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A new type of generalized quasi-Einstein manifold

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Abstract: In this paper, a new type of generalized quasi-Einstein manifold is defined. The special cases of this manifold are Einstein manifold, quasi-Einstein manifold and nearly quasi-Einstein manifold. We have shown the existence of this new type of generalised quasi-Einstein manifold by a suitable example.

Keywords: Einstein manifold, quasi-Einstein manifold, nearly quasi-Einstein manifold, conformal curvature tensor, concircular curvature tensor, Einstein field equation.

1 Introduction

Let (M^n, g), $(n > 2)$, be an n-dimensional Riemannian manifold. A Riemannian manifold is said to be an Einstein manifold if a non-zero Ricci tensor of the manifold satisfies relation

$$R_{ij} = \frac{R}{n} g_{ij}, \tag{1}$$

where R_{ij} , R and g_{ij} are Ricci tensor of type (0, 2), scalar curvature and Riemannian metric respectively.

If a non-zero Ricci tensor of the manifold satisfies relation

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j, \tag{2}$$

then, the manifold is called a quasi-Einstein manifold, where A_i is a unit covariant vector on $U = \left\{x \in M : R_{ij} \neq \frac{R}{n} g_{ij}\right\}$ and α, β are scalars on U. Generally an n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$. The quasi-Einstein manifold is also studied by U.C.De and G.C.Ghosh[13], *C.Özgür* and S. Sular[6] and A. A. Shaikh , D. W. Yoon and S. K. Hui[1] and [2,3,8,9]. According to U. C. De and A. K. Gazi [12], a manifold is said to be nearly quasi-Einstein manifold, if the non-zero Ricci tensor of the manifold satisfies the relation

$$R_{ij} = \alpha g_{ij} + \beta E_{ij}, \tag{3}$$

where E_{ij} is a symmetric tensor of type (0,2).

In 1969 K. Yano [10] define a new type of curvature tensor by combining conformal curvature tensor and concircular curvature tensor. M. C. Chaki and M. L. Ghosh[11] also combined conformal curvature and concircular curvature tensor and gave an experession for a quasi-conformal curvature tensor W of type (1,3) by

$$W^h_{ijk} = -(n-2)bC^h_{ijk} + [a + (n-2)b]L^h_{ijk}, \tag{4}$$

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where a, b are arbitrary constants, not simultaneously zero and C^h_{ijk} , L^h_{ijk} are conformal and concircular curvature tensor respectively.

From (4) we can say that the quasi-conformal curvature tensor will be equal to conformal curvature tensor or concircular curvature tensor according as $a = 1$ and $b = -\frac{1}{n-2}$ or $a = 1$ and $b = 0$ respectively.

We know that the conformal and concircular curvature tensors are defined by

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(R_{ij}\,\delta^h_k - R_{ik}\,\delta^h_j + R^h_k\,g_{ji} - R^h_j\,g_{ki}) + \frac{R}{(n-1)(n-2)}(g_{ij}\,\delta^h_k - \delta^h_j\,g_{ki}), \tag{5}$$

and

$$L^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}(g_{ij}\,\delta^h_k - \delta^h_j\,g_{ki}). \tag{6}$$

Putting the values of conformal and concircular curvature tensors from (5) and (6) in (4) we have an expression for quasi-conformal curvature tensor W of type (1,3), given by

$$W^h_{ijk} = aR^h_{ijk} + b(R_{ij}\,\delta^h_k - R_{ik}\,\delta^h_j + R^h_k\,g_{ji} - R^h_j\,g_{ki}) - \frac{R[\frac{a}{n-1} + 2b]}{n}(g_{ij}\,\delta^h_k - \delta^h_j\,g_{ki}). \tag{7}$$

2 A new type of generalized quasi-Einstein manifold

Now we define a manifold and called it new type of generalized quasi-Einstein manifold in which a non-zero Ricci tensor satisfies a different type of relation. Generalized in sense that special cases of this manifold are an Einstein manifold, quasi-Einstein manifold and nearly quasi-Einstein manifold.

