynomials

$$u(x) \cong u_N^{(\zeta,\eta)}(x) = \sum_{n=0}^{N} a_n P_n^{(\zeta,\eta)}(x) \tag{11}$$

where $P_n^{(\zeta,\eta)}(x)$, $n = 0,\ 1,\ 2,\ \ldots,\ N$ represent the orthogonal Jacobi polynomials described as in (9,10); N is any positive integer such that $N > n$; ζ and η are defined as arbitrary parameters such that $(\zeta > -1, \eta > -1)$. Our aim is to determine the unknown coefficients a_n, $n = 0,\ 1,\ 2,\ \ldots,\ N$.

2 Matrix representation of Jacobi Polynomials and conditions

The orthogonal Jacobi polynomials $P_n^{(\zeta,\eta)}(x)$ can be converted into matrix form in such a way that

$$\mathbf{P}^{(\zeta,\eta)}(x) = \mathbf{X}(x)\mathbf{D}^{(\zeta,\eta)} \tag{12}$$

where

$$\mathbf{P}^{(\zeta,\eta)}(x) = \left[P_0^{(\zeta,\eta)}(x)\; P_1^{(\zeta,\eta)}(x)\; \ldots\; P_N^{(\zeta,\eta)}(x)\right] \tag{13}$$

$$\mathbf{X}(x) = \left[1\; (x-1)\; (x-1)^2\; \ldots\; (x-1)^N\right] \tag{14}$$

and

$$\mathbf{D}^{(\zeta,\eta)} = \left[d_{ij}^{(\zeta,\eta)}\right]_{(N+1)\times(N+1)}, \quad 1 \le i, \quad j \le N+1 \tag{15}$$

such that $d_{ij}^{(\zeta,\eta)} = \begin{cases} 2^{1-i}\binom{i+j-2+\zeta+\eta}{i-1}\binom{j-1+\zeta}{j-i}, & i \le j \\ 0, & i > j \end{cases}$.

The assumed solution of Eq. (1) formulated in Eq. (11) can be presented as follows:

$$\left[u_N^{(\zeta,\eta)}(x)\right] = \mathbf{P}^{(\zeta,\eta)}(x)\,\mathbf{A} \tag{16}$$

where

$$\mathbf{A} = \left[a_0\; a_1\; \ldots\; a_N\right]^T \tag{17}$$

Substituting (12) into (16) yields

$$\left[u_N^{(\zeta,\eta)}(x)\right] = \mathbf{X}(x)\,\mathbf{D}^{(\zeta,\eta)}\mathbf{A} \tag{18}$$

Now we have to define the matrix form of derivatives $u^{(i)}(x)$ in terms of $u(x)$; for this purpose, the matrix $\mathbf{X}(x)$ is related with its derivatives $\mathbf{X}^{(i)}(x)$ as

$$\mathbf{X}^{(i)}(x) = \mathbf{X}(x)\,\mathbf{B}^i \tag{19}$$

where

$$\mathbf{B} = [b_{ij}]_{(N+1)\times(N+1)} \tag{20}$$

such that $b_{ij} = \begin{cases} i, & j-i=1 \\ 0, & others \end{cases}$. Using (18) and (19), we can write

$$\left[u^{(i)}(x)\right] \cong \left[\left(u_N^{(\zeta,\eta)}\right)^{(i)}(x)\right] = \mathbf{X}^{(i)}(x)\,\mathbf{D}^{(\zeta,\eta)}\mathbf{A} = \mathbf{X}(x)\,\mathbf{B}^i\mathbf{D}^{(\zeta,\eta)}\mathbf{A} \tag{21}$$

Likewise, we may define the delay form's derivatives in terms of the matrix form of $u(x)$

$$u^{(j)}(x-\tau) = \mathbf{X}(x)\,\mathbf{B}_\tau\mathbf{B}^j\mathbf{D}^{(\zeta,\eta)}\mathbf{A} \tag{22}$$

where $\mathbf{B}_\tau = \left[\widetilde{b}_{ij}\right]_{(N+1)\times(N+1)}$ such that

$$\widetilde{b}_{ij} = \begin{cases} \binom{j-1}{i-1}(-\tau)^{j-i}, & i \le j \\ 0, & i > j \end{cases} \tag{23}$$

The differential part of Eq. (1) can be put in the following matrix form using (21)

$$\sum_{i=0}^{J} P_i(x)\,u^{(i)}(x) = \sum_{i=0}^{J} P_i(x)\,\mathbf{X}(x)\,\mathbf{B}^i\mathbf{D}^{(\zeta,\eta)}\mathbf{A} \tag{24}$$

