

# Asymptotics of eigenvalues for Sturm-Liouville problem with eigenparameter-dependent boundary conditions

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**Abstract:** In this paper, we obtain asymptotic estimates of eigenvalues for regular Sturm-Liouville problems having the eigenvalue parameter in all boundary conditions with the potential that is continuous and its differentiation exists and is integrable.

**Keywords:** Sturm-Liouville Problems, integrable potential, eigenvalues, asymptotics.

## 1 Introduction

In this paper, we consider the boundary value problem

$$y''(t) + \{\lambda - q(t)\}y(t) = 0, \quad t \in [a, b], \quad (1)$$

$$a_1y(a) + a_2y'(a) = \lambda [a'_1y(a) + a'_2y'(a)], \quad (2)$$

$$b_1y(b) + b_2y'(b) = \lambda [b'_1y(b) + b'_2y'(b)], \quad (3)$$

where  $\lambda$  is a real parameter;  $q(t)$  is a real-valued function;  $a_i, a'_i, b_i, b'_i \in \mathbb{R}, i = 0, 1$ . Also we assume that  $q(t)$  is continuous, its differentiation exists and is integrable. This problem differs from the usual regular Sturm-Liouville problem in the sense that eigenvalue parameter  $\lambda$  is contained in the boundary condition at  $a$ . Problems of this type arise from the method of separation of variables applied to mathematical models for certain physical problems including heat conduction and wave propagation, etc. [8]. It is shown by Walter [15] that this problem is self-adjoint problem. The purpose of this paper is to obtain asymptotic approximations for the eigenvalues of (1)-(3).

Approximations of this type have been derived before. We mention in particular [7, 8] and [2]. Fulton's approach in [7] is based on an iteration of the usual Volterra integral equation, producing an asymptotic expansion of the solution in higher powers of  $1/\lambda^{1/2}$  as  $\lambda \rightarrow \infty$  and in [8] is based on the analysis of [14] for regular Sturm-Liouville problems on a finite closed interval and involves some operator-theoretical results of [15]. The approach used in [2] is based on an iterative procedure solving the associated Riccati equation and producing an asymptotic expansion of the solution in the higher powers of  $1/\lambda^{1/2}$  as  $\lambda \rightarrow \infty$  for smooth  $q(t)$ . There is also a vast amount of literature dealing with asymptotic estimates of eigenvalues for standart Sturm-Liouville problems with regular endpoints [3, 4, 5, 6, 9, 10, 11, 13, 14]. Here we follow the similar approach in [4, 10, 12]. We assume without loss of generality, that  $q(t)$  has mean value zero. That is  $\int_a^b q(t) dt = 0$ .

## 2 Conclusion

**Theorem 1.** The eigenvalues  $\lambda_n$  of (1)-(3) satisfy as  $\lambda \rightarrow \infty$ ,

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(i) if  $a'_2 \neq 0$  and  $b'_2 \neq 0$ ,

$$\begin{aligned} \lambda_n^{1/2} = & \frac{(n+1)\pi}{(b-a)} + \frac{1}{(n+1)\pi} \left\{ \frac{a'_2 b'_1 - a'_1 b'_2}{a'_2 b'_2} - \frac{b-a}{4(n+1)\pi} \int_a^b \left( \sin \frac{2(n+1)\pi(x-a)}{b-a} \right) q'(x) dx \right. \\ & - \frac{(b-a)^2}{(n+1)^2 \pi^2} \left[ \frac{-3b'_1 b'_2 b_2 + 3b_1 (b'_2)^2 + (b'_1)^3}{3(b'_2)^3} \right] \\ & + \frac{(b-a)^2}{(n+1)^2 \pi^2} \left[ \frac{-3a'_1 a'_2 a_2 + 3a_1 (a'_2)^2 + (a'_1)^3}{3(a'_2)^3} \right] + \frac{(b-a)^2}{2(n+1)^2 \pi^2} \frac{a'_1}{a'_2} [q(b) - q(a)] \\ & \left. - \frac{(b-a)^2}{2(n+1)^2 \pi^2} \frac{a'_1}{a'_2} \int_a^b \left( \cos \frac{2(n+1)\pi(x-a)}{b-a} \right) q'(x) dx \right\} + O(n^{-4}\eta(n)) + O(n^{-3}\eta^2(n)), \end{aligned}$$

(ii) if  $a'_2 \neq 0$  and  $b'_2 = 0$ ,

$$\begin{aligned} \lambda_n^{1/2} = & \frac{(2n+3)\pi}{2(b-a)} + \frac{2}{(2n+3)\pi} \left\{ \frac{a'_2 b_2 - a'_1 b'_1}{a'_2 b'_1} - \frac{(b-a)}{2(2n+3)\pi} \int_a^b \left( \sin \frac{(2n+3)\pi(x-a)}{b-a} \right) q'(x) dx \right. \\ & + \frac{4(b-a)^2}{(2n+3)^2 \pi^2} \left[ \frac{3b'_1 b_1 b_2 - b_2^3}{3(b'_1)^3} \right] + \frac{4(b-a)^2}{(2n+3)^2 \pi^2} \left[ \frac{-3a'_1 a'_2 a_2 + 3a_1 (a'_2)^2 + (a'_1)^3}{3(a'_2)^3} \right] \\ & - \frac{2(b-a)^2}{(2n+3)^2 \pi^2} \frac{a'_1}{a'_2} [q(b) + q(a)] \\ & \left. - \frac{2(b-a)^2}{(2n+3)^2 \pi^2} \frac{a'_1}{a'_2} \int_a^b \left( \cos \frac{(2n+3)\pi(x-a)}{b-a} \right) q'(x) dx \right\} + O(n^{-4}\eta(n)) + O(n^{-3}\eta^2(n)). \end{aligned}$$

**Theorem 2.** The eigenvalues  $\lambda_n$  of (1)-(3) satisfy as  $\lambda \rightarrow \infty$ ,

(i) if  $a'_2 = 0$  and  $b'_2 \neq 0$ ,

$$\begin{aligned} \lambda_n^{1/2} = & \frac{(2n+3)\pi}{2(b-a)} + \frac{2}{(2n+3)\pi} \left\{ \frac{a'_1 b'_1 - a_2 b'_2}{a'_1 b'_2} - \frac{b-a}{2(2n+3)\pi} \int_a^b \left( \sin \frac{(2n+3)\pi(x-a)}{b-a} \right) q'(x) dx \right. \\ & - \frac{4(b-a)^2}{(2n+3)^2 \pi^2} \left[ \frac{3b_1 (b'_2)^2 - 3b'_1 b'_2 b_2 + (b'_1)^3}{3(b'_2)^3} \right] \\ & + \frac{4(b-a)^2}{(2n+3)^2 \pi^2} \left[ \frac{-3a'_1 a_1 a_2 + a_2^3}{3(a'_1)^3} \right] + \frac{2(b-a)^2}{(2n+3)^2 \pi^2} \frac{a_2}{a'_1} [q(b) + q(a)] \\ & \left. + \frac{2(b-a)^2}{(2n+3)^2 \pi^2} \frac{a_2}{a'_1} \int_a^b \left( \cos \frac{(2n+3)\pi(x-a)}{b-a} \right) q'(x) dx \right\} + O(n^{-4}\eta(n)) + O(n^{-3}\eta^2(n)), \end{aligned}$$

