New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.282

On base conformal warped product manifolds with conjugate connections

Sema Kazan

Inonu University, Faculty of Science and Arts, Department of Mathematics, Malatya, Turkey

Received: 23 May 2017, Accepted: 23 October 2017
Published online: 2 May 2018.

Abstract: In this paper, we investigate base conformal warped product manifolds with generalized conjugate connections. We prove that the generalized conjugate connection defined on a base conformal warped product manifold induces generalized conjugate connections on the base and the fiber manifolds. Conversely generalized conjugate connections on the base and the fiber manifolds induce a generalized conjugate connection defined on a base conformal warped product manifold. Also, we investigate base conformal warped product manifolds with product conjugate connections and give some results.

Keywords: Conjugate connection, base conformal warped product, fiber manifold.

1 Introduction

A natural generalization of geometry of Levi-Civita connections from Riemannian manifolds theory gives geometry of conjugate connections. Since conjugate connections arise from affine differential geometry and from geometric theory of statistical inferences, many studies have been carried out in the recent 20 years [7]. In that study, the authors have defined the dual or conjugate connection as follows.

Let (M,g) a Riemannian manifold and ∇ a affine connection on M. Affine connection ∇' is said to be dual or conjugate of ∇ w.r.t. the metric g if,

$$X.g(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla'_X Z)\ ,\ \forall X,Y,Z \in TM.$$

Given an affine connection ∇ on a Riemannian manifold (M,g), there exists an unique affine connection dual of ∇ w.r.t. g, denoted by ∇^*. Also, in [4], the geometry of product conjugate connections has been studied. Furthermore there are many studies deal with this subject as [1], [3], [5]. On the other hand, base conformal warped product manifolds have been studied in [11]. They have defined the concept of a base conformal warped product of two pseudo-Riemannian manifolds in this study.

In the third section of the present paper, we study generalizations of conjugate connections on the base conformal warped product manifolds. In the fourth section, we investigate product conjugate connections on the base conformal warped product manifolds and give some results.

2 Preliminaries

In this section, let us recall some general notions about base conformal warped product manifolds by [11]. Also we statement the notions generalized conjugate connection and product conjugate connection.

* Corresponding author e-mail: sema.bulut@inonu.edu.tr

Let (B,g_B) and (F,g_F) be m and k dimensional pseudo-Riemannian manifolds, respectively. Then $M = B\times F$ is an $(m+k)$-dimensional pseudo-Riemannian manifold with $\pi : B\times F \to B$ and $\sigma : B\times F \to F$ the usual projection maps. Throughout this paper we use the natural product coordinate system on the product manifold $B\times F$, namely. Let (p_0,q_0) be a point in M and coordinate charts (U,x) and (V,y) on B and F, respectively such that $p_0 \in B$ and $q_0 \in F$. Then we can define a coordinate chart (W,z) on M such that W is an open subset in M contained in $U\times V$, $(p_0,q_0)\in W$ and for all (p,q) in W, $z(p,q) = (x(p),y(q))$, where $x = (x^1,...,x^m)$ and $y = (y^{m+1},...,y^{m+k})$. Clearly, the set of all (W,z) defines an atlas on $B\times F$. Here, for our convenience, we call the j-th component of y as y^{m+j} for all $j \in \{1,...,k\}$.

Let $\varphi : B \to \mathbb{R} \in C^\infty(B)$ then the lift of φ to $B\times F$ is $\tilde{\varphi} = \varphi\circ\pi \in C^\infty(B\times F)$, where $C^\infty(B)$ is the set of all smooth real-valued functions on B. Moreover, one can define lifts of tangent vectors as: Let $X_p \in T_p(B)$ and $q\in F$ then the lift $\tilde{X}(p,q)$ of X_p is the unique tangent vector in $T(p,q)(B\times q)$ such that $d\pi(p,q)(\tilde{X}(p,q)) = X_p$ and $d\sigma(p,q)(\tilde{X}(p,q)) = 0$. We will denote the set of all lifts of all tangent vectors of B by $L(p,q)(B)$. Similarly, we can define lifts of vector fields. Let $X\in\chi(B)$ then the lift of X to $B\times F$ is the vector field $\tilde{X}\in\chi(B\times F)$ whose value at each (p,q) is the lift of X_p to (p,q). We will denote the set of all lifts of all vector fields of B by $\mathscr{L}(B)$.

