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The exponential and trigonometric cubic B-spline methods for second order matrix differential equations

K. R. Raslan[1] *A. R. Hadhoud*[2] *and M. A. Shaalan*[3]

[1] Faculty of science, Al-Azhar University, Cairo, Egypt
[2] Faculty of Science, Menoufia University, Shebein El-Koom, Egypt
[3] Higher Technological Institute, Tenth of Ramadan City, Egypt

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Abstract: The goal of the present paper is to present numerical treatments for solving matrix differential equations of second order using exponential and trigonometric cubic B-splines. Efficiency and accuracy of the proposed methods are illustrated by calculating the maximum errors. The results of numerical experiments shown by these methods are convenient to be implemented and effective numerical technique for solving matrix differential equations.

Keywords: Matrix differential equations, exponential cubic B-spline, trigonometric cubic B-spline, kronecker product, frobenius norm.

1 Introduction

Given matrix boundary value problems

$$\left.\begin{array}{l} U''(t)=f(t,U(t),U'(t)) \\ U(a)=U_a, U(b)=U_b \end{array}\right\}, a\leq t\leq b, [a,b]\subset \Re, \tag{1}$$

where matrices $U_a, U_b, U(t)\in C^{m\times n}$ and matrix function $f:[a,b]\times C^{m\times n}\times C^{m\times n}\to C^{m\times n}$, are recurrent in various phenomena in physics and engineering. Equation (1) is considered as the statement of Newton's law of motion for coupled mechanical system. Usually models of this kind recurrently appear in molecular dynamics, quantum mechanics and for scattering methods, where one solves scalar or vectorial problems subject to boundary value conditions [1,6].

We define the Kronecker product of $Y\in C^{m\times n}$ and$X\in C^{p\times q}$, denoted by$Y\otimes X$ [7]

$$Y\otimes X=\begin{pmatrix} y_{11}X & \cdots & y_{1n}X \\ \vdots & \ddots & \vdots \\ y_{m1}X & \cdots & y_{mn}X \end{pmatrix}, \tag{2}$$

* Corresponding author e-mail: abozeid87@yahoo.com

The column vector operator on a matrix $Y \in C^{n\times m}$ is given by [7].

$$Vec(Y) = \begin{bmatrix} Y_{\bullet 1} \\ \vdots \\ Y_{\bullet m} \end{bmatrix} where Y_{\bullet k} = \begin{bmatrix} Y_{1k} \\ \vdots \\ Y_{mk} \end{bmatrix}. \tag{3}$$

Also, the derivative of a matrix $U \in C^{m\times n}$ with respect to a matrix $V \in C^{p\times q}$ is defined by [7].

$$\frac{\partial U}{\partial V} = \begin{pmatrix} \frac{\partial U}{\partial v_{11}} & \cdots & \frac{\partial U}{\partial v_{1q}} \\ \vdots & \ddots & \vdots \\ \frac{\partial U}{\partial v_{p1}} & \cdots & \frac{\partial U}{\partial v_{pq}} \end{pmatrix}, where\ \frac{\partial U}{\partial v_n} = \begin{pmatrix} \frac{\partial U_{11}}{\partial v_n} & \cdots & \frac{\partial U_{1n}}{\partial v_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial U_{m1}}{\partial v_n} & \cdots & \frac{\partial U_{mn}}{\partial v_n} \end{pmatrix}. \tag{4}$$

The derivative of a matrix product $V \in C^{p\times q}$ and $U \in C^{q\times v}$ with respect to another matrix $W \in C^{m\times n}$ is given by [7].

$$\frac{\partial VU}{\partial W} = \frac{\partial V}{\partial W}[I_n \otimes U] + [I_m \otimes V]\frac{\partial U}{\partial W}, \tag{5}$$

where the identity matrices of dimensions m and n denoted by I_m and I_n respectively. The chain rule and derivative of a Kronecker product of matrices $V \otimes U$with respect to a matrix W are given by [7].

