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On $\mathscr{I}_\sigma$-convergence of folner sequence on amenable semigroups

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Abstract: In this paper, the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$ of positive integers and corresponding $\mathscr{I}_\sigma$-convergence of functions defined on discrete countable amenable semigroups were introduced. Furthermore, for any Folner sequence inclusion relations between $\mathscr{I}_\sigma$-convergence and invariant convergence also $\mathscr{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence were given. We introduce the concept of $\mathscr{I}_\sigma$-statistical convergence and $\mathscr{I}_\sigma$-lacunary statistical convergence of functions defined on discrete countable amenable semigroups. In addition to these definitions, we give some inclusion theorems. Also, we make a new approach to the notions of $[V,\lambda]$-summability, σ-convergence and λ-statistical convergence of Folner sequences by using ideals and introduce new notions, namely, $\mathscr{I}_\sigma$-$[V,\lambda]$-summability, $\mathscr{I}_\sigma$-λ-statistical convergence of Folner sequences. We mainly examine the relation between these two methods as also the relation between $\mathscr{I}_\sigma$-statistical convergence and $\mathscr{I}_\sigma$-λ-statistical convergence of Folner sequences introduced by the author recently.

Keywords: Folner sequence, amenable group, inferior, superior, $\mathscr{I}$-convergence.

1 Introduction

Statistical convergence of sequences of points was introduced by Fast [5]. Schoenberg [27] established some basic properties of statistical convergence and also studied the concept as a summability method.

The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in K\}|,$$

where $|k \le n : k \in K|$ denotes the number of elements of K not exceeding n.

A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

In this case we write $st - \lim x_k = L$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = L$, then $st - \lim x_k = L$. The converse does not hold in general.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

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The concept of lacunary statistical convergence was defined by Fridy and Orhan [6]. Also, Fridy and Orhan gave the relationships between the lacunary statistical convergence and the Cesàro summability.

A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$ the set

$$K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has lacunary density zero, i.e. $\delta_\theta(K_\varepsilon) = 0$. In this case we write $S_\theta - \lim x_k = L$ or $x_k \to L(S_\theta)$. That is,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \left(\sigma^{m-1}(n)\right)$, $m = 1,2,3,\ldots$. A continuous linear functional Φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

(1) $\Phi(x) \geq 0$, for all sequences $x = (x_n)$ with $x_n \geq 0$ for all n;
(2) $\Phi(e) = 1$, where $e = (1,1,1,\ldots)$;
(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x)$ for all $x \in l_\infty$.

The mapping Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus, Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x) = \lim x$, for all $x \in c$. In case σ is translation mapping $\sigma(n) = n+1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

It can be shown that

$$V_\sigma = \left\{x = (x_n) : \lim_n t_{mn}(x) = L, \text{ uniformly in } m,\ L = \sigma - \lim x\right\},$$

where

$$t_{mn}(x) = \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \ldots + x_{\sigma^n(m)}}{n+1}$$

The concept of strongly σ-convergence was defined by Mursaleen in [18].

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{\infty} \left|x_{\sigma^k(n)} - L\right| = 0$$

uniformly in n. In this case we will write $x_k \to L[V_\sigma]$.

Savaş and Nuray [22] introduced the concepts of σ–statistically convergence and lacunary σ-statistically convergence and gave some inclusion relations.

A family of sets $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(1) $\emptyset \in \mathscr{I}$,
(2) For each $A, B \in \mathscr{I}$ we have $A \cup B \in \mathscr{I}$,
(3) For each $A \in \mathscr{I}$ and each $B \subseteq A$ wehave $B \in \mathscr{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathscr{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$.
A family of sets $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is a filter if and only if

(i) $\emptyset \notin \mathscr{F}$,
(ii) For each $A, B \in \mathscr{F}$ we have $A \cap B \in \mathscr{F}$,
(iii) For each $A \in \mathscr{F}$ and each $B \supseteq A$ we have $B \in \mathscr{F}$.

If $\mathscr{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathscr{I}$), then the family of sets

$$\mathscr{F}(\mathscr{I}) = \{M \subset \mathbb{N} : \exists A \in \mathscr{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ it is called the filter associated with the ideal. Filter is a dual notion of ideal.

The notion of ideal convergence was introduced first by Kostyrko et al. [10] as a generalization of statistical convergence [11,5]. More applications of ideals can be found in [12,13].

In another direction the idea of λ-statistical convergence was introduced and studied by Mursaleen [17] as an extension of the $[V,\lambda]$ summability of [14].