(ii) if  $a'_2 = 0$  and  $b'_2 = 0$ ,

$$\begin{aligned} \lambda_n^{1/2} = & \frac{(n+2)\pi}{(b-a)} + \frac{1}{(n+2)\pi} \left\{ \frac{a'_1 b_2 - a_2 b'_1}{a'_1 b'_1} + \frac{b-a}{4(n+2)\pi} \int_a^b \left( \sin \frac{2(n+2)\pi(x-a)}{b-a} \right) q'(x) dx \right. \\ & - \frac{(b-a)^2}{(n+2)^2 \pi^2} \left[ \frac{3b'_1 b_1 b_2 - b_2^3}{3(b'_1)^3} \right] + \frac{(b-a)^2}{(n+2)^2 \pi^2} \left[ \frac{-3a'_1 a_1 a_2 + a_2^3}{3(a'_1)^3} \right] \\ & - \frac{(b-a)^2}{(n+2)^2 \pi^2} \frac{a_2}{2a'_1} [q(b) - q(a)] \\ & \left. + \frac{(b-a)^2}{(n+2)^2 \pi^2} \frac{a_2}{2a'_1} \int_a^b \left( \cos \frac{2(n+2)\pi(x-a)}{b-a} \right) q'(x) dx \right\} + O(n^{-4}\eta(n)) + O(n^{-3}\eta^2(n)). \end{aligned}$$

### 3 The method

We associate with (1) the Riccati equation

$$v'(t, \lambda) = -\lambda + q - v^2.$$

We define

$$S(t, \lambda) = \text{Re}[v(t, \lambda)], \tag{4}$$

$$T(t, \lambda) = \text{Im}[v(t, \lambda)]. \tag{5}$$

It is shown in [3] that any real-valued solution of (1) is in the form

$$y(t, \lambda) = R(t, \lambda) \cos \theta(t, \lambda) \tag{6}$$

with

$$S(t, \lambda) = \frac{R'(t, \lambda)}{R(t, \lambda)}, \tag{7}$$

$$T(t, \lambda) = \theta'(t, \lambda). \tag{8}$$

Our approach to calculating  $\lambda_n$  is to approximate those  $\lambda$  which are such that

$$\theta(b, \lambda) - \theta(a, \lambda) = \int_a^b T(x, \lambda) dx. \tag{9}$$

We suppose that there exist functions  $A(t)$  and  $\eta(\lambda)$  so that

$$\left| \int_t^b e^{2i\lambda^{1/2}x} q'(x) dx \right| \leq A(t) \eta(\lambda), \quad t \in [a, b] \tag{10}$$

where

- (i)  $A(t) := \int_t^b |q'(x)| dx$  is a decreasing function of  $t$ ,
- (ii)  $A(\cdot) \in L[a, b]$ ,
- (iii)  $\eta(\lambda) \rightarrow 0$  as  $\lambda^{1/2} \rightarrow \infty$ .

For  $q' \in L[a, b]$  the existence of the  $A$  and  $\eta$  functions may be established for  $\lambda$  positive as follows. We note that, avoiding the trivial case  $\int_t^b |q'(x)| dx = 0$ ,  $\left| \int_t^b e^{2i\lambda^{1/2}x} q'(x) dx \right| \leq \int_t^b |q'(x)| dx < \infty$  so, if we define

$$F(t, \lambda) := \begin{cases} \left| \int_t^b e^{2i\lambda^{1/2}x} q'(x) dx \right| / \int_t^b |q'(x)| dx, & \text{if } \int_t^b |q'(x)| dx \neq 0, \\ 0, & \text{if } \int_t^b |q'(x)| dx = 0, \end{cases} \tag{11}$$

then  $0 \leq F(t, \lambda) \leq 1$  and we set  $\eta(\lambda) := \sup_{a \leq t \leq b} F(t, \lambda)$ .  $\eta(\lambda)$  is well defined by (11) and  $\lambda^{-1/2} \eta(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$  [12].

Our method of approximating a solution of  $v'(t, \lambda) = -\lambda + q - v^2$  on  $[a, b]$  is similar to [12], so we set

$$v(t, \lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(t, \lambda). \tag{12}$$

When we put this serie into the Riccati equation and solve differential equations, we hold

$$\begin{aligned} v_1(t, \lambda) &= -e^{-2i\lambda^{1/2}t} \int_t^b e^{2i\lambda^{1/2}x} q(x) dx, \\ v_2(t, \lambda) &= e^{-2i\lambda^{1/2}t} \int_t^b e^{2i\lambda^{1/2}x} v_1^2(x, \lambda) dx, \\ v_n(t, \lambda) &= e^{-2i\lambda^{1/2}t} \int_t^b e^{2i\lambda^{1/2}x} [v_{n-1}^2(x, \lambda) + 2v_{n-1}(x, \lambda) \sum_{m=1}^{n-2} v_m(x, \lambda)] dx, \quad n \geq 3. \end{aligned} \quad (13)$$

Also we found  $\theta(b, \lambda) - \theta(a, \lambda) = \int_a^b T(x, \lambda) dx$ , so with (8) and (12) we have

$$\theta(b, \lambda) - \theta(a, \lambda) = \int_a^b \left[ \lambda^{1/2} + Im \sum_{n=1}^{\infty} v_n(x, \lambda) \right] dx,$$

then

$$\theta(b, \lambda) - \theta(a, \lambda) = \lambda^{1/2}(b-a) + \sum_{n=1}^{\infty} Im \int_a^b v_n(x, \lambda) dx. \quad (14)$$

**Theorem 3.** [1] If  $v(t, \lambda)$  as in (12), as  $\lambda \rightarrow \infty$

$$v(t, \lambda) = i\lambda^{1/2} + v_1(t, \lambda) + O(\lambda^{-1}\eta^2(\lambda))$$

where

$$\begin{aligned} v_1(t, \lambda) &= -\frac{1}{2}\lambda^{-1/2}q(b) \sin 2\lambda^{1/2}(b-t) + \frac{1}{2}\lambda^{-1/2} \int_t^b (\sin 2\lambda^{1/2}(x-t)) q'(x) dx \\ &+ i \left\{ \frac{1}{2}\lambda^{-1/2}q(b) \cos 2\lambda^{1/2}(b-t) - \frac{1}{2}\lambda^{-1/2}q(t) - \frac{1}{2}\lambda^{-1/2} \int_t^b (\cos 2\lambda^{1/2}(x-t)) q'(x) dx \right\}. \end{aligned}$$

and  $\eta(\lambda)$  is defined (11).