$$\frac{\partial W}{\partial V} = \left[\frac{\partial [Vec(U)]^T}{\partial V} \otimes I_m\right] + \left[I_q \otimes \frac{\partial W}{\partial [Vec(U)]}\right], \tag{6}$$

$$\frac{\partial (V \otimes U)}{\partial W} = \frac{\partial V}{\partial W} \otimes U + [I_m \otimes U_1]\left[\frac{\partial U}{\partial W} \otimes V\right][I_n \otimes U_2], \tag{7}$$

where $V \in C^{p\times q}$, $U \in C^{u\times v}$, $W \in C^{m\times n}$ and U_1, U_2 are permutation matrices. The frobenius norm of $U \in C^{m\times n}$ is given by [8].

$$\|U\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|u_{ij}\right|^2}. \tag{8}$$

The following relationship between the 2-norm and Frobenius norm holds [8].

$$\|U\|_2 \le \|U\|_F \le \sqrt{n}\,\|U\|_2. \tag{9}$$

Cubic splines are discussed in [9,10], matrix differential equations are studied in [11,14] and exponential cubic B-splines are piecewise polynomial functions containing a free parameter and its properties are presented in [15]. The exponential and trigonometric cubic B-spline methods and spline with different basis as sextic are studied to solve numerical solutions of various ordinary and partial differential equations [16,22]. This paper is organized as follows; In section 2, we present the exponential and trigonometric cubic B-spline methods. In section 3, some numerical examples are discussed. Finally, the conclusion of this study is given in section 4.

2 Description of cubic B-spline methods

Firstly, we assume that the problem domain $[a,b]$ is equally divided into N subintervals $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, N-1$ by the knots $t_i = a + ih$ where $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ and the step size $h = \frac{b-a}{N}$.

2.1 Exponential cubic B-spline method (ECBSM)

The exponential cubic B-spline can be defined as follows

$$ECB_i(t) = \begin{cases} w_1\left[(t_{i-2}-t)-\frac{1}{\eta}\left(\sinh\left(\eta\left(t_{i-2}-t\right)\right)\right)\right] & t\in[t_{i-2},t_{i-1}], \\ w_2+w_3(t_i-t)+w_4e^{\eta(t_i-t)}+w_5e^{-\eta(t_i-t)} & t\in[t_{i-1},t_i], \\ w_2+w_3(t-t_i)+w_4e^{\eta(t-t_i)}+w_5e^{-\eta(t-t_i)} & t\in[t_i,t_{i+1}], \\ w_1\left[(t-t_{i+2})-\frac{1}{\eta}\left(\sinh\left(\eta\left(t-t_{i+2}\right)\right)\right)\right] & t\in[t_{i+1},t_{i+2}], \\ 0 & elsewhere. \end{cases} \tag{10}$$

$$(i=-1,0,\cdots,N+1),$$

, where

$$w_1=\frac{\eta}{2(\eta hC-S)}, w_2=\frac{\eta hC}{\eta hC-S}, w_3=\frac{\eta}{2}\left[\frac{C(C-1)+S^2}{(\eta hC-S)(1-C)}\right],$$
$$w_4=\frac{1}{4}\left[\frac{e^{-\eta h}(1-C)+S\left(e^{-\eta h}-1\right)}{(\eta hC-S)(1-C)}\right], w_5=\frac{1}{4}\left[\frac{e^{\eta h}(C-1)+S\left(e^{\eta h}-1\right)}{(\eta hC-S)(1-C)}\right], C=Cosh(\eta h), S=Sinh(\eta h),$$

and η is a free parameter.
We consider the spline function as interpolation to the solutions $\overset{kl}{u}(t)$ of the problem (1).

$$\overset{kl}{u}(t)=\sum_{i=-1}^{N+1}\overset{kl}{\zeta_i}(t)\,ECB_i(t)\,;1\le k\le n,\ 1\le l\le m, \tag{11}$$

where constants $\overset{kl}{\zeta_i}(t)$'s are be determined. To solve matrix boundary value problems of second order, we find ECB_i, ECB_i' andECB_i'' at the nodal points are needed. Their coefficients are summarized in Table 1.