Let (B,g_B) and (F,g_F) be pseudo-Riemannian manifolds and also let $w : B \to (0,\infty)$ and $c : B \to (0,\infty)$ be smooth functions. The base conformal warped product (briefly bcwp) is the product manifold $B\times F$ furnished with the metric tensor $g = c^2g_B \oplus w^2g_F$ defined by

$$g = (c\circ\pi)^2\pi^*(g_B)\oplus(w\circ\pi)^2\sigma^*(g_F). \tag{1}$$

We will denote this structure by $B\times_{(c;w)} F$. The function $w : B\to(0,\infty)$ is called the warping function and the function $c : B\to(0,\infty)$ is said to be the conformal factor. If $c\equiv 1$ and w is not identically 1, then we obtain a singly warped product. If both $w\equiv 1$ and $c\equiv 1$, then we have a product manifold. If neither w nor c is constant, then we have a nontrivial bcwp. If (B,g_B) and (F,g_F) are both Riemannian manifolds, then $B\times_{(c;w)} F$ is also a Riemannian manifold. We call $B\times_{(c;w)} F$ as a Lorentzian base conformal warped product if (F,g_F) is Riemannian and either (B,g_B) is Lorentzian or else (B,g_B) is a one-dimensional manifold with a negative definite metric $-dt^2$.

Also,let $\phi\in C^\infty(B)$ and $\psi\in C^\infty(B)$. Then

$$\nabla\phi = \frac{1}{c^2}\nabla^B\phi \ \ and\ \ \nabla\psi = \frac{1}{w^2}\nabla^F\psi.$$

On the other hand, the *generalized conjugate connection* $\overline{\nabla}^*$ of ∇ with respect to g by τ is defined by

$$Xg(Y,Z) = g(\nabla_XY,Z)+g(Y,\nabla^*_XZ)-\tau(X)g(Y,Z), \tag{2}$$

where τ is a 1-form on M.

Suppose that ∇ and ∇' are affine connections on a semi-Riemannian (M,g). We say that ∇ and ∇' are *projectively equivalent* if there exist a 1-form τ that

$$\nabla'_XY = \nabla_XY+\tau(Y)X+\tau(X)Y. \tag{3}$$

We say that ∇ and ∇' are *dual-projectively equivalent* if there exist a 1-form τ that

$$\nabla'_XY = \nabla_XY+g(X,Y)\tau^\sharp, \tag{4}$$

where $\tau^\sharp$ is the metrical dual vector field, i.e.,$g(X,\tau^\sharp)=\tau(X)$. Here, even if ∇ and ∇' are projectively (or dual-projectively) equivalent, their dual connections ∇^* and $(\nabla')^*$ may not be dual-projectively (or projectively) equivalent, respectively [7].

Let M a smooth, n-dimensional manifold for which we denote [4]: $C^\infty(M)$ the algebra of smooth real functions on M, $\chi(M)$ the Lie algebra of vector fields on M, $T^r_sM)$ the $C^\infty(M)$-module of tensor fields of (r,s)- type on M. Usually X,Y,Z,... will be vector fields on M and if $T\to M$ is a vector bundle over M, then $\Gamma(T)$ denotes the C^∞-module of sections of T [e.g. $\Gamma(TM)=\chi(M)$]. Let $\mathscr{C}(M)$ be the set of linear connections on M. Since the difference of two linear connections is a tensor field of $(1,2)$-type, it results that $\mathscr{C}(M)$ is a $C^\infty(M)$-affine module associated to the $C^\infty(M)$-linear module $T^1_2(M)$.

Now, let E an almost product structure on M, i.e. an endomorphism of the tangent bundle such that $E^2=I_{\chi(M)}$. Then, for the associated linear connections, $\nabla\in\mathscr{C}(M)$ is an *E-connection* if E is covariant constant with respect to ∇, namely $\nabla E=0$. Let $\mathscr{C}_E(M)$ be the set of these connections. In order to find the above set, let us consider after the maps

$$\psi_E:\mathscr{C}_E(M)\to\mathscr{C}_E(M),\ \ \psi_E(\nabla)=\frac{1}{2}(\nabla+E\circ\nabla\circ E)\chi_E:T^1_2(M)\to T^1_2(M),\ \ \chi_E(\tau)=\frac{1}{2}(\tau+E\circ\tau\circ E).$$

