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The excision theorem for digital Khalimsky spaces

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Abstract: Unlike the digital images, Khalimsky digital spaces have topological structures in addition to their adjacency relations. The construction of their digital singular homology groups is given in [23]. The functorial properties of digital singular homology theory allow us to characterize and classify the digital spaces. In algebraic topology, additivity and excision axioms are one of Eilenberg-Steenrod axioms in homology theory and in this paper, we check their validity for digital singular homology theory.

Keywords: Khalimsky topology, digital topology, singular homology, excision.

1 Introduction

The main purpose of the digital topology is to study and characterize the properties of digital images by using the concept of topological properties. The tools for this purpose are the digital algebraic notions derived from the notions in algebraic topology. The first attempt was from Rosenfeld [20]. He introduced the notion of continuity for functions between digital images. Later, Boxer [7] extends the results of Rosenfeld's by introducing a digital fundamental group based on digitally continuous functions and the digital homotopy and examines the digital fundamental group of unbounded digital images [6].

Another tool to classify the digital images is to compute their simplicial homology groups. Arslan et al.[3] introduce the digital simplicial homology groups of n-dimensional digital images and it is developed by Boxer, Karaca and Oztel [9]. The validity of Eilenberg-Steenrod axioms for the digital simplicial homology groups are investigated by Karaca and Ege [11]. The last tool is the digital singular homology theory based on a continuous functions from standard n-simplex to the digital Khalimsky space [23]. By Khalimsky topology is on $\mathbb{Z}^n$, the digital images are represented by locally finite T_0-topological spaces and it has been studied by many authors. Erik Melin [18] has given the classification for the subset of Khalimsky n-space $A \subset \mathbb{Z}^n$ where the continuous functions from A to the Khalimsky line $\mathbb{Z}$ can be extended to $\mathbb{Z}^n$. He has also studied digital manifolds which are locally homeomorphic to Khalimsky manifolds [19]. To consider the topological spaces of Khalimsky spaces by their corresponding algebraical structures, Karaca and Vergili [23] introduce the notions of the standard digital n-simplexes, digital singular chain complex and define the digital singular homology groups of Khalimsky spaces. They state that the dimension axiom holds and the digital singular homology theory is a functor from the category KDTC of KD-topological spaces to the category Ab of abelian groups.

In this paper, we develop the digital singular homology theory for Khalimsky spaces. We show that in each dimension, the digital singular homology group of a Khalimsky space is a direct sum of the digital singular homology groups of its path components. This property is known as *additivity*. Further, to check the validity of excision axiom for digital Khalimsky spaces, we construct the digital relative homology groups for the pair (X,A) where X is a digital Khalimsky space and A is a subset of X. The excision axiom allow us to compute the digital homology groups of a (X,A) in a less effort.

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2 Preliminaries

Let $\mathbb{Z}$ denote the set of integers and $X \subset \mathbb{Z}^m$ for some positive integer m. The adjacency relation for the subset X is as follows [6]: Let l, m be positive integers, $1 \leq l \leq m$ and two distinct points $p = (p_1, p_2, \ldots, p_m)$, $q = (q_1, q_2, \ldots, q_m)$ in X, p and q are *k_l-adjacent* if there are at most l distinct coordinates j for which $|p_j - q_j| = 1$, and for all other coordinates j, $p_j = q_j$. We call the pair (X, κ_ℓ), (or (X, κ) for short), as a digital image.

Let κ be an adjacency relation defined on $X \subset \mathbb{Z}^m$ and $p \in X$. Then the point in X which is κ-adjacent to p is called a *κ-neighbor* of p [14]. Let q be another point in X. Then the sequence $\{p_0, p_1, \ldots, p_s\}$ of points in X is called the κ-path between the points p and q if $p = p_0$, $q = p_s$ and p_i and p_{i+1} are κ-adjacent [14] in X.

Denote $\ell_\kappa(p,q)$ as the length of a shortest κ-path between p and q. (Take $\ell_\kappa(p,q) = \infty$ if no κ-path between the points p and q exists.)

Let

$$N_\kappa(p, \varepsilon) := \{q \in X \ : \ \ell_\kappa(p,q) \leq \varepsilon\} \cup \{p\}$$

where $\varepsilon \in \mathbb{N}$ [12].