Table 1: values of ECB_i, ECB_i' and ECB_i''

	t_{i-2}	t_{i-1}	t_i	t_{i+1}	t_{i+2}
ECB_i	0	β_1	1	β_1	0
ECB_i'	0	$-\beta_2$	0	β_2	0
ECB_i''	0	β_3	β_4	β_3	0

where

$$\beta_1=\frac{S-\eta h}{2(\eta hC-S)},\ \beta_2=\frac{\eta(C-1)}{2(\eta hC-S)},\ \beta_3=\frac{\eta^2 S}{2(\eta hC-S)},\ \beta_4=\frac{-\eta^2 S}{\eta hC-S}.$$

Using equations (10) - (11), the values of $\overset{kl}{u_i}$ and their derivatives up to second order at the knots are

$$\left.\begin{array}{l} \overset{kl}{u_i}=\beta_1\overset{kl}{\zeta}_{i-1}+\overset{kl}{\zeta_i}+\beta_1\overset{kl}{\zeta}_{i+1} \\ \overset{kl}{u_i'}=-\beta_2\overset{kl}{\zeta}_{i-1}+\beta_2\overset{kl}{\zeta}_{i+1} \\ \overset{kl}{u_i''}=\beta_3\overset{kl}{\zeta}_{i-1}+\beta_4\overset{kl}{\zeta_i}+\beta_3\overset{kl}{\zeta}_{i+1} \end{array}\right\}, i=0,1,\ldots,N. \tag{12}$$

Substituting from (12) in (1), we find

$$\beta_3\overset{kl}{\zeta}_{i-1}+\beta_4\overset{kl}{\zeta_i}+\beta_3\overset{kl}{\zeta}_{i+1}=f\left(ih,\beta_1\overset{kl}{\zeta}_{i-1}+\overset{kl}{\zeta_i}+\beta_1\overset{kl}{\zeta}_{i+1},-\beta_2\overset{kl}{\zeta}_{i-1}+\beta_2\overset{kl}{\zeta}_{i+1}\right),$$

$$i = 0, 1, ..., N,\ k = 1, 2, ..., n and l = 1, 2, ..., m, \tag{13}$$

and the boundary conditions are given as

$$\begin{aligned} \beta_1 \overset{kl}{\zeta}_{-1} + \overset{kl}{\zeta}_0 + \beta_1 \overset{kl}{\zeta}_1 = \overset{kl}{u}_a, \\ \beta_1 \overset{kl}{\zeta}_{N-1} + \overset{kl}{\zeta}_N + \beta_1 \overset{kl}{\zeta}_{N+1} = \overset{kl}{u}_b. \end{aligned} \tag{14}$$

Solving the system of equations (14) in $\overset{kl}{\zeta}_{-1}$ and $\overset{kl}{\zeta}_{N+1}$, the linear algebraic system of equations (13) can be converted to the following matrix form

$$\overset{kl}{A}\overset{kl}{\zeta} = \overset{kl}{F}, 1 \le k \le n,\ 1 \le l \le m, \tag{15}$$

where $\overset{kl}{A}$ is an $(N+1)\times(N+1)$ matrix, $\overset{kl}{\zeta}$ is an $(N+1)$ dimensional vector with components $\overset{kl}{\zeta}_i$ and the right hand side $\overset{kl}{F}$ is an $(N+1)$ dimensional vector.

$$\overset{kl}{\zeta} = \left[\overset{kl}{\zeta}_0, \overset{kl}{\zeta}_1, ..., \overset{kl}{\zeta}_N\right]^T, \overset{kl}{F} = \left[\overset{kl}{f}{}_0^*, \overset{kl}{f}_1, ..., \overset{kl}{f}_{N-1}, \overset{kl}{f}{}_N^*\right]^T. \tag{16}$$

2.2 Trigonometric cubic B-spline method (TCBSM)

The trigonometric cubic B-spline can be defined as follows

$$TCB_i(t) = \frac{1}{\rho}\begin{cases} \varphi^3(t_{i-2}) & t \in [t_{i-2}, t_{i-1}], \\ \varphi(t_{i-2})[\varphi(t_{i-2})\vartheta(t_i) + \varphi(t_{i-1})\vartheta(t_{i+1})] + \varphi^2(t_{i-1})\vartheta(t_{i+2}) & t \in [t_{i-1}, t_i], \\ \vartheta(t_{i+2})[\vartheta(t_{i+2})\varphi(t_i) + \vartheta(t_{i+1})\varphi(t_{i-1})] + \vartheta^2(t_{i+1})\varphi(t_{i-2}) & t \in [t_i, t_{i+1}], \\ \vartheta^3(t_{i+2}) & t \in [t_{i+1}, t_{i+2}], \\ 0 & elsewhere. \end{cases} \tag{17}$$

$$(i = -1, 0, 1, \cdots, N+1),$$

where

$$\rho = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right),\ \varphi(t_i) = \sin\left(\frac{t-t_i}{2}\right),\ \vartheta(t_i) = \sin\left(\frac{t_i-t}{2}\right).$$