Then, for the part $\lambda_2 \int_0^x K(x,t)\, u^{(r)}(t)\, dt$ of Eq. (1)

$$\int_0^x K(x,t)\, u^{(r)}(t)\, dt = \mathbf{Q}(x)\,\mathbf{B}^r \mathbf{D}^{(\zeta,\eta)} \mathbf{A} \tag{25}$$

where $\mathbf{Q}(x) = [q_{ij}]_{1\times(N+1)}$ such that

$$q_{ij}(x) = \int_0^x K(x,t)\,(t-1)^{j-1} dt \tag{26}$$

Finally, the matrix form of the third part, $\lambda_1 \int_0^x \frac{u^{(k)}(t)}{(x-t)^{\xi}} dt$ of Eq. (1) may be described as

$$\int_0^x \frac{u^{(k)}(t)}{(x-t)^{\xi}} dt = \mathbf{Q_S}(x)\,\mathbf{B}_1 \mathbf{B}^k \mathbf{D}^{(\zeta,\eta)} \mathbf{A} \tag{27}$$

where $\mathbf{Q_S}(x) = [\widetilde{q}_{ij}(x)]_{1\times(N+1)}$ such that

$$\widetilde{q}_{ij}(x) = \int_0^x \frac{t^{j-1}}{(x-t)^{\xi}} dt \tag{28}$$

We can compose the matrix $\mathbf{Q_S}(x)$ using the formula [1]

$$\int_0^x \frac{t^m}{(x-t)^{\xi}} dt = \frac{\Psi(1-\xi)\,\Psi(m+1)}{\Psi(m+2-\xi)} x^{m+1-\xi} \tag{29}$$

as follows

$$\widetilde{q}_{ij}(x) = \frac{\Psi(1-\xi)\,\Psi(j)}{\Psi(j+1-\xi)} x^{j-\xi} \tag{30}$$

Finally, summing up (24), (25) and (27), the matrix form of Eq. (1) turns into

$$\sum_{i=0}^{J} P_i(x)\,\mathbf{X}(x)\,\mathbf{B}^i \mathbf{D}^{(\zeta,\eta)} \mathbf{A} = \sigma(x) + \lambda_1 \mathbf{Q_S}(x)\,\mathbf{B}_1 \mathbf{B}^k \mathbf{D}^{(\zeta,\eta)} \mathbf{A} + \lambda_2 \mathbf{Q}(x)\,\mathbf{B}^r \mathbf{D}^{(\zeta,\eta)} \mathbf{A} \tag{31}$$

The mixed conditions (2) should also be converted into matrix form; in doing this, we utilize expression (24) to get

$$\sum_{k=0}^{m-1} \left[a_{jk} u^{(k)}(0) + b_{jk} u^{(k)}(\beta)\right] = \sum_{k=0}^{m-1} \left[a_{jk} \mathbf{X}(0) + b_{jk} \mathbf{X}(\beta)\right] \mathbf{B}^i \mathbf{D}^{(\zeta,\eta)} \mathbf{A} = [\varphi_j],\quad j = 0,\ 1,\ 2,\ \ldots,\ m-1 \tag{32}$$

3 Solution methodology

Start with defining the collocation points as

$$x_h = \frac{\beta}{N} h,\quad h = 0, 1,\ 2,\ \ldots,\ N \tag{33}$$

Reorganizing Eq. (31) using these collocation points brings out

$$\sum_{i=0}^{J} P_i(x_h)\,\mathbf{X}(x_h)\,\mathbf{B}^i \mathbf{D}^{(\zeta,\eta)} \mathbf{A} = \sigma(x_h)\,\mathbf{B}_1 \mathbf{B}^k \mathbf{D}^{(\zeta,\eta)} \mathbf{A} + \lambda_2 \mathbf{Q}(x_h)\,\mathbf{B}^r \mathbf{D}^{(\zeta,\eta)} \mathbf{A},\ h = 0, 1,\ 2,\ \ldots,\ N.$$

Therefore, the fundamental matrix equation develops into

$$\left\{\sum_{i=0}^{J} \mathbf{P}_i \mathbf{X}\mathbf{B}^i \mathbf{D}^{(\zeta,\eta)} - \lambda_1 \mathbf{Q_S}\mathbf{B}_1\mathbf{B}^k \mathbf{D}^{(\zeta,\eta)} - \lambda_2 \mathbf{Q}\mathbf{B}^l \mathbf{D}^{(\zeta,\eta)}\right\}\mathbf{A} = \sigma \tag{34}$$

where $\mathbf{P}_i = \begin{bmatrix} P_i(x_0) & 0 & \cdots & 0 \\ 0 & P_i(x_1) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & P_i(x_N) \end{bmatrix}$, $\sigma = \begin{bmatrix} \sigma(x_0) \\ \sigma(x_1) \\ \vdots \\ \sigma(x_N) \end{bmatrix}$, $\mathbf{Q_S} = \begin{bmatrix} \mathbf{Q_S}(x_0) \\ \mathbf{Q_S}(x_1) \\ \vdots \\ \mathbf{Q_S}(x_N) \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} \mathbf{Q}(x_0) \\ \mathbf{Q}(x_1) \\ \vdots \\ \mathbf{Q}(x_N) \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix}$.