We define a topology on $\mathbb{Z}$. For an integer n, let

$$B(n) = \begin{cases} \{n\}, & \text{if n is odd} \\ \{n-1, n, n+1\}, & \text{if n is even.} \end{cases}$$

Then the topology on $\mathbb{Z}$ generated by the basis

$$\mathscr{B} = \{B(n) \ : \ n \in \mathbb{Z}\}$$

is called *Khalimsky digital line topology* [15].

The topology on $\mathbb{Z}^m$ for $m > 1$ is considered as a product topology where the base system is

$$\mathscr{B} = \{\prod_{i=1}^{m} B(n_i) \ : \text{each } B(n_i) \text{ is a basis in } \mathbb{Z}\}$$

and for the subset (X, κ) of $\mathbb{Z}^m$, the topology of X inherited from $\mathbb{Z}^m$ has the basis

$$\mathscr{B} = \{X \cap \prod_{i=1}^{m} B(n_i) \ : \ \text{each } B(n_i) \text{ is a basis in } \mathbb{Z}\}.$$

We will denote such spaces by $(X_{m,\kappa}, \tau_X)$.

If $(X_{m_1,\kappa_1}, \tau_X)$ is a Khalimsky space and x is a point in $(X_{m,\kappa}, \tau_X)$, then a neighborhood of x is a subset U_x of X that includes an open set G containing x. Let $(X_{m_1,\kappa_1}, \tau_X)$ and $(Y_{m_2,\kappa_2}, \tau_Y)$ be Khalimsky spaces. The map $f : X \to Y$ is *continuous at x* [13] if for all open subsets $U_{f(x)}$ of Y, $f^{-1}(U_{f(x)})$ is an open subset of X containing x.

Definition 1.*[13] Let $(X_{m_1,\kappa_1}, \tau_X)$ and $(Y_{m_2,\kappa_2}, \tau_Y)$ be Khalimsky spaces and $f : X \to Y$ be a map. Then f is KD-(κ_1, κ_2) continuous at x, if*

(i) *f is continuous at x; and*

(ii) *for any* $N_{\kappa_2}(f(x),\varepsilon) \subset Y$, *there is* $N_{\kappa_1}(x,\delta) \subset X$ *such that* $f(N_{\kappa_1}(x,\delta)) \subset N_{\kappa_2}(f(x),\varepsilon)$, *where* $\varepsilon,\delta \in \mathbb{N}$,

A KD-(κ_1,κ_2) *continuous map* f *is KD-*(κ_1,κ_2) *continuous at any point* x *in* X. *Further a KD-*(κ_1,κ_2)*-continuous bijective map with a KD-*(κ_2,κ_1)*-continuous inverse is called a* KD-(κ_1,κ_2)-isomorphism *[13].*

Let $(X_{m_1,\kappa_1},\tau_X)$ be a Khalimsky space and $x_1,x_2 \in X$. If there exist an integer m and a KD-$(2,\kappa_1)$ continuous map $\gamma : [0,m]_{\mathbb{Z}} \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$, then γ is called the path between x_1 and x_2. If there exists a path function between any two points in X, then we say that X is *path connected*.

For the digital singular homology theory, we need the notion of n-simplex. Let (X,κ) be a digital image. For the nonempty subset S of X, we call the members s of S as the *simplices* of (X,κ) [22] if the following two statements hold

(i) if p and q are two distinct points of S, then they are κ-adjacent,
(ii) if $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

If the number of elements of S is $n+1$, then S is called an *n-simplex*.

We consider some special points in $\mathbb{Z}^n$ [23]. For $n \geq 0$ let $e_0 = (0,...,0)$ and for $1 < i \leq n$, let $e_i = (i_1,i_2,...,i_n)$ where components of e_i are defined by

$$i_m = \begin{cases} 1, & \text{if } m \leq i \\ 0, & \text{if } m > i. \end{cases}$$

Denote the digital standard n-simplex in $\mathbb{Z}^n$ spanned by the points $e_0,e_1,...,e_n$ as

$$\Delta^n = [e_0,e_1,...,e_n].$$

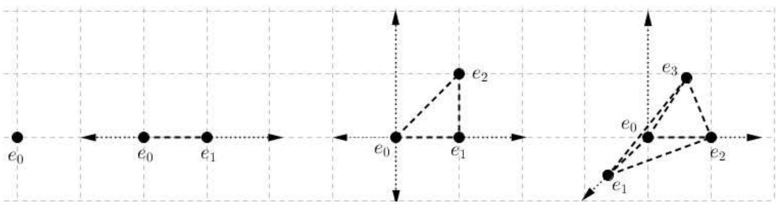

Fig. 1: The illustrations for $\Delta^0,\Delta^1,\Delta^2,\Delta^3$ respectively.