After some calculations by using the last theorem, with (4) we gain

$$\begin{aligned} S(t, \lambda) &= -\frac{1}{2}\lambda^{-1/2}q(b) \sin 2\lambda^{1/2}(b-t) + \frac{1}{2}\lambda^{-1/2} (\cos 2\lambda^{1/2}t) \int_t^b [\sin 2\lambda^{1/2}x] q'(x) dx \\ &- \frac{1}{2}\lambda^{-1/2} (\sin 2\lambda^{1/2}t) \int_t^b [\cos 2\lambda^{1/2}x] q'(x) dx + O(\lambda^{-1}\eta^2(\lambda)). \end{aligned}$$

Let define the following notations:

$$\begin{aligned} \sin \xi_t &:= \int_t^b (\cos 2\lambda^{1/2}x) q'(x) dx, \\ \cos \xi_t &:= \int_t^b (\sin 2\lambda^{1/2}x) q'(x) dx, \end{aligned}$$

thus we can write  $S(t, \lambda)$  as

$$S(t, \lambda) = -\frac{1}{2}\lambda^{-1/2}q(b) \sin 2\lambda^{1/2}(b-t) + \frac{1}{2}\lambda^{-1/2} \cos(2\lambda^{1/2}t + \xi_t) + O(\lambda^{-1}\eta^2(\lambda)). \quad (15)$$

Similarly, with (5) we find  $T(t, \lambda)$  as

$$T(t, \lambda) = \lambda^{1/2} + \frac{1}{2}\lambda^{-1/2}q(b)\cos 2\lambda^{1/2}(b-t) - \frac{1}{2}\lambda^{-1/2}q(t) - \frac{1}{2}\lambda^{-1/2}\sin\left(2\lambda^{1/2}t + \xi_t\right) + O\left(\lambda^{-1}\eta^2(\lambda)\right). \quad (16)$$

Also, by using integration by part to (13), we determine

$$\int_a^b v_1(x, \lambda) dx = \frac{i}{2\lambda^{1/2}} \int_a^b e^{2i\lambda^{1/2}(x-a)} q(x) dx$$

and again with integration by part

$$\begin{aligned} \int_a^b v_1(x, \lambda) dx &= \frac{i}{2}\lambda^{-1/2}e^{-2i\lambda^{1/2}a} \left[ \frac{q(x)e^{2i\lambda^{1/2}x}}{2i\lambda^{1/2}} \Big|_{x=a}^b - \frac{1}{2i\lambda^{1/2}} \int_a^b e^{2i\lambda^{1/2}x} q'(x) dx \right] \\ &= \frac{1}{4}\lambda^{-1}e^{2i\lambda^{1/2}(b-a)}q(b) - \frac{1}{4}\lambda^{-1}q(a) - \frac{1}{4}\lambda^{-1}e^{-2i\lambda^{1/2}a} \int_a^b e^{2i\lambda^{1/2}x} q'(x) dx \\ &= \frac{1}{4}\lambda^{-1}q(b) \left[ \cos 2\lambda^{1/2}(b-a) + i\sin 2\lambda^{1/2}(b-a) \right] - \frac{1}{4}\lambda^{-1}q(a) \\ &\quad - \frac{1}{4}\lambda^{-1} \int_a^b \left[ \cos 2\lambda^{1/2}(x-a) + i\sin 2\lambda^{1/2}(x-a) \right] q'(x) dx, \end{aligned}$$

so

$$Im \int_a^b v_1(x, \lambda) dx = \frac{1}{4}\lambda^{-1}q(b)\sin 2\lambda^{1/2}(b-a) - \frac{1}{4}\lambda^{-1}\cos\left(2\lambda^{1/2}a + \xi_a\right).$$

We also have from equation (13),

$$\int_a^b v_2(x, \lambda) dx = \frac{i}{2\lambda^{1/2}} \int_a^b \left[ 1 - e^{2i\lambda^{1/2}(x-a)} \right] v_1^2(x, \lambda) dx$$

and for  $n \geq 3$

$$\int_a^b v_n(x, \lambda) dx = \frac{i}{2\lambda^{1/2}} \times \int_a^b \left[ 1 - e^{2i\lambda^{1/2}(x-a)} \right] \left[ v_{n-1}^2(x, \lambda) + 2v_{n-1}(x, \lambda) \sum_{m=1}^{n-2} v_m(x, \lambda) \right] dx.$$

Thus, with the last equations

$$\begin{aligned} \int_a^b \sum_{n=1}^{\infty} Im \{v_n(x, \lambda)\} dx &= \sum_{n=1}^{\infty} Im \left\{ \int_a^b v_n(x, \lambda) dx \right\} \\ &= \frac{1}{4}\lambda^{-1}q(b)\sin 2\lambda^{1/2}(b-a) - \frac{1}{4}\lambda^{-1}\cos\left(2\lambda^{1/2}a + \xi_a\right) + O\left(\lambda^{-3/2}\eta^2(\lambda)\right). \end{aligned} \quad (17)$$

### 4 Proof of the main results

**Proof of theorem 1.** (i) If  $a'_2 \neq 0$  and  $b'_2 \neq 0$ , the real solution of  $y''(t) + [\lambda - q(t)]y(t) = 0$  is  $y(t, \lambda) = R(t, \lambda)\cos\theta(t, \lambda)$  from (6). We use this equation for boundary  $t = a$ , we find

$$R(a, \lambda) \left\{ \cos\theta(a, \lambda) \left[ (-a'_2\lambda + a_2) \frac{R'(a, \lambda)}{R(a, \lambda)} + (-a'_1\lambda + a_1) \right] + (a'_2\lambda - a_2) \theta'(a, \lambda) \sin\theta(a, \lambda) \right\} = 0.$$