Eq (31) denotes a system of $N+1$ algebraic equations having $N+1$ unknown coefficients. Therefore, if we determine

$$\mathbf{MA} = \sigma or \left[\mathbf{M};\sigma\right] \tag{35}$$

such that

$$\mathbf{M} = \sum_{i=0}^{J} \mathbf{P}_i \mathbf{X}\mathbf{B}^i \mathbf{D}^{(\zeta,\eta)} - \lambda_1 \mathbf{Q_S}\mathbf{B}_1\mathbf{B}^k \mathbf{D}^{(\zeta,\eta)} - \lambda_2 \mathbf{Q}\mathbf{B}^r \mathbf{D}^{(\zeta,\eta)}.$$

At the same time, from (32), one may derive the matrix form of conditions shortly as

$$\mathbf{U}_j\mathbf{A} = \varphi_j or \left[\mathbf{U}_j;\varphi_j\right], \qquad j = 0,\ 1,\ 2,\ \ldots,\ m-1 \tag{36}$$

such that

$$\mathbf{U}_j = \sum_{k=0}^{m-1}\left[a_{jk}\mathbf{X}(0) + b_{jk}\mathbf{X}(\beta)\right]\mathbf{B}^i\mathbf{D}^{(\zeta,\eta)}\mathbf{A} = \left[\varphi_j\right].$$

Subsequently, in order to deduce the solution of Eq. (1) subject to the mixed conditions (2), the matrix (36) is substituted for the last n rows of our augmented matrix (35); this yields the desired matrix form

$$\left[\widetilde{\mathbf{M}};\widetilde{\sigma}\right] \tag{37}$$

Provided $rank\widetilde{\mathbf{M}} = rank\left[\widetilde{\mathbf{M}};\widetilde{\sigma}\right] = N+1$, one may set $\mathbf{A} = \left(\widetilde{\mathbf{M}}\right)^{-1}\widetilde{\sigma}$. In this regard, the matrix $\mathbf{A}$ (so that $a_0,\ a_1,\ a_2,\ \cdots,\ a_N$) is exactly evaluated and Eq. (1) has a unique solution under the conditions (2). Therefore, we obtain the desired Jacobi polynomial solution.

4 Advanced solution by error estimation

Now, we develop a practical error estimation procedure for the basic Jacobi polynomial approximation as well as an approach to achieve an advanced (higher accuracy) solution for the problem (1, 2) by utilizing the residual correction method [24,25] and error estimation via the Tau method [26,27].

Baykuş and Sezer [28] introduced a hybrid Taylor Lucas collocation method to be able to retrieve approximations for pantograph type delay differential equations benefiting residual error function. Yüzbaşı et al. [29] presented a modified Legendre method in order to numerically solve some integro-differential equations. For the sake of determining an advanced solution, the residual function is described as

$$R_N(x) = \sum_{i=0}^{J} P_i(x)\left(u_N^{(\zeta,\eta)}\right)^{(i)}(x) - \lambda_1 \int_0^x \frac{\left(u_N^{(\zeta,\eta)}\right)^{(k)}(t)}{(x-t)^{\xi}}dt - \lambda_2 \int_0^x K(x,t)\left(u_N^{(\zeta,\eta)}\right)^{(r)}(x)\,dt - \sigma(x) \tag{38}$$

where $u_N^{(\zeta,\eta)}(x)$ is the approximate solution of Eqs. (1, 2). Thus $u_N^{(\zeta,\eta)}(x)$ satisfies

$$\begin{gathered} \sum_{i=0}^{J} P_i(x)\left(u_N^{(\zeta,\eta)}\right)^{(i)}(x) - \lambda_1 \int_0^x \frac{\left(u_N^{(\zeta,\eta)}\right)^{(k)}(t)}{(x-t)^{\xi}}dt - \lambda_2 \int_0^x K(x,t)\left(u_N^{(\zeta,\eta)}\right)^{(r)}(x)\,dt = \sigma(x) + R_N(x) \\ \sum_{k=0}^{m-1}\left[a_{jk}\left(u_N^{(\zeta,\eta)}\right)^{(k)}(0) + b_{jk}\left(u_N^{(\zeta,\eta)}\right)^{(k)}(\beta)\right] = \varphi_j,\ j=0,\ 1,\ 2,\ \ldots,\ m-1 \end{gathered} \tag{39}$$

Accordingly, Jacobi error function $e_N{}^{(\zeta,\eta)}(x)$ may be delineated as

$$e_N{}^{(\zeta,\eta)}(x) = u(x) - u_N^{(\zeta,\eta)}(x) \tag{40}$$

where $u(x)$ is the exact solution of Eqs. (1, 2). If we substitute Eq. (40) into Eqs. (1, 2) and employ (38) and (39), then we may define an error differential equation having the following homogeneous conditions:

$$\begin{gathered} \sum_{i=0}^{J} P_i(x)\left(e_N^{(\zeta,\eta)}\right)^{(i)}(x) - \lambda_1 \int_0^x \frac{\left(e_N^{(\zeta,\eta)}\right)^{(k)}(t)}{(x-t)^{\xi}}dt - \lambda_2 \int_0^x K(x,t)\left(e_N^{(\zeta,\eta)}\right)^{(r)}(x)\,dt = -R_N(x) \\ \sum_{k=0}^{m-1}\left[a_{jk}\left(u_N^{(\zeta,\eta)}\right)^{(k)}(0) + b_{jk}\left(u_N^{(\zeta,\eta)}\right)^{(k)}(\beta)\right] = 0 \end{gathered} \tag{41}$$