We take the topology on Δ^n as a Khalimsky topology inherited from $\mathbb{Z}^n$.

Definition 2.*[23] The linear ordering* $e_0 < e_1 < ... < e_n$ *on its vertices is called the orientation of* $\Delta^n = [e_0,e_1,...,e_n]$. *Also, the induced orientation of its faces defined by orienting the ith face*

$$(-1)^i[e_0,...,\widehat{e_i},...,e_n]$$

where $\widehat{e_i}$ *is considered as it is deleted and* $-[e_0,...,\widehat{e_i},...,e_n]$ *is the ith face with orientation opposite to the one with the vertices ordered as* $e_0 < e_1 < ... < e_n$.

Definition 3.*[23] Let* $(X_{m,\kappa},\tau_X)$ *be a Khalimsky digital space. The KD-*$(3^n-1,\kappa)$*-continuous map*

$$\sigma^n : \Delta^n \to X$$

is called a digital singular n-simplex in X.

The set of all digital singular n-simplexes of X form a basis for the free abelian group $S_n(X)$ for $n \geq 0$. The elements of $S_n(X)$ are called *the digital singular n-chains*.

Definition 4.[23] *Let $\varepsilon_i := \varepsilon_i^n : \Delta^{n-1} \to \Delta^n$ be the map such that it maps the vertices to vertices and preserve the orderings as:*

$$\varepsilon_0^n : (t_0, t_1, ..., t_{n-1}) \mapsto (0, t_0, ..., t_{n-1})$$

$$\varepsilon_i^n : (t_0, t_1, ..., t_{n-1}) \mapsto (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1}) \qquad i \geq 1.$$

We call ε_i as ith face map.

Let $(X_{m,\kappa}, \tau_X)$ be a Khalimsky space. Then the boundary of $\sigma^n \in S_n(X)$ is

$$\partial_n \sigma^n = \sum_{j=0}^{n} (-1)^j \sigma^n \varepsilon_j^n \in S_{n-1}^{\kappa}(X)$$

and if $n = 0$, define $\partial_0 \sigma^n = 0$. Note that for $n \geq 0$, ∂_n is a linear map and it can be extended to a unique homomorphism called *the boundary operators*

$$\partial_n : S_n(X) \to S_{n-1}(X)$$

with

$$\partial_n \sigma^n = \sum_{j=0}^{n} (-1)^j \sigma^n \varepsilon_j^n$$

for every singular digital n-simplex σ^n in X [23].

Definition 5.[23] *For each digital space $(X_{m,\kappa}, \tau_X)$, a sequence of free abelian groups and homomorphisms*

$$\cdots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$$

called the digital singular complex *of the digital space $(X_{m,\kappa}, \tau_X)$ and it is denoted by $S_*(X)$.*

Theorem 1. *[23] For all $n \geq 0$, we have $\partial_n \partial_{n+1} = 0$.*

For a digital space $(X_{m,\kappa}, \tau_X)$, the group of the digital singular *n-cyles* is

$$Z_n(X) := \text{Ker } \partial_n$$

and the group of the digital singular *n-boundaries* is

$$B_n(X) := \text{Im } \partial_{n+1}.$$

Note that $B_n(X) \subset Z_n(X) \subset S_n(X)$ for each $n \geq 0$ and therefore we can consider the quotients

$$H_n(X) := \frac{Z_n(X)}{B_n(X)}$$

called *the nth digital singular homology group* of a Khalimsky space X (see [23]).

Theorem 2. *[23] Let $X = \{x\}$ be a one point space in $\mathbb{Z}^m$. Then for all $n > 0$, $H_n(X) = 0$.*

Theorem 3. *[23] Let $X = \{a = (0,0), b = (1,0), c = (1,1)\} \subset \mathbb{Z}^2$. Then the digital singular homology groups of X up to the dimension 2 are as follows:*

$$H_0(X) \cong \mathbb{Z}, \quad H_1(X) = 0, \quad H_2(X) = 0.$$

Let $(X_{m_1,\kappa_1}, \tau_X)$ and $(Y_{m_2,\kappa_2}, \tau_Y)$ be two spaces and $f : X \to Y$ be a KD-(κ_1, κ_2)-continuous map. For $n \geq 0$, if $\sigma^n \in S_n(X)$ then $f \circ \sigma^n \in S_n(Y)$. If we extend f by linearity of singular digital n-simplexes in X, we have a homomorphism

$$f_\sharp : S_n(X) \to S_n(Y), \quad f_\sharp(\sum s_{\sigma^n}\sigma^n) = \sum s_{\sigma^n}(f \circ \sigma^n)$$

where $s_{\sigma^n} \in \mathbb{Z}$ (see [23]).