If we choose  $\alpha_1$  as

$$\begin{aligned} \sin\alpha_1 &:= (-a'_2\lambda + a_2) \frac{R'(a, \lambda)}{R(a, \lambda)} + (-a'_1\lambda + a_1), \\ \cos\alpha_1 &:= (a'_2\lambda - a_2) \theta'(a, \lambda), \end{aligned}$$

we have  $R(a, \lambda) \sin[\alpha_1 + \theta(a, \lambda)] = 0$  so  $\sin(\alpha_1 + \theta(a, \lambda)) = 0$ , or  $\theta(a, \lambda) = -\alpha_1$ . Using by equations (7) and (8) as  $S(t, \lambda) = \frac{R'(t, \lambda)}{R(t, \lambda)}$ ,  $T(t, \lambda) = \theta'(t, \lambda)$  and their asymptotic expansions (15)-(16), we calculate

$$\frac{\sin \alpha_1}{\cos \alpha_1} = \frac{-a'_1 \lambda + \frac{1}{2} \lambda^{1/2} a'_2 q(b) \sin 2\lambda^{1/2}(b-a) - \frac{1}{2} \lambda^{1/2} a'_2 \cos(2\lambda^{1/2}a + \xi_a) + a_1 - \frac{1}{2} \lambda^{-1/2} a_2 q(b) \sin 2\lambda^{1/2}(b-a) + \frac{1}{2} \lambda^{-1/2} a_2 \cos(2\lambda^{1/2}a + \xi_a) + O(\eta^2(\lambda)) + O(\lambda^{-1}\eta^2(\lambda))}{a'_2 \lambda^{3/2} + \frac{1}{2} \lambda^{1/2} a'_2 q(b) \cos 2\lambda^{1/2}(b-a) - \frac{1}{2} \lambda^{1/2} a'_2 q(a) - \frac{1}{2} \lambda^{1/2} a'_2 \sin(2\lambda^{1/2}a + \xi_a) - a_2 \lambda^{1/2} - \frac{1}{2} \lambda^{-1/2} a_2 q(b) \cos 2\lambda^{1/2}(b-a) + \frac{1}{2} \lambda^{-1/2} a_2 q(a) + \frac{1}{2} \lambda^{-1/2} a_2 \sin(2\lambda^{1/2}a + \xi_a) + O(\eta^2(\lambda)) + O(\lambda^{-1}\eta^2(\lambda))},$$

hence

$$\frac{\sin \alpha_1}{\cos \alpha_1} = \frac{-a'_1 \lambda + \frac{1}{2} \lambda^{1/2} a'_2 q(b) \sin 2\lambda^{1/2}(b-a) - \frac{1}{2} \lambda^{1/2} a'_2 \cos(2\lambda^{1/2}a + \xi_a) + a_1 - \frac{1}{2} \lambda^{-1/2} a_2 q(b) \sin 2\lambda^{1/2}(b-a) + \frac{1}{2} \lambda^{-1/2} a_2 \cos(2\lambda^{1/2}a + \xi_a) + O(\eta^2(\lambda)) + O(\lambda^{-1}\eta^2(\lambda))}{1 + \frac{1}{2} \lambda^{-1} q(b) \cos 2\lambda^{1/2}(b-a) - \frac{1}{2} \lambda^{-1} q(a) - \frac{1}{2} \lambda^{-1} \sin(2\lambda^{1/2}a + \xi_a) - \frac{a_2}{a'_2} \lambda^{-1} - \frac{1}{2} \lambda^{-2} \frac{a_2}{a'_2} q(b) \cos 2\lambda^{1/2}(b-a) + \frac{1}{2} \lambda^{-2} \frac{a_2}{a'_2} q(a) + \frac{1}{2} \lambda^{-2} \frac{a_2}{a'_2} \sin(2\lambda^{1/2}a + \xi_a) + O(\lambda^{-3/2}\eta^2(\lambda))}.$$

Then

$$\tan \alpha_1 = \left[ \begin{array}{l} -\frac{a'_1}{a'_2} \lambda^{-1/2} + \frac{1}{2} \lambda^{-1} q(b) \sin 2\lambda^{1/2}(b-a) - \frac{1}{2} \lambda^{-1} \cos(2\lambda^{1/2}a + \xi_a) + \frac{a_1}{a'_2} \lambda^{-3/2} \\ -\frac{a_2}{2a'_2} \lambda^{-2} q(b) \sin 2\lambda^{1/2}(b-a) + \frac{a_2}{2a'_2} \lambda^{-2} \cos(2\lambda^{1/2}a + \xi_a) + O(\lambda^{-3/2}\eta^2(\lambda)) \end{array} \right] \times \left[ \begin{array}{l} 1 - \frac{1}{2} \lambda^{-1} q(b) \cos 2\lambda^{1/2}(b-a) + \frac{1}{2} \lambda^{-1} q(a) \\ + \frac{1}{2} \lambda^{-1} \sin(2\lambda^{1/2}a + \xi_a) + \frac{a_2}{a'_2} \lambda^{-1} + \frac{1}{2} \lambda^{-2} \frac{a_2}{a'_2} q(b) \cos 2\lambda^{1/2}(b-a) \\ - \frac{1}{2} \lambda^{-2} \frac{a_2}{a'_2} q(a) - \frac{1}{2} \lambda^{-2} \frac{a_2}{a'_2} \sin(2\lambda^{1/2}a + \xi_a) + \frac{1}{4} \lambda^{-2} q^2(a) \\ + \frac{1}{4} \lambda^{-2} q^2(b) \cos 2\lambda^{1/2}(b-a) + \frac{1}{4} \lambda^{-2} \sin^2(2\lambda^{1/2}a + \xi_a) \\ + \left[ \frac{a_2}{a'_2} \right]^2 \lambda^{-2} - \frac{1}{2} \lambda^{-2} q(a) q(b) \cos 2\lambda^{1/2}(b-a) \\ - \frac{1}{2} \lambda^{-2} q(b) \cos 2\lambda^{1/2}(b-a) \sin(2\lambda^{1/2}a + \xi_a) \\ - \frac{a_2}{a'_2} \lambda^{-2} q(b) \cos 2\lambda^{1/2}(b-a) + \frac{1}{2} \lambda^{-2} q(a) \sin(2\lambda^{1/2}a + \xi_a) \\ + \frac{a_2}{a'_2} \lambda^{-2} \sin(2\lambda^{1/2}a + \xi_a) + \frac{a_2}{a'_2} \lambda^{-2} q(a) + O(\lambda^{-3/2}\eta^2(\lambda)) \end{array} \right],$$