The solution of Eq. (41), which can be derived in a similar manner as in the previous section, gives the approximation $e_{N,M}{}^{(\zeta,\eta)}(x)$ to $e_N{}^{(\zeta,\eta)}(x)$. This approximation may be called as the error function based upon the residual function $R_N(x)$. Note that for the approximation to be valid, $M > N$ must be satisfied. Therefore, we achieve the advanced Jacobi solution as

$$u_{N,M}^{(\zeta,\eta)}(x) = u_N^{(\zeta,\eta)}(x) + e_{N,M}{}^{(\zeta,\eta)}(x) \tag{42}$$

where, $e_{N,M}{}^{(\zeta,\eta)}(x)$ is the estimated error function. Eventually, the corrected/advanced Jacobi error function is

$$E_{N,M}^{(\zeta,\eta)}(x) = u(x) - u_{N,M}^{(\zeta,\eta)}(x). \tag{43}$$

5 Numerical examples

Here, we implement the advanced Jacobi method to four examples by using symbolic computational programing [30].

Example 1. We consider the linear WSVIDE [2]

$$\begin{gathered} u''(x) + x^2 u(x) = 6x - \tfrac{1}{2}x^4 - \tfrac{16}{5}x^{\frac{5}{2}} + \int_0^x \frac{u\prime(t)}{\sqrt{x-t}}dt - \int_0^x xtu\prime\prime(t)\,dt \\ u(0) = 1,\ u\prime(0) = 0,\ \ 0 \le x,\ t \le 1 \end{gathered} \tag{44}$$

with the exact solution $u(x) = x^3 + 1$. We assume that, for $N = 4$ and $(\zeta,\eta) = (0.5, 0)$ which are chosen arbitrarily, the Jacobi polynomial solution is

$$u(x) = a_0 P_0^{(\zeta,\eta)}(x) + a_1 P_1^{(\zeta,\eta)}(x) + a_2 P_2^{(\zeta,\eta)}(x) + a_3 P_3^{(\zeta,\eta)}(x) + a_4 P_4^{(\zeta,\eta)}(x)$$

such that

$$\mathbf{P}^{(\zeta,\eta)}(x)=\begin{bmatrix} 1 \\ \frac{1}{4}+\frac{5x}{4} \\ -\frac{5}{2}+\frac{35}{8}x+\frac{63(x-1)^2}{32} \\ -\frac{245}{32}+\frac{315}{32}x+\frac{693(x-1)^2}{64}+\frac{429(x-1)^3}{128} \\ -\frac{1995}{128}+\frac{1155}{64}x+\frac{90009(x-1)^2}{256}+\frac{6435(x-1)^3}{256}+\frac{12155(x-1)^4}{2048} \end{bmatrix}^{\mathbf{T}}$$

The collocation points are assigned as

$$\left\{x_0=0,\ x_1=\frac{1}{4},\ x_2=\frac{2}{4},\ x_3=\frac{3}{4},\ x_4=1\right\}$$

while the fundamental matrix equation for problem, due to Eq. (34), is

$$\left\{\mathbf{P}_2\mathbf{XB}^2\mathbf{D}+\mathbf{P}_1\mathbf{BXD}-\lambda_1\mathbf{Q_S}\mathbf{B}_1\mathbf{BD}-\lambda_2\mathbf{QB}^2\mathbf{D}\right\}\mathbf{A}=\sigma$$

where

$$\mathbf{Q_S}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{2}{2} & \frac{2}{12} & \frac{2}{60} & \frac{2}{280} & \frac{2}{1260} \\ \frac{\sqrt{2}}{1} & \frac{\sqrt{2}}{3} & \frac{2\sqrt{2}}{15} & \frac{2\sqrt{2}}{35} & \frac{8\sqrt{2}}{315} \\ \frac{2\sqrt{3}}{2} & \frac{2\sqrt{3}}{4} & \frac{6\sqrt{3}}{20} & \frac{54\sqrt{3}}{280} & \frac{18\sqrt{3}}{140} \\ \frac{2}{1} & \frac{4}{3} & \frac{16}{15} & \frac{32}{35} & \frac{256}{15} \end{bmatrix},\ \mathbf{Q}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{128} & \frac{-5}{768} & \frac{67}{12288} & \frac{-47}{10240} & \frac{1909}{491520} \\ \frac{1}{16} & \frac{-1}{24} & \frac{11}{384} & \frac{-13}{640} & \frac{19}{1280} \\ \frac{27}{128} & \frac{-27}{256} & \frac{243}{4096} & \frac{-189}{5120} & \frac{4077}{163840} \\ \frac{1}{2} & \frac{-1}{6} & \frac{1}{12} & \frac{-1}{20} & \frac{1}{30} \end{bmatrix}$$