Definition 1. *Let (M^n, g), $(n > 2)$, be a Riemannian manifold. If the non-zero Ricci tensor of the manifold satisfies the relation*

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + [\alpha - (n-2)\gamma]E_{ij}, \tag{8}$$

then, the manifold is called a new type of generalized quasi-Einstein manifold, where R_{ij}, E_{ij} and α, β , γ are Ricci tensor, symmetric tensor of type (0, 2) and scalars respectively.

Transvecting (8) by g^{ij}, we have

$$R = \alpha n + \beta + [\alpha - (n-2)\gamma]E, \tag{9}$$

where $E = E_{ij}g^{ij}$ and $A^j A_j = 1$ $(A^j = A_i g^{ij})$. From (8) three cases arise.

Case i. if $\beta = 0$ and $\alpha = (n-2)\gamma$ then from (8), we get

$$R_{ij} = (n-2)\gamma g_{ij}, \tag{10}$$

then, manifold becomes Einstein manifold.

Case ii. if $\alpha = (n-2)\gamma$ then from (8), we get

$$R_{ij} = (n-2)\gamma g_{ij} + \beta A_i A_j, \tag{11}$$

then, manifold becomes a quasi-Einstein manifold.

Case iii. if $\beta = 0$ and $\alpha \neq (n-2)\gamma$ then from (8), we get

$$R_{ij} = \alpha g_{ij} + [\alpha - (n-2)\gamma]E_{ij}, \tag{12}$$

then, manifold becomes a nearly quasi-Einstein manifold.

Now using (8) and (9) in (7), we obtain an expression for conformal curvature tensor in a new type of generalized quasi-Einstein manifold, given by

$$\begin{aligned} W^h_{ijk} =& aR^h_{ijk} + b\beta[\delta^h_k A_i A_j - \delta^h_j A_i A_k + A_k A^h g_{ij} - A_j A^h g_{ik}] + b[\alpha - (n-2)\gamma](\delta^h_k E_{ij} - \delta^h_j E_{ik} + E^h_k g_{ji} - E^h_j g_{ki}) \\ &- [(\alpha - (n-2)\gamma)(\frac{a+2(n-1)b}{n(n-1)})E + \frac{\beta A}{n}(\frac{a}{n-1} + 2b) + \frac{\alpha a}{(n-1)}](g_{ij}\delta^h_k - \delta^h_j g_{ki}). \end{aligned} \tag{13}$$

Now, we consider a quasi-conformally flat new type of generalized quasi-Einstein manifold i.e. $W^h_{ijk} = 0$, then from (13), we get

$$\begin{aligned} R^h_{ijk} =& -\frac{b\beta}{a}[\delta^h_k A_i A_j - \delta^h_j A_i A_k + A_k A^h g_{ij} - A_j A^h g_{ik}] - \frac{b}{a}[\alpha - (n-2)\gamma](\delta^h_k E_{ij} - \delta^h_j E_{ik} + E^h_k g_{ji} - E^h_j g_{ki}) \\ &+ \frac{1}{a}[(\alpha - (n-2)\gamma)(\frac{a+2(n-1)b}{n(n-1)})E + A\frac{\beta}{n}(\frac{a}{n-1} + 2b) + \frac{\alpha a}{(n-1)}](g_{ij}\delta^h_k - \delta^h_j g_{ki}). \end{aligned} \tag{14}$$

Equation (14) can be written as

$$\begin{aligned} R^h_{ijk} =& P[\delta^h_k A_i A_j - \delta^h_j A_i A_k + A_k A^h g_{ij} - A_j A^h g_{ik}] + Q(\delta^h_k E_{ij} - \delta^h_j E_{ik} \\ &+ E^h_k g_{ji} - E^h_j g_{ki}) + (SE + KA + U)(g_{ij}\delta^h_k - \delta^h_j g_{ki}), \end{aligned} \tag{15}$$