Let $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Valee-Pousin meaan is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be $(V,\lambda)-$summable to a number L if

$$\lim_{n \to \infty} t_n(x) = L.$$

If $\lambda_n = n$, then $(V,\lambda)-$summability reduces to $(C,1)-$summability. We write

$$[C,1] = \left\{ x = (x_n) : \exists L \in \mathbb{R},\ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \right\}$$

and

$$[V,\lambda] = \left\{ x = (x_n) : \exists L \in \mathbb{R},\ \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \right\}$$

for the sets of sequences $x = (x_k)$ which are strongly Cesaro summable and strongly $(V,\lambda)-$summable to L, i.e. $x_k \to L$ $[C,1]$ and $x_k \to L$ $[V,\lambda]$ respectively. He denoted Λ, the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$.

In [23], the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$ of positive integers and corresponding $\mathscr{I}_\sigma$-convergence were introduced. Also, inclusion relations between $\mathscr{I}_\sigma$-convergence and invariant convergence also $\mathscr{I}_\sigma$-convergence and $[V_\sigma]_p$.

Let $A \subseteq \mathbb{N}$ and

$$s_m := \min_n \left|A \cap \left\{\sigma(n), \sigma^2(n), ..., \sigma^m(n)\right\}\right|$$

$$S_m := \max_n \left|A \cap \left\{\sigma(n), \sigma^2(n), ..., \sigma^m(n)\right\}\right|.$$

If the following limits exist

$$\underline{V}(A) := \lim_{m\to\infty} \frac{s_m}{m}, \overline{V}(A) := \lim_{m\to\infty} \frac{S_m}{m}$$

then they are called a lower and an upper σ-uniform density of the set A, respectively.If $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ-uniform density of A.

Denote by $\mathscr{I}_{\sigma}$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

A sequence $x = (x_n)$ is said to be $\mathscr{I}_{\sigma}$-convergent to the number L if for every $\varepsilon > 0$

$$A(\varepsilon) := \{k : |x_k - L| \geq \varepsilon\}$$

belongs to $\mathscr{I}_{\sigma}$; i.e., $V(A_{\varepsilon}) = 0$. In this case we write $\mathscr{I}_{\sigma} - \lim x_k = L$.

In [21], they made a new approach to the notions of $[V,\lambda]$-summability and λ-statistical convergence by using ideals and introduce new notions, namely, $\mathscr{I}$-$[V,\lambda]$- summability and $\mathscr{I}$-λ-statistical convergence. They mainly examined the relation between these two new methods as also the relation between $\mathscr{I}$-λ-statistical convergence and $\mathscr{I}$-statistical convergence introduced by the authors recently.

Recently, Das, Savas and Ghosal [2] introduced new notions, namely $\mathscr{I}$-statistical convergence and $\mathscr{I}$-lacunary statistical convergence by using ideal, convergence, investigated their relationship, and made some observations about these classes.

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold, and $w(G)$ and $m(G)$ denote the spaces of all real valued functions and all bounded real functions f on G respectively. $m(G)$ is a Banach space with the supremum norm $\|f\|_{\infty} = sup\{|f(g)| : g \in G\}$. Nomika [26] showed that, if G is countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that (i) $G = \cup_{i=1}^{\infty} S_n$, (ii) $S_n \subset S_{n+1}$, $n = 1,2,3,...$, (iii) $\lim_{n\to\infty} \frac{|S_n g - \cap S_n|}{|S_n|} = 1$, $\lim_{n\to\infty} \frac{|gS_n - \cap S_n|}{|S_n|} = 1$ for all $g \in G$. Here $|A|$ denotes the number of elements in the finite set A. Any sequence of finite subsets of G satisfying (i), (ii) and (iii) is called a Folner sequence for G.

The sequence $S_n = \{0,1,2,...,n-1\}$ is a familiar Folner sequence giving rise to the classical Cesàro method of summability.

Amenable semigroups were studied by [1]. The concept of summability in amenable semigroups was introduced in [15], [16]. In [3], Douglas extended the notion of arithmetic mean to amenable semigroups and obtained a characterization for almost convergence in amenable semigroups.

In [25], the notions of convergence and statistical convergence, statistical limit point and statistical cluster point to functions on discrete countable amenable semigroups were introduced.

The purpose of the study [28] was to extend the notions of $\mathscr{I}$-convergence, $\mathscr{I}$-limit superior and $\mathscr{I}$-limit inferior, $\mathscr{I}$-cluster point and $\mathscr{I}$-limit point to functions defined on discrete countable amenable semigroups. Also, he made a new approach to the notions of $[V,\lambda]$-summability and λ-statistical convergence by using ideals and introduced new notions, namely, $\mathscr{I}$-$[V,\lambda]$-summability and $\mathscr{I}$-λ-statistical convergence to functions defined on discrete countable amenable semigroups.

2 Definitions and notations

Definition 1. [23] *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold.* $f \in w(G)$ *is said to be convergent to s, for any Folner sequence* $\{S_n\}$ *for G, if for each* $\varepsilon > 0$ *there exists* $k_0 \in \mathbb{N}$ *such that* $|f(g) - s| < \varepsilon$ *for all* $m > k_0$ *and* $g \in G \backslash S_m$.