so

$$\begin{aligned} \tan \alpha_1 = & -\frac{a'_1}{a'_2} \lambda^{-1/2} + \frac{1}{2} \lambda^{-1} q(b) \sin 2\lambda^{1/2}(b-a) - \frac{1}{2} \lambda^{-1} \cos(2\lambda^{1/2}a + \xi_a) + \frac{a_1}{a'_2} \lambda^{-3/2} \\ & - \frac{a_2}{2a'_2} \lambda^{-2} q(b) \sin 2\lambda^{1/2}(b-a) + \frac{a_2}{2a'_2} \lambda^{-2} \cos(2\lambda^{1/2}a + \xi_a) \\ & + \frac{a'_1}{2a'_2} \lambda^{-3/2} q(b) \cos 2\lambda^{1/2}(b-a) - \frac{a'_1}{2a'_2} \lambda^{-3/2} q(a) - \frac{a'_1}{2a'_2} \lambda^{-3/2} \sin(2\lambda^{1/2}a + \xi_a) - \frac{a'_1 a_2}{[a'_2]^2} \lambda^{-3/2} \\ & - \frac{1}{4} \lambda^{-2} q^2(b) \cos 2\lambda^{1/2}(b-a) \sin 2\lambda^{1/2}(b-a) + \frac{1}{4} \lambda^{-2} q(a) q(b) \sin 2\lambda^{1/2}(b-a) \\ & + \frac{1}{4} \lambda^{-2} q(b) \sin 2\lambda^{1/2}(b-a) \sin(2\lambda^{1/2}a + \xi_a) + \frac{a_2}{2a'_2} \lambda^{-2} q(b) \sin 2\lambda^{1/2}(b-a) \\ & + \frac{1}{4} \lambda^{-2} q(b) \cos 2\lambda^{1/2}(b-a) \cos(2\lambda^{1/2}a + \xi_a) - \frac{a_2}{2a'_2} \lambda^{-2} \cos(2\lambda^{1/2}a + \xi_a) \\ & - \frac{1}{4} \lambda^{-2} q(a) \cos(2\lambda^{1/2}a + \xi_a) + O(\lambda^{-3/2}\eta^2(\lambda)). \end{aligned}$$

In the last equation, by using Taylor expansion of  $\arctan x$  at  $x = 0$ , we obtain

$$\begin{aligned} \alpha_1 = & -\frac{a'_1}{a'_2} \lambda^{-1/2} + \frac{1}{2} \lambda^{-1} q(b) \sin 2\lambda^{1/2}(b-a) - \frac{1}{2} \lambda^{-1} \cos(2\lambda^{1/2}a + \xi_a) \\ & + \frac{a_1}{a'_2} \lambda^{-3/2} + \frac{1}{3} \left[ \frac{a'_1}{a'_2} \right]^3 \lambda^{-3/2} + \frac{a'_1}{2a'_2} \lambda^{-3/2} q(b) \cos 2\lambda^{1/2}(b-a) \\ & - \frac{a'_1}{2a'_2} \lambda^{-3/2} q(a) - \frac{a'_1}{2a'_2} \lambda^{-3/2} \sin(2\lambda^{1/2}a + \xi_a) - \frac{a'_1 a_2}{[a'_2]^2} \lambda^{-3/2} \\ & - \frac{(a'_1)^2}{2(a'_2)^2} \lambda^{-3/2} q(b) \sin 2\lambda^{1/2}(b-a) + \frac{1}{2} \left[ \frac{a'_1}{a'_2} \right]^2 \lambda^{-2} \sin(2\lambda^{1/2}a + \xi_a) \\ & - \frac{1}{4} \lambda^{-2} q^2(b) \cos 2\lambda^{1/2}(b-a) \sin 2\lambda^{1/2}(b-a) + \frac{1}{4} \lambda^{-2} q(a) q(b) \sin 2\lambda^{1/2}(b-a) \\ & + \frac{1}{4} \lambda^{-2} q(b) \sin 2\lambda^{1/2}(b-a) \sin(2\lambda^{1/2}a + \xi_a) + \frac{1}{4} \lambda^{-2} q(b) \cos 2\lambda^{1/2}(b-a) \cos(2\lambda^{1/2}a + \xi_a) \\ & - \frac{1}{2} \lambda^{-2} \frac{a_2}{2a'_2} \cos(2\lambda^{1/2}a + \xi_a) - \frac{1}{4} \lambda^{-2} q(a) \cos(2\lambda^{1/2}a + \xi_a) + O(\lambda^{-3/2} \eta^2(\lambda)). \end{aligned} \tag{18}$$

Similarly, when we use the form  $y(t, \lambda) = R(t, \lambda) \cos \theta(t, \lambda)$  for boundary  $t = b$ , we find

$$R(b, \lambda) \left\{ \cos \theta(b, \lambda) \left[ (-b'_2 \lambda + b_2) \frac{R'(b, \lambda)}{R(b, \lambda)} + (-b'_1 \lambda + b_1) \right] - (-b'_2 \lambda + b_2) \theta'(b, \lambda) \sin \theta(b, \lambda) \right\} = 0.$$

If we choose  $\alpha_2$  as

$$\begin{aligned} \sin \alpha_2 & := (-b'_2 \lambda + b_2) \frac{R'(b, \lambda)}{R(b, \lambda)} + (-b'_1 \lambda + b_1), \\ \cos \alpha_2 & := (-b'_2 \lambda + b_2) \theta'(b, \lambda), \end{aligned}$$

we have  $R(b, \lambda) \sin[\alpha_2 - \theta(b, \lambda)] = 0$  so  $\sin[\alpha_2 - \theta(b, \lambda)] = 0$ , or  $\theta(b, \lambda) = \alpha_2 + (n + 1)\pi$ . Using by  $S(t, \lambda) = \frac{R'(t, \lambda)}{R(t, \lambda)}$ ,  $T(t, \lambda) = \theta'(t, \lambda)$  and their asymptotic expansions (15)-(16) we can write

$$\frac{\sin \alpha_2}{\cos \alpha_2} = \frac{-\lambda b'_1 + b_1 + O(\eta^2(\lambda)) + O(\lambda^{-1} \eta^2(\lambda))}{-\lambda^{3/2} b'_2 + \lambda^{1/2} b_2 + O(\eta^2(\lambda)) + O(\lambda^{-1} \eta^2(\lambda))},$$

so

$$\begin{aligned} \tan \alpha_2 & = \left[ \lambda^{-1/2} \frac{b'_1}{b'_2} - \lambda^{-3/2} \frac{b_1}{b'_2} + O(\lambda^{-3/2} \eta^2(\lambda)) \right] \times \left[ 1 + \lambda^{-1} \frac{b_2}{b'_2} + \lambda^{-2} \frac{b_2^2}{(b'_2)^2} + O(\lambda^{-3/2} \eta^2(\lambda)) \right] \\ & = \lambda^{-1/2} \frac{b'_1}{b'_2} - \lambda^{-3/2} \frac{b_1}{b'_2} + \lambda^{-3/2} \frac{b'_1 b_2}{(b'_2)^2} + O(\lambda^{-3/2} \eta^2(\lambda)). \end{aligned}$$

