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Bernsteinn polynomials approach to determine timelike curves of constant breadth in Minkowski 3-space

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Abstract: In this study, we first show that the system of Frenet-like differential equation characterizing timelike curves of constant breadth is equivalent to a third order, linear, differential equation with variable coefficients. Then, by using a rational approximation based on Bernstein polynomials, we obtain the set of solution of the mentioned differential equation under the given initial conditions. Furthermore, we discuss that the obtained results are useable to determine timelike curves of constant breadth in Minkowski 3-space E_1^3.

Keywords: Timelike curve of constant breadth, Bernstein polynomials, Linear differential equations, Minkowski 3-space.

1 Introduction

Curves of constant breadth firstly were introduced by L. Euler in 1778 [7]. F. Reuleaux gave the obtaining method some curves of constant breadth in 1963 and led to be used in kinematics of machinery [3]. So far, in mathematics, many geometers have obtained only geometric properties of the plane curves of constant breadth, but have had a few study on space curves of constant breadth [1, 15, 16, 18]. A number of interesting properties of these curves in plane are included in the works of Mellish [2]. M. Fujivara had obtained a problem to determine whether there exist "space curve of constant breadth" or not and he defined "breadth" for space curves and obtained these curves on a surface of constant breadth [8]. Having been used the basic concepts [15] concerned with the space curves of constant breadth, a integral characterization of these curves [17, 11] has been obtained and a criterion for these curves has been determined [10]. Also the curves of constant breadth were extented to the E^4- space and some characterizations were obtained [1]. İn addition, Akdoğan and Mağden [18] extended to E^n space this kind of curves and they obtained some characterizations. Also Aydın [16] obtained differential equation characterizing curves of constant breadth in E^n and then she obtained approximate solutions of this equation by using Taylor matrix collocation method. Studies in different spaces [9, 6] on these curves are going on nowadays, currently. These curves are used in the kinematics of machinary, engineering and com design.

In this study, our first aim to establish differential equation discribing a timelike curve of constant breadth in Minkowski 3-space. The second aim is to find an approximate solution based on Bernstein polynomials of this differential equation [12, 13]. İn this study we also analyzed the role of the obtained solution in determining these curves.

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2 Preliminaries

Bernstein polynomials of nth-degree are defined by

$$B_{k,n}(x)=\binom{n}{k}\frac{x^k(R-x)^{n-k}}{R^n},\ k=0,1,\dots,n$$

where R is the maximum range of the interval $[0,R]$ over which the polynomials are defined to form a complete basis [12].

The Minkowski 3-space is real vector space R^3provided with the standart flat metric given by

$$g=-dx_1^2+dx_2^2+dx_3^2$$

where $(x_1,x_2,\ x_3)$ is a rectangular coordinate system of Minkowski 3-space E_1^3. An arbitrary vector $\overrightarrow{v}=(v_1,v_2,\ v_3)$ in E_1^3 can be timelike if $g(\overrightarrow{v},\overrightarrow{v})<0$. Similarly, an arbitrary curve $\overrightarrow{\alpha}=\overrightarrow{\alpha}(s)$ locally be timelike if all of its velocity $\overrightarrow{\alpha}'(s)$ are timelike. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively [9].

Furthermore, for a timelike curve $\overrightarrow{\alpha}(s)$ in space E_1^3, the following Frenet formulas are given

$$\begin{bmatrix}\overrightarrow{T}'\\ \overrightarrow{N}'\\ \overrightarrow{B}'\end{bmatrix}=\begin{bmatrix}0 & k_1 & 0\\ k_1 & 0 & k_2\\ 0 & -k_2 & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{T}\\ \overrightarrow{N}\\ \overrightarrow{B}\end{bmatrix}$$

where k_1 and k_2 are the curvature and torsion of a timelike curve $\overrightarrow{\alpha}$, respectively [6].

3 Differential equations characterizing timelike curves of constant breadth in E_1^3

In this section, we established differential equations characterizing the timelike curves of constant breadth. The base of our study is based on the following consepts for space curves of constant breadth, which are presented by Ö. Köse [14, 15] and M. Sezer [10, 11].