We consider the spline function as interpolation to the solutions$\overset{kl}{u}(t)$ of the problem (1)

$$\overset{kl}{u}(t) = \sum_{i=-1}^{N+1} \overset{kl}{\tau}_i(t) B_i(t) \quad ; 1 \le k \le n, 1 \le l \le m, \tag{18}$$

where constants $\overset{kl}{\tau}_i(x)$'s are be determined. To solve matrix boundary value problems of second order, we find TCB_i, TCB_i' andTCB_i'' at the nodal points are needed. Their coefficients are summarized in Table 2.

Table 2: values of TCB_i, TCB_i' andTCB_i''

	t_{i-2}	t_{i-1}	t_i	t_{i+1}	t_{i+2}
TCB_i	0	Ω_1	Ω_2	Ω_1	0
TCB_i'	0	$-\Omega_3$	0	Ω_3	0
TCB_i''	0	Ω_4	Ω_5	Ω_4	0

where

$$\Omega_1 = \frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)},\ \Omega_2 = \frac{2}{1+\cos(h)},\ \Omega_3 = \frac{3}{4\sin\left(\frac{3h}{2}\right)},\ \Omega_4 = \frac{3(1+3\cos(h))}{16\sin^2\left(\frac{h}{2}\right)\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)},$$
$$\Omega_5 = \frac{-3\cos^2\left(\frac{h}{2}\right)}{\sin^2\left(\frac{h}{2}\right)(2+4\cos(h))}.$$

Using equations (17) - (18), the values of $\overset{kl}{u_i}$ and their derivatives up to second order at the knots are

$$\left.\begin{array}{l} \overset{kl}{u_i} = \Omega_1 \overset{kl}{\tau_{i-1}} + \Omega_2 \overset{kl}{\tau_i} + \Omega_1 \overset{kl}{\tau_{i+1}} \\ \overset{kl}{u_i'} = -\Omega_3 \overset{kl}{\tau_{i-1}} + \Omega_3 \overset{kl}{\tau_{i+1}} \\ \overset{kl}{u_i''} = \Omega_4 \overset{kl}{\tau_{i-1}} + \Omega_5 \overset{kl}{\tau_i} + \Omega_4 \overset{kl}{\tau_{i+1}} \end{array}\right\}, i = 0, 1, ..., N. \tag{19}$$

Substituting from (19) in (1) we find

$$\Omega_4 \overset{kl}{\tau_{i-1}} + \Omega_5 \overset{kl}{\tau_i} + \Omega_4 \overset{kl}{\tau_{i+1}} = f\left(ih, \Omega_1 \overset{kl}{\tau_{i-1}} + \Omega_2 \overset{kl}{\tau_i} + \Omega_1 \overset{kl}{\tau_{i+1}}, -\Omega_3 \overset{kl}{\tau_{i-1}} + \Omega_3 \overset{kl}{\tau_{i+1}}\right),$$

$$i = 0, 1, ..., N,\ k = 1, 2, ..., n and l = 1, 2, ..., m, \tag{20}$$

and the boundary conditions are given as

$$\begin{array}{l} \Omega_1 \overset{kl}{\tau_{-1}} + \Omega_2 \overset{kl}{\tau_0} + \Omega_1 \overset{kl}{\tau_1} = \overset{kl}{u_a}, \\ \Omega_1 \overset{kl}{\tau_{N-1}} + \Omega_2 \overset{kl}{\tau_N} + \Omega_1 \overset{kl}{\tau_{N+1}} = \overset{kl}{u_b}. \end{array} \tag{21}$$