where $P = -\frac{b\beta}{a}$, $Q = -\frac{b}{a}[\alpha - (n-2)\gamma]$, $S = \frac{1}{a}[\alpha - (n-2)\gamma](\frac{a+2(n-1)b}{n(n-1)})$, $K = \frac{\beta}{an}[\frac{a}{n-1} + 2b]$, $a \neq 0$ and $U = \frac{\alpha}{(n-1)}$. Now, if we take

$$Q(\delta^h_k E_{ij} - \delta^h_j E_{ik} + E^h_k g_{ji} - E^h_j g_{ki}) + (SE + KA)(g_{ij}\delta^h_k - \delta^h_j g_{ki}) = 0, \tag{16}$$

then from (16) and (15), we obtain

$$R^h_{ijk} = P[\delta^h_k A_i A_j - \delta^h_j A_i A_k + A_k A^h g_{ij} - A_j A^h g_{ik}] + U(g_{ij}\delta^h_k - \delta^h_j g_{ki}). \tag{17}$$

In 1972 Chen and Yano [7] gave the concept of a manifold of quasi-constant curvature tensor and define.

Definition 2. *A Riemannian manifold* (M^n, g), $(n > 3)$ *is said to be a manifold of quasi-constant curvature if it is conformally flat and its curvature tensor* R^h_{ijk} *of type (1,3) have the form*

$$R^h_{ijk} = U[\delta^h_k A_i A_j - \delta^h_j A_i A_k + A_k A^h g_{ij} - A_j A^h g_{ik}] + V(g_{ij}\delta^h_k - \delta^h_j g_{ki}), \tag{18}$$

where A_i *is a covariant vector and U, V are scalars of which* $V \neq 0$.

Thus, from (15), (16) and (17) we conclude that

Theorem 1. *A quasi-conformally flat new type of generalized quasi-Einstein manifold will be a manifold of quasi-constant curvature if and only if*

$$\delta^h_k E_{ij} - \delta^h_j E_{ik} + E^h_k g_{ji} - E^h_j g_{ki} + (SE + KA)(g_{ij}\delta^h_k - \delta^h_j g_{ki}) = 0.$$

Now, we propose.

Corollary 1. *If a quasi-conformally flat new type of generalized quasi-Einstein manifold be a manifold of quasi-constant curvature then the symmetric tensor* E_{ij} *satisfies the relation*

$$E_{ij} = -\frac{K}{2Q+nS} A_i A_j, \tag{19}$$

Proof. From Theorem 1. it is clear that if a quasi-conformally flat new type of generalized quasi-Einstein manifold be a manifold of quasi-constant curvature then

$$Q(\delta_k^h E_{ij} - \delta_j^h E_{ik} + E_k^h g_{ji} - E_j^h g_{ki}) + (SE+KA)(g_{ij}\delta_k^h - \delta_j^h g_{ki}) = 0.$$

Contracting in h and k, we get

$$(n-1)(2Q+nS)E_{ij} + (n-1)KA_iA_j = 0, \tag{20}$$

equation (20)which implies that

$$E_{ij} = -\frac{K}{2Q+nS} A_i A_j, \tag{21}$$

Contracting (17) in h and k, we get

$$R_{ij} = P[nA_iA_j - A_iA_j + A_iA^i g_{ij} - A_jA_i] + U(ng_{ij} - g_{ji}), \tag{22}$$

equation (22)which implies that

$$R_{ij} = (n-1)Ug_{ij} + P(n-1)A_iA_j. \tag{23}$$

Taking $\nu = [(n-1)U]$ and $\mu = [P(n-1)]$, in (23), we get

$$R_{ij} = \nu g_{ij} + \mu A_i A_j, \tag{24}$$

This is a quasi-Einstein manifold. Thus we conclude that.