Let (C) be a unit speed timelike curve of the class C^3 having parallel tangents t and t* in opposite directions at the opposite points α and α^* of the curve. If the chord joining the opposite points of (C) is a double-normal, then (C) has constant breadth, and conversely, if (C) is a timelike curve of constant breadth, then every normal of (C) is a double-normal. A simple closed timelike curve (C) of constant breadth having parallel tangents in opposite directions at opposite points may be represented by the equation

$$\overrightarrow{\alpha}^*(s)=\overrightarrow{\alpha}(s)+m_1(s)\overrightarrow{T}(s)+m_2(s)\overrightarrow{N}(s)+m_3(s)\overrightarrow{B}(s) \tag{1}$$

where α and α^* are opposite points, and $\overrightarrow{\mathrm{T}},\ \overrightarrow{\mathrm{N}},\ \overrightarrow{\mathrm{B}}$ denote the unite tangent, principal normal, binormal at a generic point α, respectively. Here s denotes the arc length of (C) and $m_i(s),\ 1\le i\le 3$ are the differentiable functions of s. Differentiating this equation with respect to s and using the Frenet formulas of timelike curve we obtain

$$\frac{d\overrightarrow{\alpha}^*}{ds}=\overrightarrow{T}^*\frac{ds^*}{ds}=\left(1+\frac{dm_1}{ds}+m_2k_1\right)\overrightarrow{T}+\left(m_1k_1+\frac{dm_2}{ds}-m_3k_2\right)\overrightarrow{N}+(m_2k_2+\frac{dm_3}{ds})\overrightarrow{B}$$

Since $\overrightarrow{T} = -\overrightarrow{T}^*$ at corresponding points of (C) we have

$$\begin{aligned} 1+\frac{dm_1}{ds}+m_2k_1 &= -\frac{ds^*}{ds} \\ m_1k_1+\frac{dm_2}{ds}-m_3k_2 &= 0 \\ m_2k_2+\frac{dm_3}{ds} &= 0 \end{aligned}$$

By considering the curvature of the timelike curve defined by

$$\lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} = \frac{d\varphi}{ds} = k_1(s)$$

where $\Delta\varphi$ is the angle of contengency. Here φ denotes the angle between tangent of the curve (C) at the point $\alpha(s)$ and a given fixed direction. Also it is clear that

$$\varphi(s) = \sum_0^s k_1(s)ds$$

The distance d between the opposite points $\alpha^*(s)$ and $\alpha(s)$ of the curve is the breadth of the curve and is constant, that is

$$d^2 = d^2 = \alpha^* - \alpha^2 = -m_1^2 + m_2^2 + m_3^2 = const.$$

On the other hand, the coefficients m_1, m_2 and m_3 may be obtained by the system

$$\begin{aligned} m_1{}' &= -m_2 - f(\varphi) \\ m_2{}' &= \rho k_2 m_3 - m_1 \\ m_3{}' &= -\rho k_2 m_2 \end{aligned} \tag{2}$$

which is the system describing the timelike curves of constant breadth. $f(\varphi) = \rho + \rho^*$ and,

$$\rho = \frac{1}{k_1} \text{ and } \rho^* = \frac{1}{k_2^*}$$

denote the radii of curvatures $\alpha(s)$ and $\alpha^*(s)$, respectively. Here (‘) denotes the differentiation with respect to φ . Also, the vector $\overrightarrow{d}$ is the double normal of the curve (C) of constant breadth. First, it is clear that

$$m_2 = -m_1' - f(\varphi) \tag{3}$$

On the other hand, by using the second equation of the system (2) we obtain the following differential equation:

$$m_3 = \frac{1}{\rho k_2} m_2' + \frac{1}{\rho k_2} m_1 \tag{4}$$

By using the derivative of the equation (3), we obtain the following differential equation:

$$m_3 = -\frac{1}{\rho k_2} m_1'' + \frac{1}{\rho k_2} m_1 + \frac{1}{\rho k_2} f'(\varphi) \tag{5}$$

Also, it is clear that in the third equation of the system (2)

$$m_2 = -\frac{1}{\rho k_2} {m_3}' \tag{6}$$

Here, by using the equality of the equations (3) and (6) following equation is obtained

$$m_1' - \frac{1}{\rho k_2} m_3' + f(\varphi) = 0 \tag{7}$$

Finally, by using derivative of the equation (5), when F is as follows

$$F = -(\rho k_2) f'' + (\rho k_2)' f' - (\rho k_2)^3 f$$

we obtain the third order, linear, differential equation with variable coefficients as follows

$$(\rho k_2) m_1''' - (\rho k_2)' m_1'' + (\rho k_2)\left((\rho k_2)^2 - 1\right) m_1' + (\rho k_2)' m_1 = F \tag{8}$$

As a result, it is clearly seen that the system (2) characterizing the timelike curves of constant breadth can be reduced to the linear differential equation (8). Furthermore, we can write this equation in the general form

$$\sum_{k=0}^{m} Q_k(s) y^{(k)}(s) = F, \quad m = 2,\ 3, \ldots \tag{9}$$

where $Q_k(s)$ are continuous functions of the expression (ρk_2).