In the last equation, by using Taylor expansion of  $\arctan x$  at  $x = 0$ , we obtain

$$\alpha_2 = \lambda^{-1/2} \frac{b'_1}{b'_2} - \lambda^{-3/2} \frac{b_1}{b'_2} + \lambda^{-3/2} \frac{b'_1 b_2}{(b'_2)^2} - \lambda^{-3/2} \frac{(b'_1)^3}{(b'_2)^2} + O(\lambda^{-3/2} \eta^2(\lambda)). \tag{19}$$

Let use these findings (17), (18) and (19) in  $\theta(b, \lambda) - \theta(a, \lambda) = \lambda^{1/2}(b-a) + \sum_{n=1}^{\infty} \text{Im} \int_a^b v_n(x, \lambda) dx$ , we see that

$$\begin{aligned}
 & (n+1)\pi + \lambda^{-1/2} \frac{b'_1}{b'_2} - \lambda^{-3/2} \frac{b_1}{b'_2} + \lambda^{-3/2} \frac{b'_1 b_2}{(b'_2)^2} - \lambda^{-3/2} \frac{(b'_1)^3}{(b'_2)^2} \\
 & - \frac{a'_1}{a'_2} \lambda^{-1/2} + \frac{1}{2} \lambda^{-1} q(b) \sin 2\lambda^{1/2}(b-a) - \frac{1}{2} \lambda^{-1} \cos(2\lambda^{1/2}a + \xi_a) \\
 & + \frac{a_1}{a'_2} \lambda^{-3/2} + \frac{1}{3} \left[ \frac{a'_1}{a'_2} \right]^3 \lambda^{-3/2} + \frac{a'_1}{2a'_2} \lambda^{-3/2} q(b) \cos 2\lambda^{1/2}(b-a) \\
 & - \frac{a'_1}{2a'_2} \lambda^{-3/2} q(a) - \frac{a'_1}{2a'_2} \lambda^{-3/2} \sin(2\lambda^{1/2}a + \xi_a) - \frac{a'_1 a_2}{[a'_2]^2} \lambda^{-3/2} \\
 & - \frac{1}{4} \lambda^{-2} q^2(b) \cos 2\lambda^{1/2}(b-a) \sin 2\lambda^{1/2}(b-a) + \frac{1}{4} \lambda^{-2} q(a) q(b) \sin 2\lambda^{1/2}(b-a) \\
 & + \frac{1}{4} \lambda^{-2} q(b) \sin 2\lambda^{1/2}(b-a) \sin(2\lambda^{1/2}a + \xi_a) - \frac{1}{2} \lambda^{-2} \left[ \frac{a'_1}{a'_2} \right]^2 q(b) \sin 2\lambda^{1/2}(b-a) \\
 & + \frac{1}{4} \lambda^{-2} q(b) \cos 2\lambda^{1/2}(b-a) \cos(2\lambda^{1/2}a + \xi_a) \\
 & - \frac{1}{4} \lambda^{-2} q(a) \cos(2\lambda^{1/2}a + \xi_a) - \frac{1}{4} \lambda^{-2} q(a) \cos(2\lambda^{1/2}a + \xi_a) + O(\lambda^{-3/2} \eta^2(\lambda)) \\
 & = \lambda^{1/2}(b-a) + \frac{1}{4} \lambda^{-1} q(b) \sin 2\lambda^{1/2}(b-a) - \frac{1}{4} \lambda^{-1} \cos(2\lambda^{1/2}a + \xi_a) + O(\lambda^{-2} \eta(\lambda)).
 \end{aligned}$$

We prove the theorem by using definitions of  $\sin \xi_t$ ,  $\cos \xi_t$  and  $\eta(\lambda)$ ; also series error computation in the last equation.

**Proof of theorem 2.** (ii) If  $a'_2 = 0$  and  $b'_2 = 0$ , our problem is

$$\begin{aligned}
 y''(t) + \{\lambda - q(t)\}y(t) &= 0, \quad t \in [a, b], \\
 a_1 y(a) + a_2 y'(a) &= \lambda a'_1 y(a), \\
 b_1 y(b) + b_2 y'(b) &= \lambda b'_1 y(b).
 \end{aligned}$$

The real solution of  $y''(t) + [\lambda - q(t)]y(t) = 0$  is  $y(t, \lambda) = R(t, \lambda) \cos \theta(t, \lambda)$ . We use this equation for boundary  $t = a$ , we find

$$R(a, \lambda) \left\{ \cos \theta(a, \lambda) \left[ a_2 \frac{R'(a, \lambda)}{R(a, \lambda)} - a'_1 \lambda + a_1 \right] - a_2 \theta'(a, \lambda) \sin \theta(a, \lambda) \right\} = 0.$$

If we choose  $\alpha_3$  as

$$\begin{aligned}
 \sin \alpha_3 &:= a_2 \frac{R'(a, \lambda)}{R(a, \lambda)} - a'_1 \lambda + a_1, \\
 \cos \alpha_3 &:= -a_2 \theta'(a, \lambda),
 \end{aligned}$$

we have  $R(a, \lambda) \sin(\alpha_3 + \theta(a, \lambda)) = 0$  so  $\sin(\alpha_3 + \theta(a, \lambda)) = 0$ , or  $\theta(a, \lambda) = -\alpha_3$ . Using by  $S(t, \lambda) = \frac{R'(t, \lambda)}{R(t, \lambda)}$ ,  $T(t, \lambda) = \theta'(t, \lambda)$  and their asymptotic expansions, one writes

$$\cot \alpha_3 = \frac{-a_2 \lambda^{-1/2} - \frac{a_2}{2} \lambda^{-1/2} q(b) \cos 2\lambda^{1/2}(b-a) + \frac{a_2}{2} \lambda^{-1/2} \sin(2\lambda^{1/2}a + \xi_a) + \frac{a_2}{2} \lambda^{-1/2} q(a) + O(\lambda^{-1} \eta^2(\lambda))}{-a'_1 \lambda \left[ 1 - \frac{a_1}{a'_1} \lambda^{-1} + \frac{a_2}{2a'_1} \lambda^{-3/2} q(b) \sin 2\lambda^{1/2}(b-a) - \frac{a_2}{2a'_1} \lambda^{-3/2} \cos(2\lambda^{1/2}a + \xi_a) \right] + O(\lambda^{-2} \eta^2(\lambda))}$$