Note that

$$f_\sharp(Z_n(X)) \subset Z_n(Y) \quad \text{and} \quad f_\sharp(B_n(X)) \subset B_n(Y).$$

The KD-topological category KDTC consists of the objects as $(X_{n,\kappa}, \tau_X)$'s and morphisms as KD-(κ_1, κ_2)-continuous functions.

Theorem 4. *[23] For each $n \geq 0$, $H_n : \mathrm{KDTC} \to \mathrm{Ab}$ is a functor.*

Corollary 1. *[23] If $(X_{m_1,\kappa_1}, \tau_X)$ and $(Y_{m_2,\kappa_2}, \tau_Y)$ are KD-(κ_1, κ_2)-isomorphic, then*

$$H_n(X) \cong H_n(Y) \quad \text{for all } n \geq 0.$$

3 Additivity for digital singular homology

Theorem 5. *Let $X = \{a = (-1,0), b = (0,-1), c = (1,0), d = (0,1)\} \subset \mathbb{Z}^2$. Then,*

$$H_n(X) = \begin{cases} \mathbb{Z}^4, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

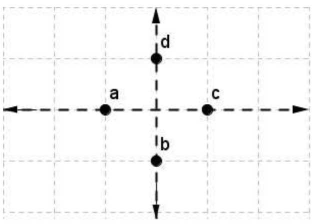

Fig. 2

Proof. The Khalimsky topology on X coincides with the discrete topology. Thus if a singular map

$$\sigma_n : \Delta^n \to X$$

is continuous, then it should be constant. Hence for all n, we get

$$S_n(X) \cong \mathbb{Z}^4$$

$$\text{Ker } \partial_n \cong \begin{cases} \mathbb{Z}^4, & \text{n is odd,} \\ 0, & \text{n is even} \end{cases} \quad \text{and} \quad \text{Im } \partial_n \cong \begin{cases} \mathbb{Z}^4, & \text{n is even,} \\ 0, & \text{n is odd.} \end{cases}$$

Hence we conclude that

$$H_n(X) = \begin{cases} \mathbb{Z}^4, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Theorem 6. *If X is a path connected Khalimsky digital space, then $H_0(X) \cong \mathbb{Z}$.*

Proof. Let X be a path connected space and $|X| = n$. From the end part of the singular complex

$$\ldots \longrightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$$

we see that $Z_0(X) \cong S_0(X) = \mathbb{Z}^n$. The proof will be completed if we show that $B_0(X) = \mathbb{Z}^{n-1}$. We assign a graph for X as follows: Consider the elements of X as a vertex set and the singular 1-simplexes σ^1's as an edge set that connects the vertexes to each other. Let's call this graph as $\mathscr{G}(X)$. Note that each singular 1-simplex $\sigma^n \in S_1(X)$ is a path in X and if X is a path connected space, then the graph of X will be a connected graph, moreover, the spanning tree (for more details, see [1]) of this graph contains $n-1$ edges. Hence $B_0(X) \cong \mathbb{Z}^{n-1}$ and $H_0(X) \cong \mathbb{Z}$.

Theorem 7. *Let X be a digital Khalimsky space and $\{X_\alpha\}$ be the set of the path components of X. Then for every $n \geq 0$,*

$$H_n(X_\alpha) \cong \sum_\alpha H_n(X_\alpha).$$

Proof. Observe that each path component X_α induces a maximal connected subgraph $\mathscr{G}(X_\alpha)$ of the graph $\mathscr{G}(X)$. Then for each $\sigma^n \in S_n(X)$ the Im σ^n is a subset of the vertex set of $\mathscr{G}(X_\alpha)$. Therefore

$$S_n X = \sum_\alpha S_n(X_\alpha).$$

$\partial_n(\sigma^n)$ is also contained in a vertex set of $\mathscr{G}(X_\alpha)$, as Imσ^n is. It follows that the boundary

$$\partial_n : S_n(X) \to S_{n-1}(X)$$

maps $S_n(X_\alpha)$ to $S_{n-1}(X_\alpha)$. Therefore we have the following direct sum decompositions

$$Z_n(X) = \sum_\alpha Z_n(X_\alpha) \quad \text{and} \quad B_n(X) = B_n(X) = \sum_\alpha B_n(X_\alpha)$$

and hence passing to the quotient groups, we get

$$H_n(X) = \sum_\alpha H_n(X_\alpha)$$

for all $n \geq 0$.