4 Bernstein series method

In this section we will explain the Bernstein series solution method for the solution of the differential equations defined as follows.

$$\sum_{k=0}^{m} P_k(s) y^{(k)}(s) = g(s), \quad 0 \le s \le R. \tag{10}$$

Let f be a solution of Eq. (10). We wish to approximate f by

$$p_n(s) = \sum_{k=0}^{n} a_k B_{k,n}(s), \quad n \ge 1 \tag{11}$$

such that $p_n(s)$ satisfies Eq. (10) on the nodes $0 < s_i < s_{i+1} < \cdots < s_{i+d} < R$. Putting $p_n(s)$ into Eq. (10), we get the system of linear equations depending on $a_0,\ a_1,\ \ldots,\ a_n$.

Assume Eq.(10) has a solution, f. Let us consider the Eq. (10) and find the matrix forms of each term in the equation. First we can convert the Bernstein series solution $y = p_n(s)$ defined by (11) and its derivatives $y^{(k)}(s)$ to matrix forms

$$y(s) = B_n(s)A \ \ and \ \ y^{(k)}(s) = B_n^k(s)A \tag{12}$$

where

$$B_n(s) = \left[B_{0,n}(s)\ B_{1,n}(s)\ \ldots\ B_{n,n}(s) \right], \quad A = \left[a_0\ a_1\ \ldots\ a_n \right]^T$$

On the other hand, it can be written $[B_n(s)]^T$ as $[B_n(s)]^T = D(S(s))^T$ or

$$B_n(s) = S(s)D^T \tag{13}$$

where

$$D = \begin{bmatrix} d_{00} & d_{01} & \cdots & d_{0n} \\ d_{10} & d_{11} & \cdots & d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n0} & d_{n1} & \cdots & d_{nn} \end{bmatrix}$$

and

$$d_{ij} = \begin{cases} \frac{(-1)^{j-i}}{R^j}\binom{n}{1}\binom{n-i}{j-i}, & i \le j \\ 0, & i > j. \end{cases}$$

It is clearly seen that that the relation between the matrix $S(s)$ and its derivative $S^{(1)}(s)$ is

$$S^{(1)}(s) = S(s)B$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & n-2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & n-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

To obtain the matrix $S^{(k)}(s)$ in terms of the matrix $S(s)$, we can use the following procedure:

$$\begin{aligned} S^{(2)}(s) &= S^{(1)}(s)B \quad = S(s)B^2 \\ &\vdots \\ S^{(k)}(s) &= S^{(k-1)}(s)B = \cdots = S(s)B^k \end{aligned} \tag{14}$$

Consequently, by substituting the matrix forms (13) and (14) into (12), we have the matrix relation.

$$y^{(k)}(s) = S(s)B^kD^TA. \tag{15}$$

Substituting the matrix relation (15) into (10) and then simplifying, we obtain the matrix equation

$$\sum_{k=0}^{m} P_k(s)S(s)B^kD^TA = g(s). \tag{16}$$

By using the nodes $\{s_i; i = 0,1,\ldots,n; 0 < s_0 < s_1 < \cdots < s_n < R\}$ in (16) we get the system of matrix equations

$$\sum_{k=0}^{m} P_k(s_j)S(s_j)B^kD^TA = g(s_j), \quad j = 0,1,\ldots,n$$

or briefly the fundamental matrix equation

$$\sum_{k=0}^{m} P_k S B^k D^T A = G \tag{17}$$

where

$$P_i = \begin{bmatrix} P_i(s_0) & 0 & 0 & \cdots & 0 & 0 \\ 0 & P_i(s_1) & 0 & \cdots & 0 & 0 \\ 0 & 0 & P_i(s_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & P_i(s_{n-1}) & 0 \\ 0 & 0 & 0 & \cdots & 0 & P_i(s_n) \end{bmatrix}, \quad G = \begin{bmatrix} g(s_0) \\ g(s_1) \\ \vdots \\ g(s_n) \end{bmatrix}, \quad S = \begin{bmatrix} S(s_0) \\ S(s_1) \\ \vdots \\ S(s_n) \end{bmatrix} = \begin{bmatrix} 1 & s_0 & s_0^2 & \cdots & s_0^n \\ 1 & s_1 & s_1^2 & \cdots & s_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & s_n^2 & \cdots & s_n^n \end{bmatrix}$$