$$\cot \alpha_3 = \frac{a_2}{a'_1} \lambda^{-1/2} + \frac{a_2}{2a'_1} \lambda^{-3/2} q(b) \cos 2\lambda^{1/2} (b-a) - \frac{a_2}{2a'_1} \lambda^{-3/2} \sin \left( 2\lambda^{1/2} a + \xi_a \right) - \frac{a_2}{2a'_1} \lambda^{-3/2} q(a) + O(\lambda^{-2} \eta^2(\lambda))$$

$$\times \left[ 1 + \frac{a_1}{a'_1} \lambda^{-1} - \frac{a_2}{2a'_1} \lambda^{-3/2} q(b) \sin 2\lambda^{1/2} (b-a) + \frac{a_2}{2a'_1} \lambda^{-3/2} \cos \left( 2\lambda^{1/2} a + \xi_a \right) \right. \\ \left. + \left[ \frac{a_1}{a'_1} \right]^2 \lambda^{-2} - \frac{a_1 a_2}{[a'_1]^2} \lambda^{-5/2} q(b) \sin 2\lambda^{1/2} (b-a) + \frac{a_1 a_2}{[a'_1]^2} \lambda^{-5/2} \cos \left( 2\lambda^{1/2} a + \xi_a \right) + O(\lambda^{-2} \eta^2(\lambda)) \right],$$

then

$$\cot \alpha_3 = \frac{a_2}{a'_1} \lambda^{-1/2} + \frac{a_2}{2a'_1} \lambda^{-3/2} q(b) \cos 2\lambda^{1/2} (b-a) - \frac{a_2}{2a'_1} \lambda^{-3/2} \sin \left( 2\lambda^{1/2} a + \xi_a \right) - \frac{a_2}{2a'_1} \lambda^{-3/2} q(a)$$

$$+ \frac{a_1 a_2}{[a'_1]^2} \lambda^{-3/2} - \frac{a_2^2}{2[a'_1]^2} \lambda^{-2} q(b) \sin 2\lambda^{1/2} (b-a) + \frac{a_1^2 a_2}{[a'_1]^3} \lambda^{-5/2}$$

$$+ \frac{a_2^2}{2[a'_1]^2} \lambda^{-2} \cos \left( 2\lambda^{1/2} a + \xi_a \right) + \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} q(b) \cos 2\lambda^{1/2} (b-a)$$

$$- \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} q(a) - \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} \sin \left( 2\lambda^{1/2} a + \xi_a \right) + O(\lambda^{-2} \eta^2(\lambda)).$$

In the last equation, by using Taylor expansion of  $\operatorname{arccot} x$  at  $x = 0$ , we obtain

$$-\theta(a, \lambda) = \alpha_3 = \frac{\pi}{2} - \frac{a_2}{a'_1} \lambda^{-1/2} - \frac{a_2}{2a'_1} \lambda^{-3/2} q(b) \cos 2\lambda^{1/2} (b-a)$$

$$+ \frac{a_2}{2a'_1} \lambda^{-3/2} q(a) + \frac{a_2}{2a'_1} \lambda^{-3/2} \sin \left( 2\lambda^{1/2} a + \xi_a \right) - \frac{a_1 a_2}{[a'_1]^2} \lambda^{-3/2}$$

$$+ \frac{a_2^3}{3[a'_1]^3} \lambda^{-3/2} + \frac{a_2^2}{2[a'_1]^2} \lambda^{-2} q(b) \sin 2\lambda^{1/2} (b-a)$$

$$- \frac{a_2^2}{2[a'_1]^2} \lambda^{-2} \cos \left( 2\lambda^{1/2} a + \xi_a \right) - \frac{a_1^2 a_2}{[a'_1]^3} \lambda^{-5/2} \tag{20}$$

$$- \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} q(b) \cos 2\lambda^{1/2} (b-a) + \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} q(a)$$

$$+ \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} \sin \left( 2\lambda^{1/2} a + \xi_a \right) - \frac{(a_2)^3}{2[a'_1]^3} \lambda^{-5/2} \sin \left( 2\lambda^{1/2} a + \xi_a \right)$$

$$- \frac{a_2^3}{2[a'_1]^3} \lambda^{-5/2} q(a) + \frac{a_1 a_2^3}{2[a'_1]^4} \lambda^{-5/2} + O(\lambda^{-2} \eta^2(\lambda)).$$

For boundary  $t = b$ , by using  $y(b, \lambda) = R(b, \lambda) \cos \theta(b, \lambda)$ , we find

$$R(b, \lambda) \left\{ \cos \theta(b, \lambda) \left[ b_2 \frac{R'(b, \lambda)}{R(b, \lambda)} - b'_1 \lambda + b_1 \right] - b_2 \theta'(b, \lambda) \sin \theta(b, \lambda) \right\} = 0.$$

If we choose  $\alpha_4$  as

$$\sin \alpha_4 := b_2 \frac{R'(b, \lambda)}{R(b, \lambda)} - b'_1 \lambda + b_1,$$

$$\cos \alpha_4 := b_2 \theta'(b, \lambda),$$

we have  $R(b, \lambda) \sin[\alpha_4 - \theta(b, \lambda)] = 0$  so  $\sin[\alpha_4 - \theta(b, \lambda)] = 0$ , or  $\theta(b, \lambda) = (n+1)\pi + \alpha_4$ . Using by  $S(t, \lambda) = \frac{R'(t, \lambda)}{R(t, \lambda)}$ ,  $T(t, \lambda) = \theta'(t, \lambda)$  and their asymptotic expansions, one writes

$$\begin{aligned} \cot \alpha_4 &= \frac{\lambda^{-1/2} b_2 + O(\lambda^{-1} \eta^2(\lambda))}{-b'_1 \lambda \left[ 1 - \frac{b_1}{b'_1} \lambda^{-1} + O(\lambda^{-2} \eta^2(\lambda)) \right]} \\ &= \left[ -\lambda^{-1/2} \frac{b_2}{b'_1} + O(\lambda^{-2} \eta^2(\lambda)) \right] \times \left[ 1 + \frac{b_1}{b'_1} \lambda^{-1} + \left[ \frac{b_1}{b'_1} \right]^2 \lambda^{-2} + O(\lambda^{-2} \eta^2(\lambda)) \right], \end{aligned}$$

then

$$\cot \alpha_4 = -\lambda^{-1/2} \frac{b_2}{b'_1} - \lambda^{-3/2} \frac{b_1 b_2}{(b'_1)^2} - \lambda^{-5/2} \frac{b_1^2 b_2}{(b'_1)^3} - \lambda^{-5/2} \frac{b_1 b_2^3}{(b'_1)^4} + O(\lambda^{-2} \eta^2(\lambda)).$$