4 Digital relative homology groups

Let A be a subspace of the Khalimsky space X and $i : A \hookrightarrow X$ be the inclusion map. Then it can easily be checked that $i_\sharp : S_n(A) \to S_n(X)$ is a monomorphism. Consider the quotient group

$$S_n(X,A) := S_n(X)/S_n(A)$$

called n dimensional chain group of the pair (X,A). Since

$$\partial_n(S_n(A)) \subset S_{n-1}(A)$$

for the boundary operator

$$\partial_n : S_n(X) \to S_{n-1}(X),$$

then there is a homomorphism between the quotient groups

$$\partial_n : S_n(X,A) \to S_{n-1}(X,A)$$

induced by the original. Note that this induced homomorphism is also a boundary operator and hence $\partial_n\partial_{n+1} = 0$. For $n \geq 0$, define

$$Z_n(X,A) := \text{Ker } \partial_n = \{s \in S_n(X,A) \ : \ \partial_n(s) = 0\}$$

called the group of n-dimensional cycles and

$$B_n(X,A) := \text{Im } \partial_{n+1} = \{\partial_{n+1}(S_{n+1}(X,A))\}$$

called the group of n-dimensional boundaries. Then the quotient

$$H_n(X,A) = Z_n(X,A)/B_n(X,A)$$

is called *the n-dimensional relative homology groups* for the pair (X,A).

Let the inclusion maps

$$i : A \hookrightarrow X \quad \text{and} \quad j : X \to (X,A)$$

induce the homomorphisms

$$i_\sharp : S_n(A) \to S_n(X) \quad \text{and} \quad j_\sharp : S_n(X) \to S_n(X,A)$$

between the chain groups and

$$i_* : S_n(A) \to S_n(X) \quad \text{and} \quad j_* : S_n(X) \to S_n(X,A)$$

between the homology groups respectively. Then the homomorphism

$$d_n : H_n(X,A) \to H_{n-1}(A)$$

defined by

$$clsz_n \longmapsto clsi_{n-1}^{-1}\partial_n j_n^{-1} z_n$$

is called *connecting homomorphism* of the pair (X,A). Then there's an exact sequence of groups and homomorphisms for any pair (X,A):

$$\ldots \xrightarrow{j_*} H_{n+1}(X,A) \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} \ldots$$

The sequence is called *the exact homology sequence* of the pair (X,A).

Theorem 8. *Let* $X = \{a = (0,0), b = (2,0), c = (1,1), d = (2,2)\} \subset \mathbb{Z}^2$*. Then the digital homology groups of* X *up to the dimension* 2 *are as follows:*

$$H_0(X) \cong \mathbb{Z}, \quad H_1(X) \cong \mathbb{Z}, \quad H_2(X) = 0\,.$$

Proof. Note that the Khalimsky topology on X is

$$\tau_X = \{\emptyset, X, \{c\}, \{a,c\}, \{c,b\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}.$$

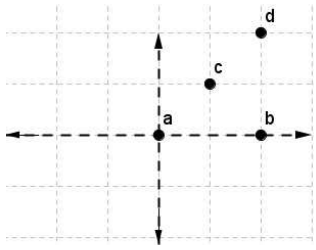

Fig. 3

Let $A = \{b,c\}$ be a subset of X. Then the Khalimsky topology on A induced by X is $\tau_A = \{\emptyset, A, \{c\}\}$.