Definition 2. [23] *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold.* $f \in w(G)$ *is said to be a Cauchy sequence for any Folner sequence* $\{S_n\}$ *for G, if for each* $\varepsilon > 0$ *there exists* $k_0 \in \mathbb{N}$ *such that* $|f(g) - f(h)| < \varepsilon$ *for all* $m > k_0$ *and* $g \in G \backslash S_m$.

Definition 3. [23] *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold.* $f \in w(G)$ *is said to be strongly summable to s, for any Folner sequence* $\{S_n\}$ *for G, if*

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{g \in S_n} |f(g) - s| = 0,$$

where $|S_n|$ *denotes the cardinality of the set* S_n.

The upper and lower Folner densities of a a set $S \subset G$ are defined by

$$\overline{\delta}(S) = \lim_{n\to\infty} \sup \frac{1}{|S_n|} |\{g \in S_n : g \in S\}|$$

and

$$\underline{\delta}(S) = \lim_{n\to\infty} \inf \frac{1}{|S_n|} |\{g \in S_n : g \in S\}|$$

respectively $\overline{\delta}(S) = \underline{\delta}(S)$, then

$$\delta(S) = \lim_{n\to\infty} \frac{1}{|S_n|} |\{g \in S_n : g \in S\}|$$

is called Folner density of S. Take $G = \mathbb{N}$, $S_n = \{0,1,2,...,n-1\}$ and S be the set of positive integers with leading digit 1 in the decimal expansion. The set S has no Folner density with respect to the Folner sequence $\{S_n\}$, since $\underline{\delta}(S) = \frac{1}{9}$, $\overline{\delta}(S) = \frac{5}{9}$. To facililate this idea we introduce the following notion: If f is function such that $f(g)$ satisfies property P for all g expect a set of Folner density zero, we say that $f(g)$ satisfies P for "almost all g", and abbreviate this by "$a.a.g$".

Definition 4. [23] *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold.* $f \in w(G)$ *is said to be statistically convergent to s, for any Folner sequence* $\{S_n\}$ *for G, if for every* $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{|S_n|} |\{g \in S_n : |f(g) - s|\}| = 0.$$

The set of all statistically convergent functions will be denoted by $S(G)$.

Definition 5. [28] *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold.* $f \in w(G)$ *is said to be* $\mathscr{I}$*-convergent to s for any Folner sequence* $\{S_n\}$ *for G, if for every* $\varepsilon > 0$;

$$\{g \in S_n : |f(g) - s| \geq \varepsilon\} \in \mathscr{I};$$

i.e., $|f(g) - s| < \varepsilon$ *a.a.g. The set of all* $\mathscr{I}$*-convergent sequences will be denoted by* $\mathscr{I}(G)$.

3 Main results

Definition 6. *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. The function $f \in w(G)$ is said to be $\mathscr{I}$-invariant convergent to s for any Folner sequence $\{S_n\}$ for G if for every $\varepsilon > 0$;*

$$\{g \in S_n : |f(g) - s| \geq \varepsilon\}$$

belongs to $\mathscr{I}_\sigma$; i.e., $V(A_\varepsilon) = 0$. The set of all $\mathscr{I}$-invariant convergent sequences will be denoted by $\mathscr{I}_\sigma(G)$.

Definition 7. *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. The function $f \in w(G)$ is said to be invariant convergent to s for any Folner sequence $\{S_n\}$ for G if*

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{1\leq k\leq |S_n| \& g\in S_n} f\left(g_{\sigma^k(m)}\right) = s, \text{ uniformly in } m.$$

In this case, we write $f \to s(V_\sigma)$.

Theorem 1. *Let $f \in w(G)$ is bounded function. If f is $\mathscr{I}_\sigma$-convergent to s, then f is invariat convergent to s for any Folner sequence $\{S_n\}$ for G.*

Proof. Let $m, n \in \mathbb{N}$ be arbitrary, $\varepsilon > 0$ and set

$$L_n = \left\{g \in S_n : \left|f\left(g_{\sigma^j(m)}\right) - s\right| \geq \varepsilon\right\}, \text{ uniformly in } m.$$

For each $f \in w(G)$, we estimate

$$t(m,n,f) := \left|\frac{f\left(g_{\sigma(m)}\right) + f\left(g_{\sigma^2(m)}\right) + \ldots + f\left(g_{\sigma^n(m)}\right)}{|S_n|} - s\right|$$

We have

$$t(m,n,f) \leq t^{(1)}(m,n,f) + t^{(2)}(m,n,f),$$

where

$$t^{(1)}(m,n,f) := \frac{1}{|S_n|} \sum_{1\leq j\leq |S_n| \& g\in L_n} \left|f\left(g_{\sigma^j(m)}\right) - s\right|$$

and

$$t^{(2)}(m,n,x) = \frac{1}{|S_n|} \sum_{1\leq j\leq |S_n| \& g\in S_n\setminus L_n} \left|f\left(g_{\sigma^j(m)}\right) - s\right|.$$