In the last equation, by using Taylor expansion of  $\operatorname{arccot} x$  at  $x = 0$ , we obtain

$$\alpha_4 = \frac{\pi}{2} + \lambda^{-1/2} \frac{b_2}{b'_1} + \lambda^{-3/2} \frac{b_1 b_2}{(b'_1)^2} - \lambda^{-3/2} \frac{b_2^3}{3(b'_1)^3} - \lambda^{-5/2} \frac{b_1^2 b_2}{(b'_1)^3} + O(\lambda^{-2} \eta^2(\lambda))$$

so

$$\begin{aligned} \theta(b, \lambda) &= (n+1)\pi + \frac{\pi}{2} + \lambda^{-1/2} \frac{b_2}{b'_1} + \lambda^{-3/2} \frac{b_1 b_2}{(b'_1)^2} - \lambda^{-3/2} \frac{b_2^3}{3(b'_1)^3} \\ &\quad - \lambda^{-5/2} \frac{b_1^2 b_2}{(b'_1)^3} - \lambda^{-5/2} \frac{b_1 b_2^3}{(b'_1)^4} + O(\lambda^{-2} \eta^2(\lambda)) \end{aligned} \quad (21)$$

Let use these findings in  $\theta(b, \lambda) - \theta(a, \lambda) = \lambda^{1/2}(b-a) + \sum_{n=1}^{\infty} \operatorname{Im} \int_a^b v_n(x, \lambda) dx$ , we estimate that

$$\begin{aligned} &(n+1)\pi + \frac{\pi}{2} + \lambda^{-1/2} \frac{b_2}{b'_1} + \lambda^{-3/2} \frac{b_1 b_2}{(b'_1)^2} - \lambda^{-3/2} \frac{b_2^3}{3(b'_1)^3} - \lambda^{-5/2} \frac{b_1^2 b_2}{(b'_1)^3} \\ &- \lambda^{-5/2} \frac{b_1 b_2^3}{(b'_1)^4} + \frac{\pi}{2} - \frac{a_2}{a'_1} \lambda^{-1/2} - \frac{a_2}{2a'_1} \lambda^{-3/2} q(b) \cos 2\lambda^{1/2}(b-a) \\ &+ \frac{a_2}{2a'_1} \lambda^{-3/2} q(a) + \frac{a_2}{2a'_1} \lambda^{-3/2} \sin(2\lambda^{1/2}a + \xi_a) - \frac{a_1 a_2}{[a'_1]^2} \lambda^{-3/2} \\ &+ \frac{a_2^3}{3[a'_1]^3} \lambda^{-3/2} + \frac{a_2^2}{2[a'_1]^2} \lambda^{-2} q(b) \sin 2\lambda^{1/2}(b-a) - \frac{a_2^2}{2[a'_1]^2} \lambda^{-2} \cos(2\lambda^{1/2}a + \xi_a) \\ &- \frac{a_1^2 a_2}{[a'_1]^3} \lambda^{-5/2} - \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} q(b) \cos 2\lambda^{1/2}(b-a) + \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} q(a) \\ &- \frac{a_2^3}{2[a'_1]^3} \lambda^{-5/2} q(a) + \frac{a_1 a_2^3}{[a'_1]^4} \lambda^{-5/2} \\ &+ \frac{a_1 a_2}{2[a'_1]^2} \lambda^{-5/2} \sin(2\lambda^{1/2}a + \xi_a) - \frac{(a_2)^3}{2[a'_1]^3} \lambda^{-5/2} \sin(2\lambda^{1/2}a + \xi_a) + O(\lambda^{-2} \eta^2(\lambda)) \\ &= \lambda^{1/2}(b-a) + \frac{1}{4} \lambda^{-1} q(b) \sin 2\lambda^{1/2}(b-a) - \frac{1}{4} \lambda^{-1} \cos(2\lambda^{1/2}a + \xi_a) \\ &+ O(\lambda^{-2} \eta^2(\lambda)) + O(\lambda^{-3/2} \eta^2(\lambda)). \end{aligned}$$

We prove the theorem by using definitions of  $\sin \xi_r$ ,  $\cos \xi_r$  and  $\eta(\lambda)$ ; also series error computation in the last equation.

Similarly, Theorem 1 (ii) follows from (14), (18) and (21); Theorem 2 (i) follows from (14), (20) and (19).

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

### References

- [1] Bařkaya, E., Regular Sturm-Liouville problems with eigenvalue parameter in the boundary conditions, PhD. Thesis, Karadeniz Technical University, The Graduate School of Natural and Applied Sciences Biology Graduate Program, Trabzon, 2013.
- [2] Coskun, H.; Bayram, N., Asymptotics of eigenvalues for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition, *J. Math. Anal. Appl.* 306 (2005), 548-566.
- [3] Coskun, H.; Harris, B. J., Estimates for the periodic and semi-periodic eigenvalues of Hill's equation, *Proc. Roy. Soc. Edinburgh (A)* 130 (2000), 991-998.
- [4] Coskun, H., On the spectrum of a second-order periodic differential equation, *Rocky Mountain J. Math.* 33 (2003), 1261-1277.
- [5] Eastham, M. S., *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh 1973.
- [6] Fix, G., Asymptotic eigenvalues of Sturm-Liouville systems, *J. Math. Anal. Appl.* 19 (1967), 519-525.
- [7] Fulton, C. T., An integral equation iterative scheme for asymptotic expansions of spectral quantities of regular Sturm-Liouville problems, *J. Integral Equations* 4 (1982), 163-172.
- [8] Fulton, C. T., Two point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh (A)* 77 (1977), 293-308.
- [9] Fulton, C. T.; Pruess, S. A., Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems, *J. Math. Anal. Appl.* 182 (1994), 297-340.
- [10] B. J. Harris, A series solution for certain Riccati equations with applications to Sturm-Liouville problems, *J. Math. Anal. Appl.* 137 (1989), 462-470.
- [11] B. J. Harris, Asymptotics of eigenvalues for regular Sturm-Liouville problems, *J. Math. Anal. Appl.* 183 (1994), 25-36.
- [12] B. J. Harris, The form of the spectral functions associated with Sturm-Liouville problems with continuous spectrum, *Mathematika* 44 (1997), 149-161.
- [13] Hochstadt, H., Asymptotic estimates for the Sturm-Liouville spectrum, *Comm. Pure Appl. Math.* 14 (1961), 749-764.
- [14] Titchmarsh, E. C., *Eigenfunction Expansions Associated with Second Order Differential Equations I*, 2 nd edn., Oxford Univ. Press, Oxford 1962.
- [15] Walter, J., Regular eigenvalue problems with eigenvalue parameter in the boundary condition, *Math. Z.* 153 (1973), 301-312.