To compute the homology groups of the pair (X,A) up to the dimension 2, we'll start with computing the homology groups of A up to the dimension 2. $S_0(A)$ has basis

$$\sigma_1^0 : e_0 \mapsto b \qquad \sigma_2^0 : e_0 \mapsto c.$$

$S_1(A)$ has basis

$$\sigma_1^1 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto b \end{array} \qquad \sigma_2^1 : \begin{array}{l} e_0 \mapsto c \\ e_1 \mapsto c \end{array} \qquad \sigma_3^1 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto c. \end{array}$$

$S_2(A)$ has basis

$$\sigma_1^2 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto b \\ e_2 \mapsto b \end{array} \qquad \sigma_2^2 : \begin{array}{l} e_0 \mapsto c \\ e_1 \mapsto c \\ e_2 \mapsto c \end{array} \qquad \sigma_3^2 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto b \\ e_2 \mapsto c \end{array} \qquad \sigma_4^2 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto c \\ e_2 \mapsto c. \end{array}$$

$S_3(A)$ has basis

$$\sigma_1^3 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto b \\ e_2 \mapsto b \\ e_3 \mapsto b \end{array} \qquad \sigma_2^3 : \begin{array}{l} e_0 \mapsto c \\ e_1 \mapsto c \\ e_2 \mapsto c \\ e_3 \mapsto c \end{array} \qquad \sigma_3^3 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto b \\ e_2 \mapsto c \\ e_3 \mapsto c \end{array} \qquad \sigma_4^3 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto c \\ e_2 \mapsto c \\ e_3 \mapsto c \end{array}$$

$$\sigma_5^3 : \begin{array}{l} e_0 \mapsto b \\ e_1 \mapsto b \\ e_2 \mapsto c \\ e_3 \mapsto c. \end{array}$$

Then it's clear that $S_0(A) \cong \mathbb{Z}^2, \quad S_1(A) \cong \mathbb{Z}^3, \quad S_2(A) \cong \mathbb{Z}^4, \quad S_3(A) \cong \mathbb{Z}^5.$

Now we'll determine the cycles and boundaries of each singular digital singular n-chains:

$$\partial_1 : S_1(A) \to S_0(A).$$

For $\sigma_i^1 \in S_1(A)$ we have a differential map

$$\partial_1(\sigma_i^1) = \sigma_i^1(e_1) - \sigma_i^1(e_0) \qquad i = 1,2,3.$$

The following are hold:

$$\partial_1(\sigma_1^1) = 0, \quad \partial_1(\sigma_2^1) = 0, \quad \partial_1(\sigma_3^1) = \sigma_2^0 - \sigma_1^0.$$

Then we get Im $\partial_1 \cong \mathbb{Z}$ and Ker $\partial_1 \cong \mathbb{Z}^2$. Consider

$$\partial_2 : S_2(A) \to S_1(A).$$

For $\sigma_i^2 \in S_2(A)$ we have a differential map

$$\partial_2(\sigma_i^2) = \sigma_i^2([e_1,e_2]) - \sigma_i^2([e_0,e_3]) + \sigma_i^2([e_0,e_1]) \qquad i = 1,2,3,4.$$

The following are observed:

$$\partial_2(\sigma_1^2) = \sigma_1^1, \quad \partial_2(\sigma_2^2) = \sigma_2^1 \quad \partial_2(\sigma_3^2) = \sigma_1^1, \quad \partial_2(\sigma_4^2) = \sigma_2^1.$$

From this observation, we get Im $\partial_2 \cong \mathbb{Z}^2$ and Ker $\partial_2 \cong \mathbb{Z}^2$. Consider

$$\partial_3 : S_3(A) \to S_2(A)$$

For $\sigma_i^3 \in S_3(A)$ we have a differential map

$$\partial_3(\sigma_i^3) = \sigma_i^3([e_1,e_2,e_3]) - \sigma_i^3([e_0,e_2,e_3]) + \sigma_i^3([e_0,e_1,e_2]) - \sigma_i^3([e_0,e_1,e_2]) \qquad i = 1,2,3,4,5.$$

It's clear that

$$\partial_3(\sigma_1^3) = 0, \quad \partial_3(\sigma_2^3) = 0, \quad \partial_3(\sigma_3^3) = \sigma_3^2 - \sigma_1^2$$
$$\partial_3(\sigma_4^3) = \sigma_2^2 - \sigma_4^2, \quad \partial_3(\sigma_5^3) = 0.$$

Thus we have

$$\text{Im } \partial_3 \cong \mathbb{Z}^2 \quad \text{and} \quad \text{Ker } \partial_3 \cong \mathbb{Z}^3.$$