Therefore, we have $t^{(2)}(m,n,f) < \varepsilon$ for each $f \in w(G)$ and for every $m = 1, 2, \ldots$. The boundedness of f is implies that there exist $M > 0$ such that

$$\left|f\left(g_{\sigma^j(m)}\right) - s\right| \leq M, \ (j = 1,2,\ldots; m = 1,2\ldots),$$

then this implies that

$$\begin{aligned} t^{(1)}(m,n,f) &\leq \tfrac{M}{|S_n|} \left|\left\{1 < j < |S_n| : \left|f\left(g_{\sigma^j(m)}\right) - s\right| \geq \varepsilon\right\}\right| \\ &\leq M.\frac{\max_m \left|\left\{1<j<|S_n|,: f_k\left(\left|f\left(g_{\sigma^j(m)}\right)-s\right|\right)\geq\varepsilon\right\}\right|}{|S_n|} \\ &= M.\tfrac{K_n}{|S_n|}. \end{aligned}$$

Hence, f is invariant convergent to s for any Folner sequence $\{S_n\}$ for G.

Definition 8. *The function* $f \in w(G)$ *is said to be* $\mathscr{I}^*$*-invariat convergent to* s *for any Folner sequence* $\{S_n\}$ *for* G*, if there exists a set*

$$M = \{m = (m_i) : m_i < m_{i+1} \text{ , } i \in \mathbb{N}\} \in \mathscr{F}(\mathscr{I}_\sigma)$$

such that

$$\lim_{k\to\infty} f\left(g_{m_k}\right) = s.$$

The set of all $\mathscr{I}^*$*-invariat convergent sequences will be denoted by* $\mathscr{I}_\sigma^*(G)$.

Theorem 2. *If the function* $f \in w(G)$ *is* $\mathscr{I}^*$*-invariat convergent to* s*, the function is* $\mathscr{I}$*-invariant convergent to* s *for any Folner sequence* $\{S_n\}$ *for* G.

Proof. By assumption, there exists a set $H \in \mathscr{I}_\sigma$ such that for $M = N \setminus H = M = \{m = (m_i) : m_i < m_{i+1} \text{ , } i \in \mathbb{N}\}$ we have

$$\lim_{k\to\infty} f\left(g_{m_k}\right) = s, \tag{1}$$

Let $\varepsilon > 0$ by (1), there exists $k_0 \in \mathbb{N}$ such that

$$\left|f\left(g_{m_k}\right) - s\right| < \varepsilon,$$

for each $k > k_0$. Then, obviously

$$\left\{k \in \mathbb{N} : \left|f\left(g_{m_k}\right) - s\right| \geq \varepsilon\right\} \subset H \cup \left\{m_1 < m_2 < ... < m_{k_0}\right\}. \tag{2}$$

Since $\mathscr{I}_\sigma$ is admissible, the set on the right-hand side of (2) belongs to $\mathscr{I}_\sigma$. So f is $\mathscr{I}$-invariant convergent to s for any Folner sequence $\{S_n\}$ for G.

Definition 9. *The function* $f \in w(G)$ *is said to be* p*-strongly invariant convergent to* s *for any Folner sequence* $\{S_n\}$ *for* G *if*

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{1\leq k\leq |S_n| \& g\in S_n} \left| f\left(g_{\sigma^k(m)}\right) - s\right|^p = 0, \text{ uniformly in } m,$$

where $0 < p < \infty$*. In this case, we write* $f \to s\,[V_\sigma]_p$.

Theorem 3. *Let* $\mathscr{I}_\sigma \subset 2^{\mathbb{N}}$ *be an admissible ideal and* $0 < p < \infty$.

(i) *If* $f \to s\,[V_\sigma]_p$*, then* $f \to s(\mathscr{I}_\sigma)$.
(ii) *If* $f \in w(G)$ *is bounded and* $f \to s(\mathscr{I}_\sigma)$*, then* $f \to s\,[V_\sigma]_p$.
(iii) *If* $f \in w(G)$*, then* f *is* $\mathscr{I}_\sigma$*-convergent if and only if* $f \to s\,[V_\sigma]_p$.