Hence, the fundamental matrix Eq. (17) corresponding to (11) can be written in the form

$$WA = G \;\; or \;\; [W;G] = A, \;\; W = [W_{kh}], \;\; k,h = 0,1,\ldots,n \tag{18}$$

where

$$W = \sum_{k=0}^{m} P_k S B^k D^T$$

Here (18) corresponds to a system of $(n+1)$ linear algebraic equations with unknown coefficients $a_0,\ a_1,\ldots,a_n$. Now let us obtain the matrix equation of the conditions by means of the relation (15), as follows

$$S(0)B^k D^T A = [\alpha_k] \qquad k = 0,1,\ldots,m-1$$

or

$$S(0)B^k D^T A = [\beta_k] \qquad k = 0,1,\ldots,\frac{m}{2}-1$$

$$S(R)B^k D^T A = [\gamma_k] \qquad k = 0,1,\ldots,\frac{m}{2}-1$$

On the other hand, the matrix forms for the conditions can be written as

$$U_k A = [\alpha_k] \;\; or \;\; [U_k\,;\alpha_k\,], \;\; k = 0,1,\ldots,m-1 \tag{19}$$

or

$$\begin{aligned} U_k A = [\beta_k] \;\; or \;\; [U_k\,;\beta_k\,], \;\; k = 0,1,\ldots,\frac{m}{2}-1 \\ V_k A = [\gamma_k] \;\; or \;\; [V_k\,;\gamma_k\,], \;\; k = 0,1,\ldots,\frac{m}{2}-1 \end{aligned} \tag{20}$$

where

$$U_k = S(0)B^k D^T = \begin{bmatrix} u_{k0} \; u_{k1} \; \cdots \; u_{kn} \end{bmatrix}$$

$$V_k = S(R)B^k D^T = \begin{bmatrix} u_{k0} \; u_{k1} \; \cdots \; u_{kn} \end{bmatrix}$$

Replacing the row matrices 19 or 20 by any m rows of the matrix 18, we get the augmented matrix $[\tilde{W};\tilde{G}]$ as

$$[\tilde{W};\tilde{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0n} & ; & g(s_0) \\ w_{10} & w_{11} & \cdots & w_{1n} & ; & g(s_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{(n-m)0} & w_{(n-m)1} & \cdots & w_{(n-m)n} & ; & g(s_{n-m}) \\ u_{00} & u_{01} & \cdots & u_{0n} & ; & \alpha_0 \\ u_{10} & u_{11} & \cdots & u_{1n} & ; & \alpha_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & \cdots & u_{(m-1)n} & ; & \alpha_{m-1} \end{bmatrix}$$

or

$$[\tilde{W};\tilde{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0n} & ; & g(s_0) \\ w_{10} & w_{11} & \cdots & w_{1n} & ; & g(s_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{(n-m)0} & w_{(n-m)1} & \cdots & w_{(n-m)n} & ; & g(s_{n-m}) \\ u_{00} & u_{01} & \cdots & u_{0n} & ; & \beta_0 \\ u_{10} & u_{11} & \vdots & u_{1n} & ; & \beta_1 \\ \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ u_{(\frac{m}{2}-1)0} & u_{(\frac{m}{2}-1)1} & \cdots & u_{(\frac{m}{2}-1)n} & ; & \beta_{\frac{m}{2}-1} \\ v_{00} & v_{01} & \cdots & v_{0n} & ; & \gamma_0 \\ v_{10} & v_{11} & \cdots & v_{1n} & ; & \gamma_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{(\frac{m}{2}-1)0} & v_{(\frac{m}{2}-1)1} & \vdots & v_{(\frac{m}{2}-1)n} & ; & \gamma_{\frac{m}{2}-1} \end{bmatrix}$$

Note that $rank\tilde{W} = rank\,[\tilde{W};\tilde{G}] = n+1$ in the case of the exact solution $f \in C^{n+1}(0,R)$. As a result we can write

$$A = (\tilde{W})^{-1}\tilde{G}$$

and hence the elements $a_0,\ a_1,\ldots,a_n$ of A are uniquely determined.

5 The solution of differential equations characterizing the timelike curves of constant breadth in E_1^3

$$\begin{aligned} &(\rho k_2) = t \\ &P_0 = t^{'}, P_1 = -t+t^3, P_2 = -t^{'}, P_3 = t \;\; and \;\; y = m_1. \end{aligned}$$

Using the above equations we can rewrite the differential equation (8) characterizing the timelike curves of constant breadth as fallows;

$$\sum_{k=0}^{m} P_k(s)y^{(k)}(s) = F(s),\ m=3,\ 0 \le s \le 2\pi \tag{21}$$

Let f be a solution of Eq. (21) . We wish to approximate f by

$$p_n(s) = \sum_{k=0}^{n} a_k B_{k,n}(s), \; n = 4 \tag{22}$$

such that $p_n(s)$ satisfies Eq. (21) on the nodes $0 \leq s_0 < s_1 < \cdots < s_4 \leq 2\pi$. Here we take n = 4 for simplicity. Putting $p_n(s)$ into Eq. (21), we get the system of linear equations depending on $a_0, \; a_1, \; \ldots, \; a_4$. Let us consider the Eq. (21) and find the matrix forms of each term in the equation. First we can convert the Bernstein series solution $y = p_n(s)$ defined by (22) and its derivatives $y^{(k)}(s)$ to matrix forms, for $n = 4$ and $k = 0,1,2,3$.