Then the digital singular homology groups of A are as follows:

$$H_0(A) \cong \mathbb{Z} \qquad H_1(A) = 0 \qquad H_2(A) = 0.$$

Now from the exact sequence of the pair (X,A),

$$0 \longrightarrow 0 \longrightarrow H_2(X,A) \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_1(X,A) \longrightarrow \mathbb{Z} \longrightarrow$$

$$\mathbb{Z} \longrightarrow H_0(X,A) \longrightarrow 0$$

we conclude that

$$H_0(X,A) \cong \mathbb{Z} \qquad H_1(X,A) \cong \mathbb{Z} \qquad H_2(X,A) = 0.$$

Theorem 9. *Let $X = \{a = (-1,0), b = (0,-1), c = (1,0), d = (0,1)\}$ as in Theorem 5 and $A = \{a = (-1,0), b = (0,-1)\} \subset X$. Then the induced topology on A is also discrete so that it can be easily verified regarding the Theorem 6 and Theorem 7 that*

$$H_n(A) = \begin{cases} \mathbb{Z}^2, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Proof. Define the continuous map

$$r : X \to A$$

$$a \longmapsto a, \quad b \longmapsto a, \quad c \longmapsto b, \quad d \longmapsto b$$

such that $r \circ i = id_A$. Then we obtain that i_* is injective. Thus the exact sequence

$$0 \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \longrightarrow 0$$

splits. We conclude that

$$H_n(X,A) = \begin{cases} \mathbb{Z}^2, & n = 0, \\ 0, & n \neq 0 \end{cases}.$$

Let X be a digital Khalimsky space and X_1, X_2 be the subsets of X. Consider the diagram such that all the maps are inclusions:

$$\begin{tikzcd} (X_1 \cap X_2, \emptyset) \arrow[r, "i_1"] \arrow[d, "i_2"'] & (X_1, \emptyset) \arrow[r, "p"] \arrow[d, "g"] & (X_1, X_1 \cap X_2) \arrow[d, "h"] \\ (X_2, \emptyset) \arrow[r, "f'"'] & (X, \emptyset) \arrow[r, "q"'] & (X, X_2) \end{tikzcd}$$

One can easily obtain analogue of Mayer-Vietoris theorem of Euclidean Topology for a digital Khalimsky space (see [21]).

Theorem 10. *(Mayer-Vietoris) Let X be a digital Khalimsky space and X_1, X_2 be the subspaces of X with $X = X_1^\circ \cup X_2^\circ$. Then there is an exact sequence*

$$\ldots \longrightarrow H_n(X_1 \cap X_2) \xrightarrow{(i_{1*}, i_{2*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{g_* - j_*} H_n(X) \xrightarrow{dh_*^{-1}q_*} H_{n-1}(X_1 \cap X_2) \longrightarrow \ldots$$

where d is the connecting homomorphism of the pair $(X_1, X_1 \cap X_2)$.

We modify Excision axiom in Algebraic Topology for a digital Khalimsky space (see [21]).

Theorem 11. *(Excision) Let X be a digital Khalimsky space and X_1, X_2 be the subspaces of X with $X = X_1^\circ \cup X_2^\circ$. Then the inclusion*

$$j : (X_1, X_1 \cap X_2) \to (X, X_2)$$

induces isomorphisms

$$j_* : H_n(X_1, X_1 \cap X_2) \to H_n(X, X_2)$$

for all n.

Example 1. Let $X = \{a = (0,0), b = (2,0), c = (1,1), d = (2,2)\} \subset \mathbb{Z}^2$ as in Example 8. Take X_1 and X_2 as $\{a,c,d\}$ and $\{b,c\}$ respectively. Then X_1 and X_2 satisfy the excision axioms, i.e., for all $n \geq 0$,

$$H_n(\{a,c,d\},\{c\}) \cong H_n(X,\{b,c\})$$

Hence the homology groups of the pair $(\{a,c,d\},\{c\})$ up to the dimension 2 are as follows:

$$H_0(X_1, X_1 \cap X_2) \cong \mathbb{Z}, \quad H_1(X_1, X_1 \cap X_2) \cong \mathbb{Z}, \quad H_2(X_1, X_1 \cap X_2) = 0.$$

5 Conclusion

In this paper, we define the digital relative singular homology groups and compute the relative homology groups of some certain Khalimsky spaces. We have seen that the validity of additivity and excision can be used to compute the digital homology groups of Khalimsky spaces with a less effort. The next work based on this paper is to investigate whether the homotopy axiom is valid or not in the digital singular homology theory, to define the digital singular cohomology and to compare the digital singular homology groups with the digital singular cohomology groups.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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