Then, ψ_E is a $C^\infty(M)$-projector on $\mathscr{C}(M)$ associated to the $C^\infty(M)$-linear projector χ_E:

$$\psi_E^2=\psi_E,\ \ \chi_E^2=\chi_E,\ \ \psi_E(\nabla+\tau)=\psi_E(\nabla)+\chi_E(\tau).$$

So, we have

$$\psi_E(\nabla)=\frac{1}{2}[(\nabla_XY+E(\nabla_XEY)]\chi_E(\tau)(X,Y)=\frac{1}{2}[\tau(X,Y)+E(\tau(X,EY))].$$

It follows that $\nabla E=0$ means $E(\nabla)=\nabla$ which gives that $\mathscr{C}_E(M)=Im_{\psi_E}$. This determines completely $\mathscr{C}_E(M)$. Let ∇_0 arbitrary in $\mathscr{C}(M)$ and ∇ in $\mathscr{C}_E(M)$. So, $\nabla=E(\nabla')$ with $\nabla'=\nabla_0+\tau$. In conclusion, $\nabla=E(\nabla_0)+\chi_E(\tau)$; in other words, $\mathscr{C}_E(M)$ is the affine submodule of $\mathscr{C}(M)$ passing through the E-connection $E(\nabla_0)$ and having the direction given by the linear submodule Im_{χ_E} of $T^1_2(M)$. Let us remark a decomposition (of arithmetic mean type) of it:

$$\psi_E(\nabla)=\frac{1}{2}(\nabla+C_E(\nabla)$$

with *the conjugation map* $C_E:\mathscr{C}(M)\to\mathscr{C}(M)$:

$$C_E(\nabla)_X=E\circ\nabla_X\circ E.$$

Then *the product conjugate connection* $C_E(\nabla)$ measures how far the connection ∇ is from being an E-connection and C_E is the affine symmetry of the affine module $\mathscr{C}(M)$ with respect to the affine submodule $\mathscr{C}_E(M)$, made parallel with the linear submodule $\ker\chi_E$.

For simplification we will denote by a superscript E the product conjugate connection of ∇

$$\nabla^{(E)}=C_E(\nabla)=\nabla+E\circ\nabla E$$

and then

$$\nabla^{(E)}_XY=\nabla_XY+E(\nabla_XEY-E(\nabla_XY))=E(\nabla_XEY). \tag{5}$$

On the other hand, a more general notion like restricting to a distribution is that of geodesically invariance: the distribution D is ∇-*geodesically invariant* if for every geodesic $\gamma : [a,b] \to M$ of ∇ with $\dot{\gamma}(a) \in D_{\gamma(a)}$ it follows $\dot{\gamma}(t) \in D_{\gamma(t)}$ for any $t \in [a,b]$. As a necessary and sufficient condition for a distribution D to be ∇-geodesically invariant: for any X and $Y \in \Gamma(D)$, the symmetric product $< X,Y >:= \nabla_X Y + \nabla_Y X$ to belong to $\Gamma(D)$ or equivalently, for any $X \in \Gamma(D)$ to have $\nabla_X X \in \Gamma(D)$.

3 Base conformal warped product manifolds with generalized conjugate connections

In this section, we investigate base conformal warped product manifolds with generalized conjugate connections.

Let (M,g) be a semi-Riemannian manifold, and ∇ an affine connection on M. We can define another affine connection ∇^* by

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla^*_X Z). \tag{6}$$

Then the triple of a semi-Riemannian metric and a pair of conjugate connection (g,∇,∇^*) satisfying (6) is called a *dualistic structure* on M [6]. Many authors have studied such structures (see [6], [8], [10]).

Let (g,∇,∇^*) be a dualistic structure on $B \times F$. For $\bar{X},\bar{Y},\bar{Z} \in \chi(B)$, $X,Y \in \mathscr{L}_{\mathscr{H}}(\mathscr{B})$ and $\bar{U},\bar{V},\bar{W} \in \chi(F)$, $U,V \in \mathscr{L}_{\mathscr{V}}(\mathscr{F})$, we get

$$\pi_*(\nabla_X Y) = {}^B\nabla_{\bar{X}}\bar{Y} \; and \; \pi_*(\nabla^*_X Y) = {}^B\nabla^*_{\bar{X}}\bar{Y}\sigma_*(\nabla_U V) = {}^F\nabla_{\bar{U}}\bar{V} \; and \; \sigma_*(\nabla^*_U V) = {}^F\nabla^*_{\bar{U}}\bar{V}.$$