Proof. **(i)** Let $f \to s\,[V_\sigma]_p$, $0 < p < \infty$. Suppose $\varepsilon > 0$. Then for every $m \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{1<j<|S_n| \& g\in S_n} \left|f\left(g_{\sigma^j(m)}\right) - s\right|^p &\geq \sum_{1<j<|S_n| \& \left|f\left(g_{\sigma^j(m)}\right)-s\right|\geq\varepsilon} \left|f\left(g_{\sigma^j(m)}\right) - s\right|^p \\
&\geq \varepsilon^p. \left|\left\{1 < j < |S_n| : \left|f\left(g_{\sigma^j(m)}\right) - s\right| \geq \varepsilon\right\}\right| \\
&\geq \varepsilon^p. \max_m \left|\left\{1 < j < |S_n| : \left|f\left(g_{\sigma^j(m)}\right) - s\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{1}{|S_n|} \sum_{1<j<|S_n| \& g\in S_n} \left|f\left(g_{\sigma^j(m)}\right) - s\right|^p &\geq \varepsilon^p. \tfrac{\max_m \left|\left\{1<j<|S_n| \& g\in S_n : \left|f\left(g_{\sigma^k(m)}\right)-s\right|\geq\varepsilon\right\}\right|}{|S_n|} \\
&= \varepsilon^p. \tfrac{K_n}{|S_n|}
\end{aligned}$$

for every $m = 1,2,\ldots$. This implies $\lim_{n\to\infty}\frac{K_n}{|S_n|} = 0$ and so $f \to s(\mathscr{I}_\sigma)$.

(ii) Now suppose that $f \in w(G)$ is bounded and $f \to s(\mathscr{I}_\sigma)$. Let $0 < p < \infty$ and $\varepsilon > 0$. By assumption, we have $V(A_\varepsilon) = 0$. The boundedness of $f \in w(G)$ implies that there exist $M > 0$ such that

$$\left|f\left(g_{\sigma^j(m)}\right) - s\right| \leq M,\ j = 1,2,\ldots; m = 1,2,\ldots.$$

Observe that for every $n \in \mathbb{N}$ we have that

$$\frac{1}{|S_n|}\sum_{1<j<|S_n|\&g\in S_n}\left|f\left(g_{\sigma^j(m)}\right)-s\right|^p = \frac{1}{|S_n|}\sum_{1<j<|S_n|\&\left|f\left(g_{\sigma^j(m)}\right)-s\right|\geq\varepsilon}\left|f\left(g_{\sigma^j(m)}\right)-s\right|^p + \sum_{1<j<|S_n|\&\left|f\left(g_{\sigma^j(m)}\right)-s\right|<\varepsilon}\left|f\left(g_{\sigma^j(m)}\right)-s\right|^p$$

$$\leq M.\frac{\max_m\left|\left\{1\leq j\leq|S_n|: \left|f\left(g_{\sigma^j(m)}\right)-s\right|\geq\varepsilon\right\}\right|}{|S_n|} + \varepsilon^p$$

$$\leq M.\frac{K_n}{|S_n|} + \varepsilon^p,$$

for each $f \in w(G)$.

Hence, we obtain

$$\lim_{n\to\infty}\frac{1}{|S_n|}\sum_{1<j<|S_n|\&g\in S_n}\left|f\left(g_{\sigma^j(m)}\right)-s\right|^p = 0, \text{ uniformly in } m.$$

(iii) This is immediate consequence of **(i)** and **(ii)**.

Definition 10. *The function $f \in w(G)$ is said to be $\mathscr{I}$-lacunary invariant statistically convergent to s for any Folner sequence $\{S_n\}$ for G for each $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r\&g \in S_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathscr{I}_\sigma, \textit{ uniformly in } m.$$

In this case we write $f \to s(S_{\sigma\theta}(\mathscr{I}))$.

Definition 11. *The function $f \in w(G)$ is said to be strongly $\mathscr{I}$-lacunary invariant convergent to s for any Folner sequence $\{S_n\}$ for G for each $\varepsilon > 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r\&g\in S_n}\left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\} \in \mathscr{I}_\sigma, \textit{ uniformly in } m.$$

In this case we write $f \to s(N_{\sigma\theta}(\mathscr{I}))$.

We shall denote by $S_{\sigma\theta}(\mathscr{I})$, $N_{\sigma\theta}(\mathscr{I})$ the collections of all $\mathscr{I}$-lacunary invariant statistically convergent and strongly $\mathscr{I}$-lacunary invariant functions for the function $f \in w(G)$, respectively.

Theorem 4. *Let $\theta = \{k_r\}$ be a lacunary sequence and $f \in w(G)$ be a function in S.*

(i) *If $f \to s(N_{\sigma\theta}(\mathscr{I}))$ then $f \to s(S_{\sigma\theta}(\mathscr{I}))$.*
(ii) *If $f \in w(G)$ is bounded function and $f \to s(S_{\sigma\theta}(\mathscr{I}))$ then $f \to s(N_{\sigma\theta}(\mathscr{I}))$.*

Proof. **(i)** Let $\varepsilon > 0$ and $f \to s(N_{\sigma\theta}(\mathscr{I}))$. Then we can write

$$\sum_{k\in I_r\&g\in S_n}\left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \sum_{k\in I_r, g\in S_n\&\left|f\left(g_{\sigma^k(m)}\right)-s\right|\geq\varepsilon}\left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon.\left|\left\{k \in I_r\&g \in S_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right|.$$

So for given $\delta > 0$,

$$\frac{1}{h_r}\left|\left\{k \in I_r \& g \in S_n : \left|f\left(g_{\sigma^k(m)}\right)-s\right| \geq \varepsilon\right\}\right| \geq \delta \Longrightarrow \frac{1}{h_r}\sum_{k\in I_r, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| \geq \varepsilon.\delta,$$

i.e.