Solving the system of equations (21) in $\overset{kl}{\tau_{-1}}$ and $\overset{kl}{\tau_{N+1}}$, the linear algebraic system of equations (20) can be converted to the following matrix form

$$\overset{kl}{A}\overset{kl}{\tau} = \overset{kl}{F}, 1 \le k \le n,\ 1 \le l \le m, \tag{22}$$

where $\overset{kl}{A}$ is an $(N+1)\times(N+1)$ matrix, $\overset{kl}{\tau}$ is an $(N+1)$ dimensional vector with components $\overset{kl}{\tau_i}$ and the right hand side $\overset{kl}{F}$ is an $(N+1)$ dimensional vector,

$$\overset{kl}{\tau} = \left[\overset{kl}{\tau_0}, \overset{kl}{\tau_1}, ..., \overset{kl}{\tau_N}\right]^T, \overset{kl}{F} = \left[\overset{kl}{f_0^*}, \overset{kl}{f_1}, ..., \overset{kl}{f_{N-1}}, \overset{kl}{f_N^*}\right]^T. \tag{23}$$

3 Numerical examples

In this section, three examples of matrix differential equations of second order are presented to show efficiency and accuracy of the proposed methods using Frobenius norm of the difference between approximate solution and exact solution at each point in the interval $[0,1]$ taking $h = 0.1$and the results are generated with Mathematical using Find Root function to solve the emerging algebraic equations.

Example 1. We examine non-linear differential vector system [13].

$$\left.\begin{array}{l} u_1''(t) = 1-\cos(t)+\sin(u_2'(t))+\cos(u_2'(t)) \\ u_2''(t) = \frac{1}{4+u_1(t)^2} - \frac{1}{5-\sin^2(t)} \\ u_1(0)=1, \qquad u_1(1)=\cos(1), \\ u_2(0)=0, \qquad u_2(1)=\pi \end{array}\right\} \quad 0 \le t \le 1. \tag{24}$$

This example has an exact solution $U(t) = \begin{pmatrix} \cos(t) \\ \pi t \end{pmatrix}$. Thus, we can contrast our numerical estimates with the exact solution to get the exact errors of the approximation which briefed in Table 3 and figure 1. For free equilibrium points $u_j \equiv 0, (j = 1,2,...,4)$, we find that the Jacobian matrix of (24).

$$\begin{pmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{pmatrix}, \tag{25}$$

and its eigenvalues are evaluated from the equation

$$\lambda^4 = 0$$

where $\lambda_j = 0$, then the equilibrium points $u_j = 0$ of (24) are unstable $(j = 1,2,...,4)$.

Table 3: Comparison of maximum absolute errors for Example 1.

t	Exponential Cubic B-spline errors (ECBSM)	Trigonometric Cubic B-spline errors (TCBSM)	Cubic spline errors (CSM) [13]
0	0	1.11022×10^{-16}	0
0.1	6.82552×10^{-5}	3.63629×10^{-5}	4.16114×10^{-6}
0.2	1.19923×10^{-4}	6.77517×10^{-5}	1.66032×10^{-5}
0.3	1.55258×10^{-4}	9.32094×10^{-5}	3.72028×10^{-5}
0.4	1.74684×10^{-4}	1.11637×10^{-4}	6.57658×10^{-5}
0.5	1.78781×10^{-4}	1.21817×10^{-4}	1.02012×10^{-4}
0.6	1.68287×10^{-4}	1.22433×10^{-4}	1.45563×10^{-4}
0.7	1.44083×10^{-4}	1.12078×10^{-4}	1.96167×10^{-4}
0.8	1.07188×10^{-4}	8.92682×10^{-5}	2.52912×10^{-4}
0.9	5.87418×10^{-5}	5.24493×10^{-5}	3.15643×10^{-4}
1.0	4.44089×10^{-16}	0	3.83638×10^{-4}

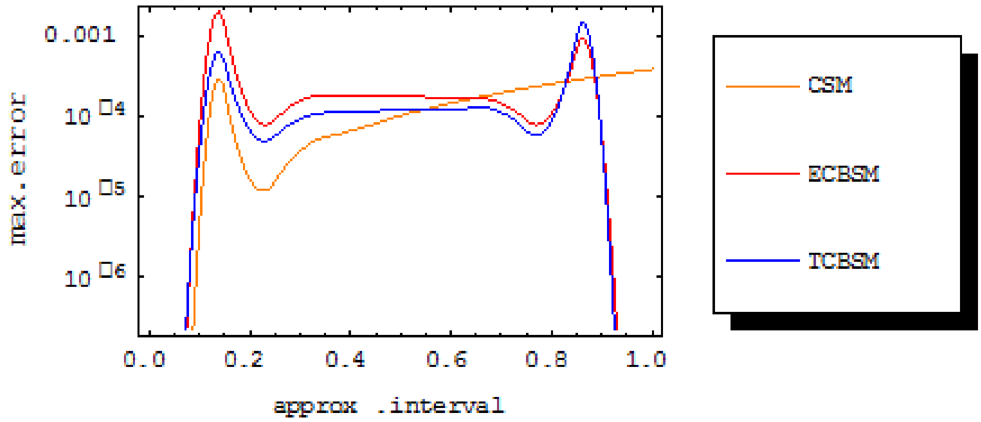

Fig. 1: Comparison of maximum absolute errors for non-linear differential vector system in the interval $[0, 1]$ with step size $h = 0.1$.