$$y(s) = B_4(s)A \;\; and \;\; y^{(k)}(s) = B_4^k(s)A \tag{23}$$

where

$$B_4(s) = \begin{bmatrix} B_{0,0}(s) & B_{1,0}(s) & \cdots & B_{4,0}(s) \\ B_{0,1}(s) & B_{1,1}(s) & \cdots & B_{4,1}(s) \\ \vdots & \vdots & \ddots & \vdots \\ B_{0,4}(s) & B_{1,4}(s) & \cdots & B_{4,4}(s) \end{bmatrix}, \; A = \begin{bmatrix} a_0 & a_1 & \ldots & a_4 \end{bmatrix}^T$$

On the other hand, it can be written $[B_4(s)]^T$ as $[B_4(s)]^T = D(S(s))^T$ or

$$B_4(s) = S(s)D^T. \tag{24}$$

Where for

$$d_{ij} = \begin{cases} \frac{(-1)^{j-i}}{R^j}\binom{n}{1}\binom{n-i}{j-i}, & i \leq j \\ 0, & i > j \end{cases}$$

the matrix D is calculated as follows

$$D = \begin{bmatrix} 1 & -2/\pi & 3/2\pi^2 & -1/2\pi^3 & 1/16\pi^4 \\ 0 & 2/\pi & -3/\pi^2 & 3/2\pi^3 & -1/4\pi^4 \\ 0 & 0 & 3/2\pi^2 & -3/2\pi^3 & 3/8\pi^4 \\ 0 & 0 & 0 & 1/2\pi^3 & -1/4\pi^4 \\ 0 & 0 & 0 & 0 & 1/16\pi^4 \end{bmatrix}$$

It is clearly seen that that the relation between the matrix $S(s)$ and its derivative $S^{(1)}(s)$ is

$$S^{(1)}(s) = S(s)B$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S(s) = \begin{bmatrix} 1 & s & s^{s2} & s^3 & s^4 \end{bmatrix}.$$

To obtain the matrix $S^{(k)}(s)$ in terms of the matrix $S(s)$, we can use the following procedure:

$$\begin{aligned} S^{(2)}(s) &= S^{(1)}(s)B = S(s)B^2 \\ &\vdots \\ S^{(3)}(s) &= S^{(2)}(s)B = \cdots = S(s)B^3 \end{aligned} \tag{25}$$

where

$$B^2 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B^3 = \begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consequently, by substituting the matrix forms (24) and (25) into (23), we have the matrix relation.

$$\begin{aligned} y(s) &= S(s)D^TA \\ y^{(1)}(s) &= S(s)BD^TA \\ y^{(2)}(s) &= S(s)B^2D^TA \\ y^{(3)}(s) &= S(s)B^3D^TA \end{aligned} \tag{26}$$

Substituting the matrix relation 26 into 21 and then simplifying, we obtain the matrix equation

$$\sum_{k=0}^{m} P_k(s)S(s)B^kD^TA = g(s) \tag{27}$$

By using the nodes $\{s_i; i = 0,1,\ldots,4; 0 \le s_0 < s_1 < \cdots < s_4 \le 2\pi\}$ in (27) we get the system of matrix equations

$$\sum_{k=0}^{m=3} P_k(s_i)S(s_i)B^kD^TA = g(s_i), \quad i = 0,1,\ldots,4$$

where $s_0 = 0, s_1 = \frac{\pi}{2}, s_2 = \pi, s_3 = \frac{3\pi}{2}, s_4 = 2\pi$ and

$$P_0(s_i) = \begin{bmatrix} P_0(0) & 0 & 0 & 0 & 0 \\ 0 & p_0(\pi/2) & 0 & 0 & 0 \\ 0 & 0 & P_0(\pi) & 0 & 0 \\ 0 & 0 & 0 & P_0(3\pi/2) & 0 \\ 0 & 0 & 0 & 0 & P_0(2\pi) \end{bmatrix}$$

$$P_1(s_i) = \begin{bmatrix} P_1(0) & 0 & 0 & 0 & 0 \\ 0 & P_1(\pi/2) & 0 & 0 & 0 \\ 0 & 0 & P_1(\pi) & 0 & 0 \\ 0 & 0 & 0 & P_1(3\pi/2) & 0 \\ 0 & 0 & 0 & 0 & P_1(2\pi) \end{bmatrix}$$

$$P_2(s_i) = \begin{bmatrix} P_2(0) & 0 & 0 & 0 & 0 \\ 0 & P_2(\pi/2) & 0 & 0 & 0 \\ 0 & 0 & P_2(\pi) & 0 & 0 \\ 0 & 0 & 0 & P_2(3\pi/2) & 0 \\ 0 & 0 & 0 & 0 & P_2(2\pi) \end{bmatrix}$$