Theorem 2. *If a quasi-conformally flat new type of generalized quasi-Einstein manifold be a manifold of quasi-constant curvature then this becomes a quasi-Einstein manifold.*

3 Einstein field equation in a new type of generalized quasi-Einstein manifold

The Einstein field equation with a cosmological term is given by [8]

$$R_{ij} - R\frac{1}{2}g_{ij} + \wedge g_{ij} = kT_{ij}, \tag{25}$$

where $\wedge$, k and T_{ij} are cosmological constant , Gravitational constant and energy momentum tensor respectively.
Using (8) and (9) in (25), we have

$$\alpha g_{ij} + \beta A_iA_j + [\alpha-(n-2)\gamma]E_{ij} - \frac{1}{2}(\alpha n + \beta A + [\alpha-(n-2)\gamma]E)g_{ij} + \wedge g_{ij} = kT_{ij}, \tag{26}$$

equation (26) which implise that

$$(\alpha(1-\frac{n}{2}) + \wedge)g_{ij} + \beta\frac{1}{2}A_iA_j + (1-\frac{n}{2})[\alpha-(n-2)\gamma]E_{ij} = kT_{ij}, \tag{27}$$

which is required Einstein field equation in a new type of generalized quasi-Einstein manifold. Taking covariant derivative of (27) and suppose $\nabla_j A_i = 0$, we have

$$(\alpha(1-\frac{n}{2})+\wedge)\nabla_h g_{ij}+(1-\frac{n}{2})[\alpha-(n-2)\gamma]\nabla_h E_{ij}=k\nabla_h T_{ij}, \tag{28}$$

equation (28)which implies that

$$(1-\frac{n}{2})[\alpha-(n-2)\gamma]\nabla_h E_{ij}=k\nabla_h T_{ij}. \tag{29}$$

Thus, we conclude.

Theorem 3. *In a new type of generalized quasi-Einstein manifold, if A_i be covariant constant then*

(i) *Symmetric tensor E_{ij} is covariant constant if the energy momentum tensor is covariant constant,*

(ii) *Symmetric tensor E_{ij} is recurrent if the energy momentum tensor is recurrent.*

Taking covariant derivative of (27) and A_i be a covariant constant, we get

$$(\alpha(1-\frac{n}{2})+\wedge)g_{ij,k}+(1-\frac{n}{2})[\alpha-(n-2)\gamma]E_{ij,k}=kT_{ij,k}, \tag{30}$$

Interchanging i, j and k in cyclic order in (30), we have

$$(\alpha(1-\frac{n}{2})+\wedge)g_{jk,i}+(1-\frac{n}{2})[\alpha-(n-2)\gamma]E_{jk,i}=kT_{jk,i}, \tag{31}$$

and

$$(\alpha(1-\frac{n}{2})+\wedge)g_{ki,j}+(1-\frac{n}{2})[\alpha-(n-2)\gamma]E_{ki,j}=kT_{ki,j}. \tag{32}$$

Adding (30), (31) and (32), we get

$$(1-\frac{n}{2})[\alpha-(n-2)\gamma]\left(E_{ij,k}+E_{jk,i}+E_{ki,j}\right)=k\left(T_{ij,k}+T_{jk,i}+T_{ki,j}\right). \tag{33}$$

Now, if symmetric tensor E_{ij} satisfies the Bianchi second identity, then

$$E_{ij,k}+E_{jk,i}+E_{ki,j}=0, \tag{34}$$

therefore from (33), we get

$$T_{ij,k}+T_{jk,i}+T_{ki,j}=0. \tag{35}$$

i.e. the energy momentum tensor satisfies the Bianchi second identity. Thus, we conclude that.

Theorem 4. *In a new type of generalized quasi-Einstein manifold if A_i be covariant constant then the symmetric tensor E_{ij} satisfies the Bianchi second identity if and only if energy momentum tensor satisfies the Bianchi second identity.*

4 An example of a new type of generalized quasi-Einstein manifold

Now, we take a manifold (M, g) such that $M = R^4$ and the metric g in R^4 is given by

$$ds^2 = g_{ij}dx^i dx^j = f(x^4)[(dx^1)^2+(dx^2)^2+(dx^3)^2]+(dx^4)^2, \tag{36}$$

the only non-vanishing components of christoffel symbols and the curvature tensor are given by

$$\Gamma^4_{11} = \Gamma^4_{33} = \Gamma^4_{22} = -\frac{1}{2} f'(x^4),\ \Gamma^1_{14} = \Gamma^3_{34} = \Gamma^2_{24} = \frac{1}{2}\Big(\frac{f'(x^4)}{f(x^4)}\Big), \tag{37}$$

and

$$R_{1441} = R_{2442} = R_{4334} = \frac{1}{2} f''(x^4) - \frac{1}{4}\frac{\big(f'(x^4)\big)^2}{f(x^4)}, R_{2112} = R_{3113} = R_{2332} = \frac{1}{4}\big(f'(x^4)\big)^2. \tag{38}$$