Thereupon, the matrix $\mathbf{M}$ is shaped as

$$\mathbf{M}=\begin{bmatrix} 0 & 0 & \frac{63}{16} & \frac{99}{64} & -\frac{4719}{512} \\ 0 & -\frac{85}{64} & \frac{22883}{8192} & \frac{2566821}{327680} & -\frac{40305309}{58720256} \\ 0 & -\frac{5}{16}-\frac{5\sqrt{2}}{4} & \frac{3731}{1024}-\frac{7\sqrt{2}}{4} & \frac{24297}{2048}-\frac{63\sqrt{2}}{640} & \frac{3482941}{262144}+\frac{3267\sqrt{2}}{1792} \\ 0 & \frac{45}{64}-\frac{10\sqrt{3}}{8} & \frac{25137}{8192}-\frac{77\sqrt{3}}{32} & \frac{1103805}{65536}-\frac{2601\sqrt{3}}{1280} & \frac{322028223}{8388608}-\frac{8041\sqrt{3}}{14336} \\ 0 & -\frac{15}{4} & -\frac{525}{128} & \frac{30429}{2560} & \frac{3427457}{57344} \end{bmatrix}$$

and

$$\sigma=\begin{bmatrix} 0 & \frac{3579}{2560} & \frac{95}{32}-\frac{2\sqrt{2}}{5} & \frac{2223}{512}-\frac{18\sqrt{3}}{20} & \frac{23}{10} \end{bmatrix}^{\mathbf{T}}$$

Using (36), we may pose the matrix form of our initial conditions as

$$\begin{bmatrix}\mathbf{U}_0;\varphi_0\\ \mathbf{U}_1;\varphi_1\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{4} & -\frac{17}{32} & -\frac{23}{128} & \frac{827}{2048} & ; 1\\ 0 & \frac{5}{4} & \frac{7}{16} & -\frac{225}{128} & -\frac{341}{512} & ; 0\end{bmatrix}$$

Consequently, to derive the solution of Eq. (44), we replace the third and fourth rows of augmented matrix $[\mathbf{M};\sigma]$ by the matrix $[\mathbf{U};\varphi]$, and find out the augmented matrix $\left[\widetilde{\mathbf{M}};\widetilde{\sigma}\right]$. Solving this augmented matrix yields the Jacobi polynomial coefficient matrix

$$\mathbf{A}=\begin{bmatrix}\frac{92}{105} & \frac{76}{165} & -\frac{32}{273} & \frac{128}{429} & 0\end{bmatrix}^T$$

Therefore the Jacobi polynomial solution can be deduced using Eq. (11) as $u_4^{(0.5,0)}(x)=2+3(x-1)+3(x-1)^2+(x-1)^3$; this is the exact solution for the problem. Likewise, different values of ζ and η can be utilized to extract the same solution for $N=5$.

Example 2. We consider the WSVIDE [1,13]

$$u''(x)+u(x)+\frac{1}{\sqrt{\pi}}\int_0^x\frac{u''(t)}{\sqrt{x-t}}dt=f(x),u(0)=u'(0)=1 \tag{45}$$

Case I. In Refs. [1,13], $f(x)$ was chosen as $3+x+x^2+\frac{4\sqrt{x}}{\sqrt{\pi}}$, and the exact solution was $1+x+x^2$. For $\zeta=\eta=-\frac{1}{2}$ (Chebyshev base) and $N=3$, we obtain Jacobi characteristic matrix $\mathbf{D}$ and the Jacobi polynomial coefficient matrix $\mathbf{A}$ as follows

$$\mathbf{D}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}=\begin{bmatrix}1 & \frac{4}{5} & \frac{18}{25} & \frac{84}{125}\\ 0 & \frac{4}{5} & \frac{117}{50} & \frac{567}{125}\\ 0 & 0 & \frac{117}{100} & \frac{1449}{250}\\ 0 & 0 & 0 & \frac{483}{250}\end{bmatrix},\ \mathbf{A}=\begin{bmatrix}\frac{18}{13} & \frac{5}{4} & \frac{100}{117} & 0\end{bmatrix}^T \tag{46}$$

From Eqs. (18) and (46) the Jacobi polynomial solution of the problem is determined as $u_3^{(-0.5,-0.5)}(x)=3+3(x-1)+(x-1)^2$; this is the exact solution of Eq. (45).