$$P_3(s_i)=\begin{bmatrix} P_3(0) & 0 & 0 & 0 & 0 \\ 0 & P_3(\pi/2) & 0 & 0 & 0 \\ 0 & 0 & P_3(\pi) & 0 & 0 \\ 0 & 0 & 0 & P_3(3\pi/2) & 0 \\ 0 & 0 & 0 & 0 & P_3(2\pi) \end{bmatrix}$$

$$S(s_i)=\begin{bmatrix} S(s_0) \\ S(s_1) \\ S(s_2) \\ S(s_3) \\ S(s_4) \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & (\pi/2) & (\pi/2)^2 & (\pi/2)^3 & (\pi/2)^4 \\ 1 & (\pi) & (\pi)^2 & (\pi)^3 & (\pi)^4 \\ 1 & (3\pi/2) & (3\pi/2)^2 & (3\pi/2)^3 & (3\pi/2)^4 \\ 1 & (2\pi) & (2\pi)^2 & (2\pi)^3 & (2\pi)^4 \end{bmatrix}, F(s_i)=\begin{bmatrix} f(0) \\ f(\pi/2) \\ f(\pi) \\ f(\pi/2) \\ f(2\pi) \end{bmatrix}$$

The fundamental matrix equation can be written briefly as

$$\sum_{k=0}^{m} P_k S B^k D^T A = G. \tag{28}$$

Hence, the fundamental matrix Eq. (28) corresponding to (22) can be written in the form

$$WA=F \quad \text{or} \quad [W;F]=A,\ W=[W_{kh}],\ k,h=0,1,\ldots,4 \tag{29}$$

where

$$W=\sum_{k=0}^{m} P_k S B^k D^T$$

Here, the Eq. 29 corresponds to a matrix of type (5x5). Now let us obtain the matrix equation of the conditions by means of the relation 26, as follows

$$S(0)B^k D^T A=[\alpha_k],\ k=0,1,2$$

Firstly, the matrix forms for the conditions can be written as

$$U_k A=[\alpha_k] \quad \text{or} \quad [U_k\,;\alpha_k\,],\ k=0,1,2 \tag{30}$$

where for

$$\begin{aligned} U_0=S(0)D^T &= \begin{bmatrix} u_{00}\ u_{01}\ \cdots\ u_{04} \end{bmatrix}=\begin{bmatrix} 1\ 0\ 0\ 0\ 0 \end{bmatrix} \\ U_1=S(0)B^1D^T &= \begin{bmatrix} u_{10}\ u_{11}\ \cdots\ u_{14} \end{bmatrix}=\begin{bmatrix} -2/\pi\ 2/\pi\ 0\ 0\ 0 \end{bmatrix} \\ U_2=S(0)B^2D^T &= \begin{bmatrix} u_{20}\ u_{21}\ \cdots\ u_{24} \end{bmatrix}=\begin{bmatrix} 12/\pi^2\ -6/\pi^2\ 3/\pi^2\ 0\ 0 \end{bmatrix} \end{aligned}$$

Replacing the row matrices 30 by any m rows of the matrix 29, we get the augmented matrix $[\tilde{W};\tilde{G}]$ as

$$[\tilde{W};\tilde{G}]=\begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & w_{04} & ; & f(0) \\ w_{10} & w_{11} & w_{12} & w_{13} & w_{14} & ; & f(\pi/2) \\ u_{00} & u_{01} & u_{02} & u_{03} & u_{04} & ; & \alpha_0 \\ u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & ; & \alpha_1 \\ u_{20} & u_{21} & u_{22} & u_{23} & u_{24} & ; & \alpha_2 \end{bmatrix}$$

where, $w_{ij}\ \ (i=0,1\ \ j=0,1,\ldots,4)$ obtained as follows;