Also, we take [9]:

$$\bar{X}g(\bar{Y},\bar{Z})\circ\pi = Xg(Y,Z) \; and \; \bar{U}g(\bar{V},\bar{W})\circ\sigma = Ug(V,W).$$

Since ∇ and ∇^* are affine connections on $B \times F$ and π and σ are the projections of $B \times F$ on B and F respectively, ${}^B\nabla$ and ${}^B\nabla^*$ are affine connections on B and ${}^F\nabla$ and ${}^F\nabla^*$ are affine connections on F. Then we can give results for bcwp-manifold at the following proposition :

Proposition 1. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold. Then the triple $(g_B, {}^B\nabla, {}^B\nabla^*)$ is a dualistic structure on B and the triple $(g_F, {}^F\nabla, {}^F\nabla^*)$ is a dualistic structure on F.

Proof. The proof is obvious by [6].

Proposition 2. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold. Then the triple (g,∇,∇^*) is a dualistic structure on M.

Proof. The proof is obvious by [6].

Now, we can give a main result at following:

Proposition 3. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold. If $\bar{\nabla}^*$ is a generalized conjugate connection of a affine connection ∇ on M, then ${}^B\bar{\nabla}^*$ and ${}^F\bar{\nabla}^*$ are generalized conjugate connections of affine connections ${}^B\bar{\nabla}$ on B and ${}^F\bar{\nabla}$ on F, respectively.

Proof. Let $\bar{X},\bar{Y},\bar{Z} \in \chi(B)$ and $X,Y,Z \in \mathscr{L}_{\mathscr{H}}(B)$ be their corresponding horizontal lifts respectively. Then we obtain

$$\begin{aligned}
\bar{X}g_B(\bar{Y},\bar{Z})\circ\pi &= (c\circ\pi)^{-2}Xg(Y,Z)\\
&= (c\circ\pi)^{-2}[g(\nabla_X Y,Z) + g(Y,\bar{\nabla}^*_X Z) - \tau(X)g(Y,Z)]\\
&= (c\circ\pi)^{-2}[(c\circ\pi)^2 g_B(\pi_*(\nabla_X Y),\pi_*(Z))\circ\pi + (c\circ\pi)^2 g_B(\pi_*(Y),\pi_*(\bar{\nabla}^*_X Z))\circ\pi\\
&\quad - (c\circ\pi)^2\tau(\pi_*(X))g_B(\pi_*(Y),\pi_*(Z))\circ\pi].
\end{aligned} \tag{7}$$

Then, from (7), we have

$$\bar{X}g_B(\bar{Y},\bar{Z}) = g_B({}^B\bar{\nabla}_{\bar{X}}\bar{Y},\bar{Z}) + g_B(\bar{Y},{}^B\bar{\nabla}^*_{\bar{X}}\bar{Z}) - \tau(\bar{X})g_B(\bar{Y},\bar{Z}).$$

Therefore, ${}^B\bar{\nabla}^*$ is generalized conjugate connection of affine connection ${}^B\bar{\nabla}$ with respect to g_B. Similarly, let $\bar{U},\bar{V},\bar{W} \in \chi(F)$ and $U,V,W \in \mathscr{L}_{\mathscr{V}}(F)$ be their corresponding horizontal lifts respectively. Then we obtain,

$$\begin{aligned}\bar{U}g_F(\bar{V},\bar{W})\circ\sigma &= (w\circ\pi)^{-2}Ug(V,W)\\ &= (w\circ\pi)^{-2}[g(\nabla_U V,W) + g(V,\bar{\nabla}^*_U W) - \tau(U)g(V,W)]\\ &= (w\circ\pi)^{-2}[(w\circ\pi)^2 g_F(\sigma_*(\nabla_U V),\sigma_*(W))\circ\sigma + (w\circ\pi)^2 g_F(\sigma_*(V),\sigma_*(\bar{\nabla}^*_U W))\circ\sigma\\ &- (w\circ\pi)^2\tau(\sigma_*(U))g_F(\sigma_*(V),\sigma_*(W))\circ\sigma].\end{aligned} \tag{8}$$

Hence, from (8), we have

$$\bar{U}g_F(\bar{V},\bar{W}) = g_F({}^F\bar{\nabla}_{\bar{U}}\bar{V},\bar{W}) + g_F(\bar{V},{}^F\bar{\nabla}^*_{\bar{U}}\bar{W}) - \tau(\bar{U})g_F(\bar{V},\bar{W}).$$

Then, ${}^F\bar{\nabla}^*$ is generalized conjugate connection of affine connection ${}^F\bar{\nabla}$ with respect to g_F.