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r \& g \in S_n : \left|f\left(g_{\sigma^k(m)}\right)-s\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| \geq \varepsilon.\delta\right\}.$$

Since $f \to s\left(N_{\sigma\theta}\left(\mathscr{I}\right)\right)$, the set on the right-hand side belongs to $\mathscr{I}_\sigma$ and so it follows that $f \to s\left(S_{\sigma\theta}\left(\mathscr{I}\right)\right)$.

(ii) Suppose that $f \in w(G)$ is bounded function and $f \to s\left(S_{\sigma\theta}\left(\mathscr{I}\right)\right)$. Then we can assume that

$$\left|f\left(g_{\sigma^k(m)}\right)-s\right| \leq M$$

for each $k \in I_r$ and $g \in S_n$.

Given $\varepsilon > 0$, we get

$$\frac{1}{h_r}\sum_{k\in I_r, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| = \frac{1}{h_r}\sum_{\substack{k\in I_r, g\in S_n \\ \left|f\left(g_{\sigma^k(m)}\right)-s\right|\geq\varepsilon}} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) + \frac{1}{h_r}\sum_{\substack{k\in I_r, g\in S_n \\ \left|f\left(g_{\sigma^k(m)}\right)-s\right|<\varepsilon}} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)$$

$$\leq \frac{M}{h_r}\left|\left\{k \in I_r, g \in S_n : \left|f\left(g_{\sigma^k(m)}\right)-s\right| \geq \varepsilon\right\}\right| + \varepsilon.$$

Note that

$$A(\varepsilon) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r, g \in S_n : \left|f\left(g_{\sigma^k(m)}\right)-s\right| \geq \varepsilon\right\}\right| \geq \frac{\varepsilon}{M}\right\} \in \mathscr{I}_\sigma.$$

If $r \in (A(\varepsilon))^c$ then

$$\frac{1}{h_r}\sum_{k\in I_r, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| < 2\varepsilon.$$

Hence

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| \geq 2\varepsilon\right\} \subset A(\varepsilon)$$

and so belongs to $\mathscr{I}_\sigma$. This shows that $f \to s\left(N_{\sigma\theta}\left(\mathscr{I}\right)\right)$.This completes the proof.

Definition 12. *The function $f \in w(G)$ is said to be $\mathscr{I}_\sigma$-statistically convergent to s for any Folner sequence $\{S_n\}$ for G if for each $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{g \in S_n : \frac{1}{|S_n|}\left|\left\{k \leq |S_n| : \left|f\left(g_{\sigma^k(m)}\right)-s\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathscr{I}_\sigma$$

In this case we write $f \to s\left(S\left(\mathscr{I}_\sigma\right)\right)$.

Theorem 5. *If $\theta = \{k_r\}$ be a lacunary sequence with $\liminf_r q_r > 1$, then*

$$f \to s\left(S\left(\mathscr{I}_\sigma\right)\right) \Rightarrow f \to s\left(S_{\sigma\theta}\left(\mathscr{I}\right)\right).$$

Proof. Suppose first that $\liminf_r q_r > 1$, then there exists a $\alpha > 0$ such that $q_r \geq 1+\alpha$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}.$$

If $f \to s(S(\mathscr{I}_\sigma))$, then for every $\varepsilon > 0$, for each $x \in X$ and for sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{g \in S_{k_r} : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \geq \frac{1}{k_r}\left|\left\{k \in I_r, g \in S_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{\alpha}{1+\alpha}\frac{1}{h_r}\left|\left\{k \in I_r, g \in S_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right|;$$

Then for any $\delta > 0$, we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r}\left|\left\{g \in S_{k_r} : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \geq \frac{\delta\alpha}{(\alpha+1)}\right\} \in \mathscr{I}_\sigma.$$

This completes the proof.

Theorem 6. *If $\theta = \{k_r\}$ be a lacunary sequence with $\limsup_r q_r < \infty$, then*

$$f \to s(S_{\sigma\theta}(\mathscr{I})) \Rightarrow f \to s(S(\mathscr{I}_\sigma)).$$

Proof. If $\limsup_r q_r < \infty$ then without any loss of generality we can assume that there exists a $K > 0$ such that $q_r < K$ for all $r \geq 1$. Let $f \to s(S_{\sigma\theta}(\mathscr{I}))$ and for $\delta > 0$. Then there exists $B > 0$ and $\varepsilon > 0$ such that for every $j \geq B$

$$M_j = \frac{1}{h_j}\left|\left\{k \in I_j, g \in S_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| < \delta.$$