The only non-zero Rici tensors are given by

$$R_{11} = R_{22} = R_{33} = -\frac{1}{2} f''(x^4) - \frac{1}{2}\frac{(f'(x^4))^2}{f(x^4)},\quad R_{44} = -\frac{3f''(x^4)}{2f(x^4)} + \frac{3}{4}\Big(\frac{f'(x^4)}{f(x^4)}\Big)^2, \tag{39}$$

assuming

$$\alpha = -\frac{f''(x^4)}{2f(x^4)} - \frac{1}{2}\Big(\frac{f'(x^4)}{f(x^4)}\Big)^2, \beta = \frac{3}{2f(x^4)}\left\{\frac{f''(x^4)}{f(x^4)} - \Big(\frac{f'(x^4)}{f(x^4)}\Big)^2\right\}, \gamma = -\Big(\frac{f'(x^4)}{f(x^4)}\Big)^2 + \frac{1}{2}\frac{f''(x^4)}{f(x^4)},$$

$$A_i = \begin{cases} \sqrt{f(x^4)}, & i = 1 \\ 0, & i = 2,3,4, \end{cases}$$

and

$$E_{ij} = \begin{cases} 1, & i = j = 1 \\ 0, & i \neq j \text{ and } i = j = 2,3 \\ \frac{1}{6}\Big(\frac{4f''(x^4)f(x^4) - 5f'(x^4)^2}{f''(x^4)f(x^4) - f'(x^4)^2}\Big), & i = j = 4. \end{cases}$$

Now, using above value, we have

$$\begin{aligned} \alpha g_{11} + \beta A_1 A_1 + (\alpha - 2\gamma)E_{11} &= -\frac{1}{2} f''(x^4) - \frac{1}{2}\frac{(f'(x^4))^2}{f(x^4)}, \\ \alpha g_{22} + \beta A_2 A_2 + (\alpha - 2\gamma)E_{22} &= -\frac{1}{2} f''(x^4) - \frac{1}{2}\frac{(f'(x^4))^2}{f(x^4)}, \\ \alpha g_{33} + \beta A_3 A_3 + (\alpha - 2\gamma)E_{33} &= -\frac{1}{2} f''(x^4) - \frac{1}{2}\frac{(f'(x^4))^2}{f(x^4)}, \\ \alpha g_{44} + \beta A_4 A_4 + (\alpha - 2\gamma)E_{44} &= -\frac{3f''(x^4)}{2f(x^4)} + \frac{3}{4}\Big(\frac{f'(x^4)}{f(x^4)}\Big)^2. \end{aligned} \tag{40}$$

Now, from equation (39) and (40), we have

(1) $R_{11} = \alpha g_{11} + \beta A_1 A_1 + (\alpha - 2\gamma)E_{11}$,
(2) $R_{22} = \alpha g_{22} + \beta A_2 A_2 + (\alpha - 2\gamma)E_{22}$,
(3) $R_{33} = \alpha g_{33} + \beta A_3 A_3 + (\alpha - 2\gamma)E_{33}$,
(4) $R_{44} = \alpha g_{44} + \beta A_4 A_4 + (\alpha - 2\gamma)E_{44}$.

We shall now show that the 1-forms are unit

$$g^{ij}A_iA_j = g^{11}A_1A_1 + g^{22}A_2A_2 + g^{33}A_3A_3 + g^{44}A_4A_4 = 1, \tag{41}$$

this shows that (R^4, g) is a new type of generalized quasi-Einstein manifold.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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