Proposition 4. Let $(M = B\times_{(c,w)} F, g)$ be a base conformal warped product manifold. If ${}^B\bar{\nabla}^*$ and ${}^F\bar{\nabla}^*$ are generalized conjugate connections of affine connections ${}^B\bar{\nabla}$ on B and ${}^F\bar{\nabla}$ on F, respectively. Then $\bar{\nabla}^*$ is a generalized conjugate connection of a affine connection ∇ on M.

Proof. Let $\bar{X},\bar{Y},\bar{Z} \in \chi(B)$ and $X,Y,Z \in \mathscr{L}_{\mathscr{H}}(B)$ be their corresponding horizontal lifts respectively. Then we obtain

$$\begin{aligned}Xg(Y,Z) &= c^2\bar{X}g_B(\bar{Y},\bar{Z})\circ\pi\\ &= c^2[g_B({}^B\nabla_{\bar{X}}\bar{Y},\bar{Z}) + g_B(\bar{Y},{}^B\bar{\nabla}^*_{\bar{X}}\bar{Z}) - \tau(\bar{X})g_B(\bar{Y},\bar{Z})]\circ\pi\\ &= c^2[g_B(\pi_*(\nabla_X Y),\pi_*(Z))\circ\pi + g_B(\pi_*(Y),\pi_*(\bar{\nabla}^*_X Z))\circ\pi\\ &- \tau(\pi_*(X))g_B(\pi_*(Y),\pi_*(Z))\circ\pi]\\ &= g(\nabla_X Y,Z) + g(Y,\bar{\nabla}^*_X Z) - \tau(X)g(Y,Z).\end{aligned} \tag{9}$$

On the other hand, let $\bar{U},\bar{V},\bar{W} \in \chi(F)$ and $U,V,W \in \mathscr{L}_{\mathscr{V}}(F)$ be their corresponding horizontal lifts respectively. Then we obtain,

$$\begin{aligned}Ug(V,W) &= w^2\bar{U}g_F(\bar{V},\bar{W})\circ\sigma\\ &= w^2[g_F({}^F\nabla_{\bar{U}}\bar{V},\bar{W}) + g_F(\bar{V},{}^F\bar{\nabla}^*_{\bar{U}}\bar{W}) - \tau(\bar{U})g(\bar{V},\bar{W})]\circ\sigma\\ &= w^2[g_F(\sigma_*(\nabla_U V),\sigma_*(W))\circ\sigma + g_F(\sigma_*(V),\sigma_*(\bar{\nabla}^*_U W))\circ\sigma\\ &- \tau(\sigma_*(U))g_F(\sigma_*(V),\sigma_*(W))\circ\sigma]\\ &= g(\nabla_U V,W) + g(V,\bar{\nabla}^*_U W) - \tau(U)g(V,W).\end{aligned} \tag{10}$$

Terefore, the proof is complete.

Theorem 1. Let $(M = B\times_{(c,w)} F, g)$ be a base conformal warped product manifold, ∇ an affine connection on M and ∇^* the standard conjuge connection of ∇ with respect to g. Assume that an affine connection ∇' is projectively equivalent to ∇ by τ. Then the generalized conjugate connection $\bar{\nabla'}^*$ of ∇' by τ is dual-projectively equivalent to ∇^* by τ with respect to g.

Proof. Let ∇' be an affine connection on M, for $X,Y,Z \in \chi(M)$, from (2), we have

$$Xg(Y,Z) = g(\nabla'_X Y,Z) + g(Y,\bar{\nabla'}^*_X Z) - \tau(X)g(Y,Z).$$

Here using (3), we obtain

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) + \tau(Y)g(X,Z).$$

Using (6) in the last equation, we have

$$g(Y, \overline{\nabla'}^*_X Z) = g(Y, \nabla^*_X Z) + \tau(Y)g(X,Z). \tag{11}$$

Then, for all $Y \in \chi(M)$, the proof is complete.