Case II. If $f(x)$ is chosen $2e^x+erf(\sqrt{x})e^x$, the exact solution is given as $u(x)=e^x$ in Refs. [1,13]. The approximate solution of the WSVIDE obtained by the basic Jacobi method is

$$\begin{aligned}u_4^{(-0.5,-0.5)}(x)=&0.1437720452e-1+2.701787249x+1.308375018(x-1)^2\\&+0.3760461068(x-1)^3+0.5329388356e-1(x-1)^4\end{aligned}$$

and the error estimate function is

$$\begin{aligned}e_{4,5}^{(-0.5,-0.5)}(x)=&-0.1297755894e-1+0.1495374255e-1x+0.4422748836e-1(x-1)^2+0.06272184490(x-1)^3\\&+0.4269527684e-1(x-1)^4++0.01122336138(x-1)^5\end{aligned}$$

Finally, we forecast the advanced solution function by adding up the approximate solution and the error estimate function as follows:

$$u_{4,5}^{(-0.5,-0.5)}(x) = u_4^{(-0.5,-0.5)}(x) + e_{4,5}^{(-0.5,-0.5)}(x) = 0.139964558e-2 + 2.716740992x + 1.352602506(x-1)^2 + 0.4387679517(x-1)^3 + 0.09598916040(x-1)^4 + 0.1122336138e-1(x-1)^5$$

By taking $\zeta = -0.5$ and $\eta = -0.5$, we then compare the absolute errors of the basic Jacobi solution for $N = 4$ and those of the advanced solution for $M = 5,\ 6$ with the error values of the Taylor expansion method [13] and the Bernstein method [1,13] in Table 1. The absolute error values of the basic Jacobi solution are the same as those of the Bernstein method; anyhow, the basic solution yields better results than Taylor expansion method. Furthermore, our advanced method gives better results than Taylor expansion, Bernstein and also Jacobi polynomial solution methods.

Table 1: Comparison of the absolute errors of the advanced Jacobi method with some other numerical methods for Example 2

	Taylor expansion method [13]	Bernstein method [1]	Present method $(\zeta = -0.5,\ \eta = -0.5)$		
x_i	$N = 4$	$N = 4$	$\left\lvert e_4^{(\zeta,\eta)}(x)\right\rvert$	$\left\lvert E_{4,5}^{(\zeta,\eta)}(x)\right\rvert$	$\left\lvert E_{4,6}^{(\zeta,\eta)}(x)\right\rvert$
0	0	0	0	0	0
0.2	$1.372e-3$	$1.452e-5$	$1.452e-5$	$9.874e-7$	$5.405e-8$
0.4	0.0015	$3.665e-5$	$3.665e-5$	$1.835e-6$	$9.458e-8$
0.6	0.0063	$3.186e-5$	$3.186e-5$	$3.087e-6$	$1.438e-7$
0.8	0.0172	$3.220e-4$	$3.220e-4$	$4.541e-6$	$3.160e-7$
1	0.0369	$2.117e-3$	$2.117e-3$	$1.411e-4$	$8.798e-6$

In Table 2, the actual absolute error values are compared with the estimated absolute error values derived by the error estimation algorithm for $N = 4$ and $M = 5,\ 6,\ 7,\ 8$. It is seen that the error estimation algorithm gives almost the same values as the actual absolute error.

Table 2: Actual absolute error and estimated absolute errors comparison for Example 2

	Actual Absolute Error	Estimated Absolute Errors			
x_i	$\left\lvert e_4^{(\zeta,\eta)}(x)\right\rvert$	$\left\lvert e_{4,5}^{(\zeta,\eta)}(x)\right\rvert$	$\left\lvert e_{4,6}^{(\zeta,\eta)}(x)\right\rvert$	$\left\lvert e_{4,7}^{(\zeta,\eta)}(x)\right\rvert$	$\left\lvert e_{4,8}^{(\zeta,\eta)}(x)\right\rvert$
0	0	0	0	0	0
0.2	$1.452e-5$	$1.551e-5$	$1.446e-5$	$1.452e-5$	$1.452e-5$
0.4	$3.665e-5$	$3.849e-5$	$3.656e-5$	$3.666e-5$	$3.665e-5$
0.6	$3.186e-5$	$3.495e-5$	$3.172e-5$	$3.187e-5$	$3.186e-5$
0.8	$3.220e-4$	$3.174e-4$	$3.217e-4$	$3.220e-4$	$3.220e-4$
1	$2.117e-3$	$1.976e-2$	$2.108e-3$	$2.116e-3$	$2.117e-3$

Figure 1 compares the basic Jacobi solution $u_N(x)$ for $N = 2$ and the advanced Jacobi solutions $u_{N,\,M}(x)$ for $N = 2$ and $M = 3,\ 4,\ 5$ with the exact solution $u(x)$ of Eq. (45). We can see from the figure that the advanced Jacobi polynomial solutions provide more convergent results than Jacobi polynomial solution. When M increases, advanced Jacobi polynomial solution yields better results.