$$
\begin{aligned}
w_{00} &= t^{'}(0)-\frac{2}{\pi}[-t(0)+t^{3}(0)]+\frac{3}{\pi^{2}}[-t^{'}(0)]-\frac{3}{\pi^{3}}t(0)\\
w_{01} &= \frac{2}{\pi}\left[-t(0)+t^{3}(0)\right]-\frac{6}{\pi^{2}}\left[-t^{'}(0)\right]+\frac{9}{\pi^{3}}t(0)\\
w_{02} &= \frac{3}{\pi^{2}}\left[-t^{'}(0)\right]-\frac{9}{\pi^{3}}t(0)\\
w_{03} &= \frac{3}{\pi^{3}}t(0),\quad w_{04}=0,\\
w_{10} &= \frac{81}{256}t^{'}\left(\frac{\pi}{2}\right)-\frac{27}{32\pi}\left[-t\left(\frac{\pi}{2}\right)+t^{3}\left(\frac{\pi}{2}\right)\right]+\frac{27}{16\pi^{2}}[-t^{'}\left(\frac{\pi}{2}\right)]-\frac{9}{4\pi^{3}}t\left(\frac{\pi}{2}\right)\\
w_{11} &= \frac{27}{64}t^{'}\left(\frac{\pi}{2}\right)-\frac{9}{4\pi^{2}}[-t^{'}\left(\frac{\pi}{2}\right)]+\frac{6}{\pi^{3}}t\left(\frac{\pi}{2}\right)\\
w_{12} &= \frac{27}{128}t^{'}\left(\frac{\pi}{2}\right)+\frac{9}{16\pi}\left[-t\left(\frac{\pi}{2}\right)+t^{3}\left(\frac{\pi}{2}\right)\right]-\frac{3}{8\pi^{2}}[-t^{'}\left(\frac{\pi}{2}\right)]-\frac{9}{2\pi^{3}}t\left(\frac{\pi}{2}\right)\\
w_{13} &= \frac{3}{64}t^{'}\left(\frac{\pi}{2}\right)+\frac{1}{4\pi}\left[-t\left(\frac{\pi}{2}\right)+t^{3}\left(\frac{\pi}{2}\right)\right]+\frac{3}{4\pi^{2}}[-t^{'}\left(\frac{\pi}{2}\right)]\\
w_{14} &= \frac{1}{256}t^{'}\left(\frac{\pi}{2}\right)+\frac{1}{32\pi}\left[-t\left(\frac{\pi}{2}\right)+t^{3}\left(\frac{\pi}{2}\right)\right]+\frac{3}{16\pi^{2}}[-t^{'}\left(\frac{\pi}{2}\right)]+\frac{3}{4\pi^{3}}t\left(\frac{\pi}{2}\right).
\end{aligned}
$$

As a result we can write

$$
A=(\tilde{W})^{-1}\tilde{G}=\begin{bmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & \pi/2 & 0\\ 0 & 0 & -2 & \pi & \pi^{2}/3\\ R & 0 & T & K & V\\ Y & Z & Q & L & C \end{bmatrix}\begin{bmatrix} f(0)\\ f(\pi/2)\\ \alpha_0\\ \alpha_1\\ \alpha_2 \end{bmatrix}
$$

where

$$
\begin{aligned}
&R=1/w_{03},T=2w_{02}-w_{00}-w_{01}/w_{03},\ \ K=-\pi(w_{01}+2w_{02})/2w_{03}\\
&V=-\pi^{2}w_{02}/3w_{03},Y=-w_{13}/w_{03}w_{14},\ \ Z=1/w_{14}\\
&Q=-w_{13}(w_{10}+w_{11}-2w_{12})+w_{13}(w_{00}+w_{01}-2w_{02})/w_{03}w_{14}\\
&L=-\pi w_{03}(w_{11}+w_{12})+\pi w_{13}(w_{01}+2w_{02})/2w_{03}w_{14}\\
&C=\pi^{2}(w_{13}w_{02}-w_{03}w_{12})/w_{03}w_{14}
\end{aligned}
$$

and hence the elements $a_0,\ a_1,\ldots,a_4$ of A are uniquely determined as follow

$$
\begin{aligned}
a_0 &= \alpha_0\\
a_1 &= \alpha_0+\frac{\pi}{2}\alpha_1\\
a_2 &= -2\alpha_0+\pi\alpha_1+\frac{\pi^{2}}{3}\alpha_2\\
a_3 &= Rf(0)+T\alpha_0+K\alpha_1+V\alpha_2\\
a_4 &= Yf(0)+Zf\left(\frac{\pi}{2}\right)+Q\alpha_0+L\alpha_1+C\alpha_2.
\end{aligned}
$$

If we put this a_4 unknowns in equation (22), we obtain the Bernstein series solution $y=p_n(s)=m_1$ of the Eq. (21).

6 The probed of differential equations characterizing timelike curves of constant breadth in E_1^3

We found that the expression is $y = m_1$ coefficient which is determined the timelike curves of constant breadth in E_1^3. Also m_2 coefficient is finded with method similar under the same initial conditions. For this first, it is clear that in the second equation of the system (2)

$$m_1 = \rho k_2 m_3 - {m_2}'. \tag{31}$$

We used where the first equation of the system (2), the derivative of the equation (31)

$$-{m_2}'' + (\rho k_2)' m_3 + (\rho k_2) m_3' = -m_2 - f \tag{32}$$

Also, it is clear that in the second equation of the system (2)

$$m_3 = \frac{{m_2}' + m_1}{\rho k_2} \tag{33}$$

By using the third equation of the system (2) and the equation (33) in the equation (32), we obtain the following differential equation:

$$\rho k_2 {m_2}'' - (\rho k_2)' {m_2}' + [(\rho k_2)^3 - \rho k_2] m_2 - (\rho k_2)' m_1 - \rho k_2 f = 0$$