Corollary 1. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold, ∇ an affine connection on M and ∇^* the standard conjuge connection of ∇ with respect to g. Assume that an affine connection ∇' is projectively equivalent to ∇ by τ. If the generalized conjugate connection $\overline{\nabla'}^*$ of ∇' by τ is dual-projectively equivalent to ∇^* by τ with respect to g, then the generalized conjugate connections ${}^B\overline{\nabla'}^*$ of ${}^B\nabla'$ and ${}^F\overline{\nabla'}^*$ of ${}^F\nabla'$ are dual-projectively equivalent to ${}^B\overline{\nabla}^*$ on B and ${}^F\overline{\nabla}^*$ on F, respectively.

Proof. For $X,Y,Z \in \chi(M)$, from (11), we obtain

$$\begin{aligned}(c\circ\pi)^2 g_B(\pi_*(Y), \pi_*(\overline{\nabla'}^*_X Z))\circ\pi = (c\circ\pi)^2 g_B(\pi_*(Y), \pi_*(\nabla^*_X Z))\circ\pi \\ + \tau(\pi_*(Y))(c\circ\pi)^2 g_B(\pi_*(X), \pi_*(Z))\circ\pi.\end{aligned}$$

Hence, for $\overline{X}, \overline{Y}, \overline{Z} \in \chi(B)$, we have

$$g_B(\overline{Y}, \overline{\nabla'}^*_{\overline{X}} \overline{Z}) = g_B(\overline{Y}, \nabla^*_{\overline{X}} \overline{Z}) + \tau(\overline{Y}) g_B(\overline{X}, \overline{Z}).$$

Then, for all $\overline{Y}$, we obtain $\overline{\nabla'}^*_{\overline{X}} \overline{Z} = \nabla^*_{\overline{X}} \overline{Z} + g_B(\overline{X}, \overline{Z})\rho$, where ρ is a metrical dual vector field on M. Besides, we can give a similar result for the fiber manifolds on the base conformal warped product manifold as above. Therefore, the proof is complete.

4 Base conformal warped product manifolds with product conjugate connections

In this section, we investigate base conformal warped product with product conjugate connections and give some results.

Let (M,g) be an paracompact and differentiability semi-Riemannian manifold and D be a distribution on M. If D' is a complementary distribution to D in TM, then TM has the decomposition

$$TM = D \oplus D'.$$

Now, denote by h and v the projection morphisms of TM on D and D' respectively. Then we have

$$h^2 = h, \; v^2 = v, \; hv = vh = 0, \; h + v = I \tag{12}$$

where I is the identity morphism on TM. Now, we define the tensor field E of type $(1,1)$ by

$$E = h - v. \tag{13}$$

It follows that E is an *almost product structure* on M, that is, E satisfies $E^2 = I$. For this reason we call (M, D, D') an *almost product manifold*. Next, from (12) and (13), we deduce that

$$h = \frac{1}{2}(I + E) \; and \; v = \frac{1}{2}(I - E). \tag{14}$$

We say that E is *parallel* with respect to a linear connection ∇ on M if its covariant derivative with respect to ∇ vanishes, i.e., for $\forall X,Y \in \Gamma(TM)$, we have

$$(\nabla_X E)Y = \nabla_X EY - E(\nabla_X Y) = 0. \tag{15}$$

On the other than, let D be an n-distribution on an $(n+p)$- dimensional manifold M. A linear connection ∇ on M is said to be adapted to D if

$$\nabla_X U \in \Gamma(D), \tag{16}$$

where $\forall X \in \Gamma(TM), U \in \Gamma(D)$ [2].

Now, we can give the relation between the product conjugate connection of base conformal warped product and the affine connections of its base and fiber manifolds by following proposition:

Proposition 5. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold with an almost product structure E and let ∇, ${}^B\nabla$ and ${}^F\nabla$ be affine connections on M, B and F respectively, then we have

(i) $\nabla^{(E)}_{\overline{X}}\overline{Y} = h({}^B\nabla_{\overline{X}} h\overline{Y})$,
(ii) $\nabla^{(E)}_{\overline{U}}\overline{V} = v({}^F\nabla_{\overline{U}} v\overline{V})$,

where $\overline{X},\overline{Y} \in \chi(B)$, $\overline{U},\overline{V} \in \chi(F)$ and $E = h - v$ is an almost product structure on M.

*Proof.*Suppose that $(M = B \times_{(c,w)} F, g)$ is a base conformal warped product manifold with an almost product structure E and $\nabla^{(E)}$ is a product conjugate connection of ∇.