Also we can find $A > 0$ such that $M_j < A$ for all $j = 1, 2, \ldots$. Now, let $n \in \mathbb{N}$ be an integer satisfying $k_{r-1} < |S_n| \leq k_r$ where $r > B$. Then, we can write

$$\frac{1}{|S_n|}\left|\left\{k \leq |S_n| : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \leq \frac{1}{k_{r-1}}\left|\left\{k \leq |S_{k_r}| : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{k_{r-1}}\left|\left\{k \in I_1 : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| + \frac{1}{k_{r-1}}\left|\left\{k \in I_2 : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| + \ldots$$

$$+ \frac{1}{k_{r-1}}\left|\left\{k \in I_r : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| = \frac{k_1}{k_{r-1}k_1}\left|\left\{k \in I_1 : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right|$$

$$+ \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}\left|\left\{k \in I_2 : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| + \ldots + \frac{k_B - k_{B-1}}{k_{r-1}(k_B - k_{B-1})}\left|\left\{k \in I_B : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right|$$

$$+ \ldots + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})}\left|\left\{k \in I_r : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| = \frac{k_1}{k_{r-1}}M_1 + \frac{k_2 - k_1}{k_{r-1}}M_2 + \ldots + \frac{k_B - k_{B-1}}{k_{r-1}}M_B$$

$$+ \ldots + \frac{k_r - k_{r-1}}{k_{r-1}}M_r \leq \left\{\sup_{i\geq 1} M_i\right\}\tfrac{k_B}{k_{r-1}} + \left\{\sup_{i\geq B} M_i\right\}\tfrac{k_r - k_B}{k_{r-1}} \leq A\tfrac{k_B}{k_{r-1}} + \delta K.$$

This completes the proof of the theorem.

Combining Theorem 5 and Theorem 6 we have

Theorem 7. *If $\theta = \{k_r\}$ be a lacunary sequence with $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then*

$$f \to s\ (S_{\sigma\theta}(\mathscr{I})) \Leftrightarrow f \to s(S(\mathscr{I}_\sigma))$$

Proof. This is an immediate consequence of Theorem 5 and Theorem 6.

Definition 13. *The function $f \in w(G)$ is said to be strongly Cesàro $\mathscr{I}_\sigma$-summable to s for any Folner sequence $\{S_n\}$ for G if for each $\varepsilon > 0$,*

$$\left\{g \in S_n : \frac{1}{|S_n|} \sum_{1 \leq k \leq |S_n| \& g \in S_n} \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon \right\} \in \mathscr{I}_\sigma$$

uniformly in m. (denoted by $f \to s[C_1(\mathscr{I}_\sigma)]$).

Definition 14. *The function $f \in w(G)$ is said to be strongly $\lambda_{\mathscr{I}}$-invariant convergent to s for any Folner sequence $\{S_n\}$ for G if for each $\varepsilon > 0$,*

$$\left\{g \in S_n : \frac{1}{\lambda_n} \sum_{k \in I_n, g \in S_n} \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon \right\} \in \mathscr{I}_\sigma$$

uniformly in m, where $I_n = [n - \lambda_n + 1, n]$. (denoted by $f \to s(V_\lambda(\mathscr{I}_\sigma))$).

Definition 15. *The function $f \in w(G)$ is said to be $\mathscr{I}_\sigma$-λ-statistically convergent to s for any Folner sequence $\{S_n\}$ for G if for each $\varepsilon > 0$, for each $\delta > 0$,*

$$\left\{g \in S_n : \frac{1}{\lambda_n} \left| \left\{k \in I_n : \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon \right\}\right| \geq \delta \right\} \in \mathscr{I}_\sigma$$

uniformly in m. (denoted by $f \to s(S_\lambda(\mathscr{I}_\sigma))$).

Theorem 8. *Let $\lambda \in \Lambda$ and $\mathscr{I}_\sigma$ is an admissible ideal in $\mathbb{N}$. If $f \to s(V_\lambda(\mathscr{I}_\sigma))$, then $f \to s(S_\lambda(\mathscr{I}_\sigma))$.*

Proof. Assume that $f \to s(V_\lambda(\mathscr{I}_\sigma))$ and $\varepsilon > 0$. Then,

$$\sum_{k \in I_n, g \in S_n} \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \sum_{\substack{k \in I_n, g \in S_n \\ \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon}} \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon . \left| \left\{k \in I_n, g \in S_n : \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon \right\}\right|$$

and so,

$$\frac{1}{\varepsilon\lambda_n} \sum_{k \in I_n, g \in S_n} \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \frac{1}{\lambda_n} \left| \left\{k \in I_n, g \in S_n : \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon \right\}\right|.$$

Then for any $\delta > 0$,

$$\left\{g \in S_n : \frac{1}{\lambda_n} \left| \left\{k \in I_n : \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon \right\}\right| \geq \delta \right\}$$
$$\subseteq \left\{g \in S_n : \frac{1}{\lambda_n} \sum_{k \in I_n, g \in S_n} \left| f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\delta \right\}.$$

Since right hand belongs to $\mathscr{I}_\sigma$ then left hand also belongs to $\mathscr{I}_\sigma$ and this completes the proof.