Here, m_1 is conjugated and then by using derivative of the expression obtained, we obtain the following differential equation;

$$\begin{aligned} {m_1}' = &\frac{\rho k_2}{(\rho k_2)'} {m_2}''' + \left[\left(\frac{\rho k_2}{(\rho k_2)'} \right)' - 1 \right] {m_2}'' + \left[\frac{(\rho k_2)^3 - \rho k_2}{(\rho k_2)'} \right] {m_2}' + \left[\frac{(\rho k_2)^3 - \rho k_2}{(\rho k_2)'} \right]' m_2 \\ &- \left(\frac{\rho k_2}{(\rho k_2)'} \right)' f - \frac{\rho k_2}{(\rho k_2)'} f' \end{aligned} \tag{34}$$

By using the equation (34) and the first equation of the system (2) following equation is obtained

$$\begin{aligned} &\frac{\rho k_2}{(\rho k_2)'} {m_2}''' + \left[\left(\frac{\rho k_2}{(\rho k_2)'} \right)' - 1 \right] {m_2}'' + \left[\frac{(\rho k_2)^3 - \rho k_2}{(\rho k_2)'} \right] {m_2}' + [(\frac{(\rho k_2)^3 - \rho k_2}{(\rho k_2)'})' + 1] m_2 - [\left(\frac{\rho k_2}{(\rho k_2)'} \right)' - 1] f \\ &- \frac{\rho k_2}{(\rho k_2)'} f' \end{aligned} \tag{35}$$

Finally, while $(\rho k_2) = \mathrm{t}$ and F as follows:

$$F = \left[\left(\frac{\rho k_2}{(\rho k_2)'} \right)' - 1 \right] f + \frac{\rho k_2}{(\rho k_2)'} f'$$

we obtain the third order, linear, differential equation with variable coefficients as follows

$$\frac{\mathrm{t}}{\mathrm{t}'} {m_2}''' + \left[\left(\frac{\mathrm{t}}{\mathrm{t}'} \right)' - 1 \right] {m_2}'' + \left[\frac{t^3 - \mathrm{t}}{\mathrm{t}'} \right] {m_2}' + \left[\left(\frac{t^3 - \mathrm{t}}{\mathrm{t}'} \right)' + 1 \right] m_2 = F \tag{36}$$

This equation is differential equation with unknown m_2 characterizing timelike curves of constant breadth in E_1^3.

Also, m_3 coefficient is finded with method similar under the same initial conditions. First, it is clear that in the third equation of the system (2)

$$m_2 = -\frac{1}{\rho k_2} m_3{}' \tag{37}$$

We used where the second equation of the system (2), the derivative of the equation (37)

$$m_1 = -\frac{1}{\rho k_2} m_3{}'' + \left(-\frac{1}{\rho k_2}\right)' m_3{}' - (\rho k_2)\, m_3 \tag{38}$$

By using the derivative of the equation (38) in the first equation of the system (2), we obtain the following differential equation:

$$\frac{1}{\rho k_2} m_3{}''' + 2\left(\frac{1}{\rho k_2}\right)' m_3{}'' + \left[\left(-\frac{(\rho k_2)'}{(\rho k_2)^2}\right)' + \frac{1}{\rho k_2} + \rho k_2\right] m_3{}' + (\rho k_2)' m_3 = f \tag{39}$$

Finally, while $(\rho k_2) = \mathrm{t}$ we obtain the third order, linear, differential equation with variable coefficients as follows:

$$\frac{1}{\mathrm{t}} m_3{}''' + 2\left(\frac{1}{\mathrm{t}}\right)' m_3{}'' + \left[\left(-\frac{\mathrm{t}'}{\mathrm{t}^2}\right)' + \frac{1}{\mathrm{t}} + \mathrm{t}\right] m_3{}' + \mathrm{t}' m_3 = f \tag{40}$$

This equation is differential equation with unknown m_3 characterizing the timelike curves of constant breadth in E_1^3.

7 Corollary

By using Bernstein series solution method, the solutions of these equations (36) and (40) are approximately obtained. If we use these m_i, $(i = 1,2,3)$ coefficients, which we have calculated, in equation $d^2 = -m_1^2 + m_2^2 + m_3^2$, we get the constant value of the breadth of the curve in E_1^3.

Thus, we obtain general expression connected with torsion and curvature of a timelike curve of constant breadth in E_1^3.

Also, in this work, the motion point is a system of differential equations like Frenet. With the similar idea, the equation characterizing timelike curves of constant breadth in 4-dimensional Minkowski space can be obtained. And the equation obtained by the same solution method can be examined.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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