Case (1): For $\overline{X},\overline{Y},\overline{Z} \in \chi(B)$ and $X,Y,Z \in \mathscr{L}_{\mathscr{H}}(B)$, we have

$$\begin{aligned} g_B(\nabla^{(E)}_{\overline{X}}\overline{Y},\overline{Z}) \circ \pi &= (c \circ \pi)^{-2} g(\nabla^{(E)}_X Y, Z) \\ &= (c \circ \pi)^{-2} g(E(\nabla^{(E)}_X EY), Z) \\ &= g_B(\pi_*(E(\nabla^{(E)}_X EY)), \pi_*(Z)) \circ \pi. \end{aligned}$$

Here, using $E = h - v$, we obtain $\nabla^{(E)}_{\overline{X}}\overline{Y} = h({}^B\nabla_{\overline{X}} h\overline{Y})$.

Case (2): For $\overline{U},\overline{V},\overline{W} \in \chi(F)$ and $U,V,W \in \mathscr{L}_{\mathscr{V}}(F)$, we have

$$\begin{aligned} g_F(\nabla^{(E)}_{\overline{U}}\overline{V},\overline{W}) \circ \sigma &= (w \circ \pi)^{-2} g(\nabla^{(E)}_U V, W) \\ &= (w \circ \pi)^{-2} g(E(\nabla^{(E)}_U EV), W) \\ &= g_F(\sigma_*(E(\nabla^{(E)}_U EV)), \sigma_*(W)) \circ \sigma. \end{aligned}$$

Using $E = h - v$ in the above equation, we obtain $\nabla^{(E)}_{\overline{U}}\overline{V} = v({}^F\nabla_{\overline{U}} v\overline{V})$. Therefore, the proof is complete.

Then we can give the product conjugate connection $\nabla^{(E)}$ as follows, if the affine connection ∇ is a Levi-Civita connection.

Lemma 1. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold with an almost product structure E and $\nabla^{(E)}$ be a product conjugate connection of a affine connection ∇ on M. For $X,Y \in \Gamma(TB)$ and $U,V \in \Gamma(TF)$, if ∇ is a Levi-Civita connection, then we have

(i) $\nabla^{(E)}_X Y = h({}^B\nabla_X hY) + \frac{X(c)}{c} hY + \frac{Y(c)}{c} hX - \frac{g_B(X,hY)}{c} h({}^B\nabla c)$,
(ii) $\nabla^{(E)}_U X = -\frac{hX(w)}{w} vU$,

(iii) $\nabla^{(E)}_U V = v({}^F\nabla_U vV) - \frac{w}{c^2} g_F(U, vV) v({}^B\nabla w)$,

where ${}^B\nabla$ and ${}^F\nabla$ are Levi-Civita connections of B and F, respectively.

Proof. Since ∇ is affine connection we have $\nabla^{(E)}_X Y = E(\nabla_X EY)$ and $E = h - v$. In particular, if ∇ is a Levi-Civita connection, then case (1)-(3) are hold.

Theorem 2. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold with an almost product structure E and $\nabla^{(E)}$ be a product conjugate connection of a affine connection ∇ on M. If $\nabla^{(E)}$ is torsion-free, then the distributions D_h and D_v of B and F, respectively are involutive distributions.

Proof. For $X, Y \in \Gamma(TM)$, we have

$$E[X,Y] = h[X,Y] - v[X,Y] = \nabla^{(E)}_X Y - \nabla^{(E)}_Y X. \tag{17}$$

On the other hand, the product conjugate connection of ∇ is

$$\nabla^{(E)}_X Y = h(\nabla_X hY - \nabla_X vY) - v(\nabla_X hY - \nabla_X vY). \tag{18}$$

Thus, using (18) in (17), we obtain $h[X,Y] = h(\nabla_X hY) - h(\nabla_X vY)$ and $v[X,Y] = v(\nabla_Y vX) - v(\nabla_X vY)$. In the first equation, taking $X \to vX$ and $Y \to vY$, we have $h[vX, vY] = 0$ and in the second equation, taking $X \to hX$ and $Y \to hY$, we have $v[hX, hY] = 0$. Then, the proof is complete.