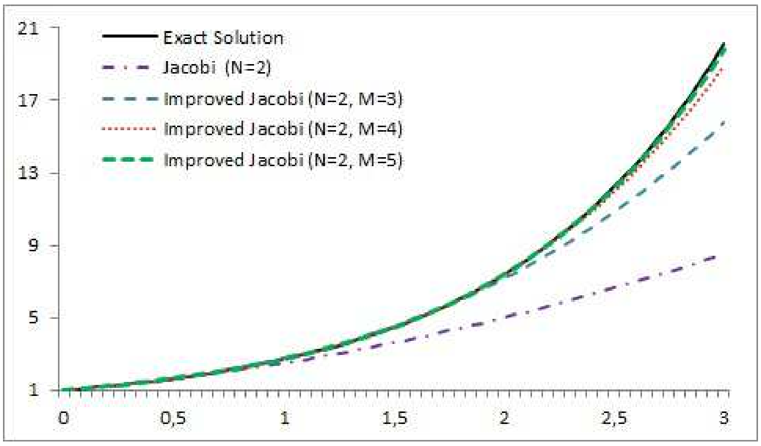

Fig. 1: Comparison of the basic Jacobi and the advanced Jacobi polynomial solutions with the exact solution for Example 2

Example 3. Consider the WSVIE [11],

$$u(x) = e^x\left(1+\sqrt{\pi}\, erf\left(\sqrt{x}\right)\right) - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt,\ x\in[0,1] \tag{47}$$

where $erf(x)$ is the error function and the exact solution is $u(x) = e^x$. In this example, Jacobi parameters are taken arbitrarily as $\zeta = -0.2$, $\eta = 0.3$. When $N = 3$ is chosen, the Jacobi polynomial solution is obtained as follows.

$$u_3^{(-0.2,0.3)}(x) = 0,0208136421 + 2,697072576x + 1,251781926 + 0,2725955734(x-1)^3$$

By using error estimation algorithm given in Section 4, the error estimation function is obtained as;

$$\begin{aligned} e_{3,5}^{(-0.2,0.3)}(x) = &-0,02069494154 + 0,0210895476x + 0,1061028506(x-1)^2 + 0,1754317228(x-1)^3 \\ &+ 0,1037098965(x-1)^4 + 0,01368607729(x-1)^5 \end{aligned}$$

The absolute error estimation function that we found above coincides with the absolute error function of the Jacobi polynomial method, as seen in Figure 2. By adding the error estimation function to the basic Jacobi solution, we obtain the advanced Jacobi polynomial solution as follows,

$$\begin{aligned} u_{3,5}^{(-0.2,0.3)}(x) = &u_3^{(-0.2,0.3)}(x) + e_{3,5}^{(-0.2,0.3)}(x) = 0,00011870056 + 2,718162124x + 1,357884777(x-1)^2 \\ &+ 0,4480272962(x-1)^3 + 0,1037098965(x-1)^4 + 0,01368607729(x-1)^5 \end{aligned}$$

Better results are achieved by this advanced Jacobi polynomial solution when compared with the basic Jacobi solution method, taking into consideration the same number of terms. Figure 3 illustrates a comparison of absolute errors of the

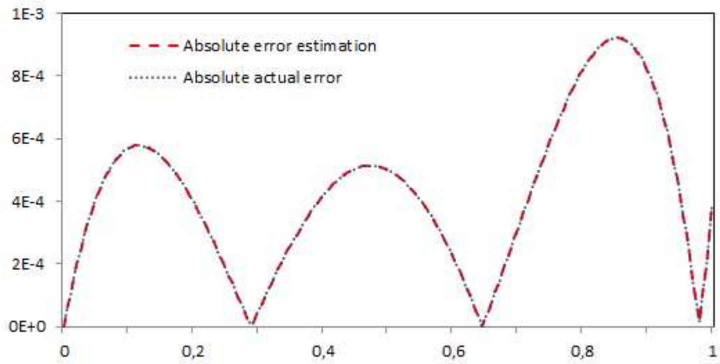

Fig. 2: Comparison of the absolute actual error $\left|e_3^{(-0.2,0.3)}(x)\right|$ with the absolute error estimation $\left|e_{3,5}^{(-0.2,0.3)}(x)\right|$ for Example 3

basic and advanced Jacobi methods.

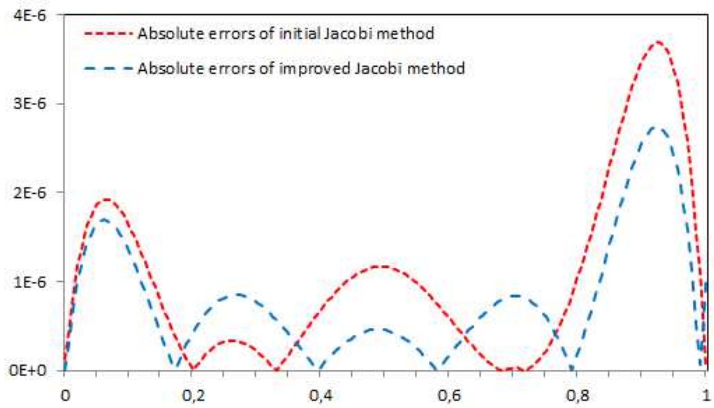

Fig. 3: Comparison of the absolute actual errors of the basic Jacobi solution $\left|e_5^{(-0.2,0.3)}(x)\right|$ with the advanced Jacobi solution $\left|E_{3,5}^{(-0.2,0.3)}(x)\right|$ for Example 3

The maximum error for the basic Jacobi method $u_N^{(\zeta,\eta)}$ can be assessed as,

$$E_N^{(\zeta,\eta)} = ||u_N^{(\zeta,\eta)}(x) - u(x)||_\infty = max\left\{\left|u_N^{(\zeta,\eta)}(x) - u(x)\right|, a \leq x \leq \beta\right\} \tag{48}$$

Table 3 exhibits the maximum errors $E_N^{(\zeta,\eta)}$ for specific values of N where the maximum error reduces rapidly as N increases.