Example 2. We investigate incomplete differential system of second order [13].

$$U''(t) + AU(t) = 0, 0 \leq t \leq 1. \tag{26}$$

Where $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and corresponding exact solution $U(t) = \begin{pmatrix} \sin(t) & 0 \\ t\cos(t) & \sin(t) \end{pmatrix}$. Thus, we can observe the exact errors of the approximation which briefed in Table 4 and figure 2.

For the only equilibrium point $u_j = 0, (j = 1, 2, ..., 8)$, we find that the Jacobian matrix of (26) is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{27}$$

and its eigenvalues are evaluated from the equation

$$\lambda^8 + 4\lambda^6 + 6\lambda^4 + 4\lambda^2 + 1 = 0,$$

where $\lambda_j = \pm i$, then (26) has a unique equilibrium point $u_j = 0$ is an stable $(j = 1, 2, ..., 8)$.

Example 3. We consider the following problem of second order polynomial matrix equation [13].

$$U''(t) + A_0 U'(t) + A_1 U(t) = 0, 0 \leq t \leq 1. \tag{28}$$

Table 4: Comparison of maximum absolute errors for Example 2.

t	Exponential Cubic B-spline errors (ECBSM)	Trigonometric Cubic B-spline errors (TCBSM)	Cubic spline errors (CSM) [13]
0	9.03349×10^{-16}	3.88578×10^{-15}	0
0.1	1.28910×10^{-4}	3.06544×10^{-5}	1.0072×10^{-6}
0.2	2.48964×10^{-4}	5.91219×10^{-5}	6.3032×10^{-6}
0.3	3.51531×10^{-4}	8.32875×10^{-5}	2.0059×10^{-5}
0.4	4.28431×10^{-4}	1.01178×10^{-4}	4.6213×10^{-5}
0.5	4.72145×10^{-4}	1.11029×10^{-4}	8.8359×10^{-5}
0.6	4.76020×10^{-4}	1.11351×10^{-4}	1.4964×10^{-4}
0.7	4.34449×10^{-4}	1.00979×10^{-4}	2.3267×10^{-4}
0.8	3.43038×10^{-4}	7.91308×10^{-5}	3.3941×10^{-4}
0.9	1.98746×10^{-4}	4.54413×10^{-5}	4.7114×10^{-4}
1.0	1.92296×10^{-16}	1.11022×10^{-16}	6.2838×10^{-4}

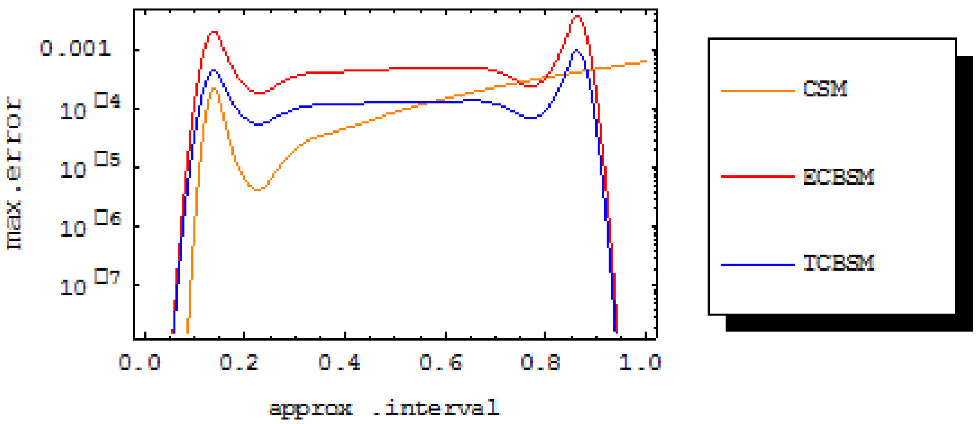

Fig. 2: Comparison of maximum absolute errors for incomplete differential system of second order in the interval [0, 1] with step size $h = 0.1$.