Theorem 3. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold with an almost product structure E and $\nabla^{(E)}$ be a product conjugate connection of a affine connection ∇ on M. If ∇ adapted to B and F, also projections h and v are parallel with respect to affine connections ${}^B\nabla$ and ${}^F\nabla$ respectively, then E is parallel with respect to product conjugate connection $\nabla^{(E)}$.

Proof. For $X, Y \in \Gamma(TM)$, we have

$$(\nabla^{(E)}_X E)Y = \nabla^{(E)}_X EY - E(\nabla^{(E)}_X Y). \tag{19}$$

Hence, using $\nabla^{(E)}_X EY = E(\nabla_X Y)$ and $E(\nabla^{(E)}_X Y) = \nabla_X EY$ (see [4]) in (19), we obtain $(\nabla^{(E)}_X E)Y = -(\nabla_X E)Y$. Here, taking $E = h - v$ and using ∇ adapted to B and F, we have $(\nabla^{(E)}_X E)Y = ({}^F\nabla_X v)Y - ({}^B\nabla_X h)Y$. Then, the proof is complete.

Proposition 6. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold with an almost product structure E and $\nabla^{(E)}$ be a product conjugate connection of a affine connection ∇ on M. Suppose that $\bar{M}$ is a submanifold of $M = B \times_{(c,w)} F$. If $\bar{M}$ is invariant (E-invariant) and ∇ adapted to $\bar{M}$, then $\nabla^{(E)}$ also adapted to $\bar{M}$.

Proof. Suppose that $\bar{M}$ is an invariant submanifold of M then, for $X \in \Gamma(T\bar{M})$, we have $EX \in \Gamma(T\bar{M})$. Also, since the affine connection ∇ adapted to $\bar{M}$, for any $Y \in \Gamma(TM)$, $\nabla_Y X \in \Gamma(T\bar{M})$. Thus, from the definition of product conjugate connection $\nabla^{(E)}$, we have $\nabla^{(E)}_Y X = E(\nabla_Y EX) \in \Gamma(T\bar{M})$. Therefore, the proof is complete.

Corollary 2. Let $(M = B \times_{(c,w)} F, g)$ be a base conformal warped product manifold with an almost product structure E and $\nabla^{(E)}$ be a product conjugate connection of a affine connection ∇ on M. If $\bar{M} \subset M$ is invariant and ∇ adapted to $\bar{M}$, then $\bar{M}$ is ∇-geodesically invariant for $\nabla^{(E)}$.

Proof. The proof is obvious from the definition of ∇-geodesically invariant.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Aghasi M., Dodson C.T.J, Galanis G.N. and Suri A., Conjugate connections and differential equations on indefinite dimensional manifolds, Manchester Institute for Mathematical Sciences School of Mathematics,The University of Manchester. 2006.

[2] Bejancu A. and Farran H.R., Foliations and Geometric Structures, Springer, 2006.

[3] Blaga A.M. and Crasmareanu M., The geometry of complex conjugate connections, Hacettepe Journal of Mathematics and Statistics. 2012; 41(1), 119-126.

[4] Blaga A.M. and Crasmareanu M., The geometry of product conjugate connections, ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.) MATEMATICĂ, Tomul LIX, 2013, f.1, DOI: 10.2478/v10157-012-0026-7.

[5] Blaga A.M. and Crasmareanu M., The geometry of tangent conjugate connections, Hacettepe Journal of Mathematics and Statistics. 2015; 44(4).

[6] Diallo A.S, Dualistic Structures on Doubly Warped Product Manifolds, International Electronic Journal of Geometry, 2013; 6(1), 41-45.

[7] Clain O., Matsuzoe H. and Zhang J., Generalizations of conjugate connections, Proceedings of 9 th International Workshop on Complex Structures, Integrability, and Vector Fields. 2009; 26-34.

[8] Diallo A.S. and Todjihounde L., Dualistic Structures on Twisted Product Manifolds, Global Journal of Advanced Research on Classical and Modern Geometries, 2015; 4(1), 35-43.

[9] O'Neill, B., Semi-Riemannian geometry, Academic Press, New-York, 1983.

[10] Todjihounde L, Dualistic Structures on Warped Product Manifolds, Differential Geometry-Dynamical Systems, Balkan Society of Geometers, Geometry Balkan Press, 2006; 8, 278-284.

[11] Ünal B. and Dobarro F., Curvature in Special Base Conformal Warped Products, Acta Appl Math., 2008; 104, 1-46, DOI:10.1007/s10440-008-9239-x.