Theorem 9. *If $\liminf \dfrac{\lambda_n}{|S_n|} > 0$ then $f \to s\ (S(\mathscr{I}_\sigma))$ implies $f \to s(S_\lambda(\mathscr{I}_\sigma))$.*

Proof. Assume that $\liminf \frac{\lambda_n}{|S_n|} > 0$ there exists a $\delta > 0$ such that $\frac{\lambda_n}{|S_n|} \geq \delta$ for sufficiently large n. For given $\varepsilon > 0$ we have,

$$\frac{1}{|S_n|}\left\{k \leq |S_n| : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\} \supseteq \frac{1}{|S_n|}\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}.$$

Therefore,

$$\begin{aligned}\tfrac{1}{|S_n|}\left|\left\{k \leq |S_n| : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| &\geq \tfrac{1}{|S_n|}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \\ &\geq \frac{\lambda_n}{|S_n|}\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \\ &\geq \delta.\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right|\end{aligned}$$

then for any $\eta > 0$ we get

$$\begin{aligned}&\left\{g \in S_n : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \geq \eta\right\} \\ &\quad\subseteq \left\{g \in S_n : \frac{1}{|S_n|}\left|\left\{k \leq |S_n| : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \geq \eta\delta\right\} \in \mathscr{I}_\sigma\end{aligned}$$

and this completes the proof.

Theorem 10. *If $\lambda = (\lambda_n) \in \Delta$ be such that $\lim_{n\to\infty} \frac{\lambda_n}{|S_n|} = 1$, then $S_\lambda(\mathscr{I}_\sigma) \subset S(\mathscr{I}_\sigma)$.*

Proof. Let $\delta > 0$ be given. Since $\lim_{n\to\infty} \frac{\lambda_n}{|S_n|} = 1$, we can choose $m \in N$ such that $\left|\frac{\lambda_n}{|S_n|} - 1\right| < \frac{\delta}{2}$, for all $n \geq m$. Now observe that, for $\varepsilon > 0$

$$\begin{aligned}\frac{1}{|S_n|}\left|\left\{k \leq |S_n| : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| &= \frac{1}{|S_n|}\left|\left\{k \leq |S_n| - \lambda_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \\ &+ \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \\ &\leq \frac{|S_n| - \lambda_n}{|S_n|} + \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \\ &\leq 1 - \left(1 - \frac{\delta}{2}\right) + \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \\ &= \frac{\delta}{2} + \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right|,\end{aligned}$$

for all $n \geq m$. Hence

$$\begin{aligned}&\left\{g \in S_n : \frac{1}{|S_n|}\left|\left\{k \leq |S_n| : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\quad\subset \left\{g \in S_n : \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|f\left(g_{\sigma^k(m)}\right) - s\right| \geq \varepsilon\right\}\right| \geq \tfrac{\delta}{2}\right\} \cup \{1, 2, ..., m\}\end{aligned}$$

If f is $\mathscr{I}_\sigma$-λ-statistically convergent to s, then the set on the right hand side belongs to $\mathscr{I}_\sigma$ and so the set on the left hand side also belongs to $\mathscr{I}_\sigma$. This shows that f is $\mathscr{I}_\sigma$-statistically convergent to s.

Theorem 11. *If* $f \to s\left(V_\lambda\left(\mathscr{I}_\sigma\right)\right)$ *is then* $f \to s\left[C_1\left(\mathscr{I}_\sigma\right)\right]$.

Proof. Assume that $f \to s\left(V_\lambda\left(\mathscr{I}_\sigma\right)\right)$ and $\varepsilon > 0$. Then,

$$\begin{aligned}
\frac{1}{|S_n|}\sum_{g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| &= \frac{1}{|S_n|}\sum_{k=1}^{|S_n|-\lambda_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| + \frac{1}{|S_n|}\sum_{k\in I_n, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| \\
&\le \frac{1}{\lambda_n}\sum_{k=1}^{|S_n|-\lambda_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| + \frac{1}{\lambda_n}\sum_{k\in I_n, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| \\
&\le \frac{2}{\lambda_n}\sum_{k\in I_n, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right|
\end{aligned}$$

and so,

$$\left\{g\in S_n : \frac{1}{|S_n|}\sum_{g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| \ge \varepsilon\right\} \subseteq \left\{g\in S_n : \frac{1}{\lambda_n}\sum_{k\in I_n, g\in S_n}\left|f\left(g_{\sigma^k(m)}\right)-s\right| \ge \frac{\varepsilon}{2}\right\}$$

belongs to $\mathscr{I}_\sigma$. Hence $f \to s\left[C_1\left(\mathscr{I}_\sigma\right)\right]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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