Where $A_0 = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and the exact solution $U(t) = \begin{pmatrix} e^t & -1+e^t-te^t \\ 0 & e^t \end{pmatrix}$. Thus, we are illustrated the exact errors at each point in Table 5 and figure 3.

For free equilibrium points $u_j \equiv 0, (j = 1, 2, ..., 8)$, we find that the Jacobian matrix of (28) is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (29)$$

and its eigenvalues are evaluated from the equation

$$\lambda^8 - 6\lambda^7 + 15\lambda^6 - 20\lambda^5 + 15\lambda^4 - 6\lambda^3 + \lambda^2 = 0$$

where $\lambda_{1,2} = 0$ and $\lambda_j = 1, (j = 3, ..., 8)$, then the equilibrium points $u_j \equiv 0$ of (28) are unstable.; $(j = 1, 2, ..., 8)$.

Table 5: Comparison of maximum absolute errors for Example 3.

t	Exponential Cubic B-spline errors (ECBSM)	Trigonometric Cubic B-spline errors (TCBSM)	Cubic spline errors (CSM) [13]
0	2.04743×10^{-15}	2.22044×10^{-16}	0
0.1	9.31479×10^{-5}	3.81878×10^{-4}	1.53895×10^{-5}
0.2	1.76710×10^{-4}	7.44554×10^{-4}	6.67523×10^{-5}
0.3	2.47641×10^{-4}	1.07301×10^{-3}	1.63924×10^{-4}
0.4	3.02360×10^{-4}	1.34804×10^{-3}	3.18789×10^{-4}
0.5	3.36670×10^{-4}	1.54528×10^{-3}	5.45654×10^{-4}
0.6	3.45675×10^{-4}	1.63419×10^{-3}	8.61682×10^{-4}
0.7	3.23672×10^{-4}	1.57674×10^{-3}	1.28740×10^{-3}
0.8	2.64039×10^{-4}	1.32589×10^{-3}	1.84731×10^{-3}
0.9	1.59105×10^{-4}	8.23872×10^{-4}	2.57055×10^{-3}
1.0	4.57756×10^{-16}	4.44089×10^{-16}	3.49171×10^{-3}

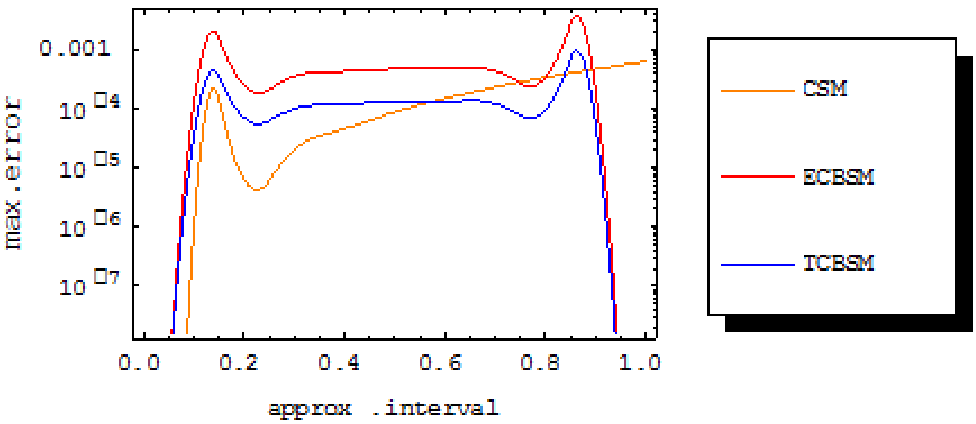

Fig. 3: Comparison of maximum absolute errors for second order polynomial matrix equation in the interval [0, 1] with step size $h = 0.1$.

4 Conclusion

In this article, we have examined scheme treat numerically with the second-order matrix differential equations by exponential and trigonometric cubic B-splines. From the computational results, we can see that the exponential and trigonometric cubic B-splines as summarized in tables (3-5) and figures (1-3) enjoy high accuracy and easy to be implemented by finding Frobenius norm and are compared with [13].

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