By modifying Eq. (48), we define maximum error for advanced Jacobi method $u_{N,M}^{(\zeta,\eta)}$ as,

$$E_{N,M}^{(\zeta,\eta)} = ||u_{N,M}^{(\zeta,\eta)}(x) - u(x)||_\infty == max\left\{\left|u_{N,M}^{(\zeta,\eta)}(x) - u(x)\right|, a \leq x \leq \beta\right\} \tag{49}$$

Table 3: The maximum absolute error of basic Jacobi method $E_N^{(-0.2,0.3)}$ for Example 3

N	5	10	15	20	25	30
$E_N^{(-0.2,0.3)}$	$3.51e-06$	$1.74e-12$	$7.56e-18$	$2.62e-23$	$8.99e-29$	$2.21e-34$

Table 4 displays the maximum errors $E_{N,M}^{(\zeta,\eta)}$ for specific values of M and a constant value of N.
The maximum errors reduce rapidly as M increases. Comparing the values in Tables 3 and 4, it seen that the maximum errors of advanced Jacobi polynomial solution are less than those of the basic Jacobi solution.

Table 4: The maximum error of advanced Jacobi method $E_{N,M(=N+2)}^{(-0.2,0.3)}$ for Example 3

N	3	8	13	18	23	27
$E_{N,(N+2)}^{(\zeta,\eta)}$	$2.74e-06$	$5.15e-14$	$1.75e-21$	$3.89e-30$	$2.79e-39$	$8.06e-49$

The absolute errors of the basic and advanced Jacobi methods are compared with Tau method in Table 5.

Table 5: Comparison of the absolute errors of the advanced Jacobi method with other numerical method for Example 3

	Tau method [11]			Present method	
n	Standart base	Chebysev base	Legendre base	Jacobi base $(\zeta=-0.2,\eta=0.3)$ $E_n^{(-0.2,0.3)}$	$E_{n-2,n}^{(-0.2,0.3)}$
5	$6.73e-04$	$3.21e-04$	$3.13e-04$	$3.51e-06$	$2.74e-06$
10	$3.81e-07$	$2.09e-07$	$2.22e-07$	$1.74e-12$	$5.15e-14$
15	$5.41e-10$	$3.54e-10$	$2.98e-10$	$7.56e-18$	$1.75e-21$
20	$6.48e-12$	$4.11e-12$	$4.59e-12$	$2.62e-23$	$3.89e-30$
25	$7.14e-14$	$6.01e-14$	$5.86e-14$	$8.99e-29$	$2.79e-39$
30	$4.02e-15$	$2.61e-15$	$2.44e-15$	$2.21e-34$	$8.06e-49$

Example 4. Consider the linear VIDE [2]

$$u^{(4)}(x)-u(x)=-\frac{2}{3}+\frac{1}{3}x+\frac{11}{3}e^x-\frac{1}{9}x^3e^x+\frac{1}{3}\int_0^x te^{x-t}u(t)\,dt,\ 0\le x,t\le 1 \tag{50}$$

subject to the initial conditions
$u(0)=1,\ u\prime(0)=1,\ u\prime\prime(0)=2,\ u\prime\prime\prime(0)=3.$

with the exact solution $u(x)=1+xe^x$. Eq. (50) is solved by the advanced Jacobi method under the given initial conditions. Table 6 compares the results with those of the Bessel collocation method.

Table 6: Comparison of the absolute maximum errors of Bessel collocation method and advanced Jacobi method ($\zeta = \eta = 0$)

N	8	9	10	11	12
Bessel Base [2]	$1.05e-5$	$1.05e-5$	$2.80e-8$	$1.21e-9$	$8.76e-12$
Jacobi Base	$1.64e-6$	$7.30e-8$	$3.66e-9$	$1.28e-10$	$5.24e-12$

6 Conclusion

We advanced the previously used Jacobi matrix method for WSVIDEs. Thereupon, we transformed the orthogonal Jacobi polynomials from algebraic form to matrix form and substituted them into the WSVIDE together with the mixed conditions. Hence, the matrix form of WSVIDE was obtained, and the desired approximate solution was extracted by performing various matrix operations. Furthermore, an efficient error estimation algorithm is developed and it is employed to obtain a corrected/advanced solution.

The majority of the former research deals with approximations using Gegenbauer, Chebyshev, and Legendre polynomials. On the contrary, the present study introduces an advanced Jacobi polynomial solution which comprehends these polynomial solutions entirely.

Numerical test examples are given to illustrate the accuracy and the implementation of the method; the results support the claim. The accuracy of our advanced solution can be further improved exploiting the proposed error estimation algorithm based on residual function.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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