New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.280

On $\alpha_{(\gamma,\gamma')}$-open sets in topological spaces

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Received: 12 January 2018, Accepted: 18 March 2018
Published online: 26 April 2018.

Abstract: The purpose of this paper is to introduce the concept of $\alpha_{(\gamma,\gamma')}$-open sets in topological spaces and study some of their properties. It also studies the class of $\alpha_{(\gamma,\gamma')}$-generalized closed set in topological spaces and some of their properties.

Keywords: Bioperation, α-open set, $\alpha_{(\gamma,\gamma')}$-open set, $\alpha_{(\gamma,\gamma')}$-g.closed sets.

1 Introduction

In 1965, Njastad [4] initiated and explored a new class of generalized open sets in a topological space called α-open sets. In 1979, S. Kasahara [3] defined the concept of an operation on topological spaces and introduced α-closed graphs of an operation. H. Ogata [5] called the operation α as γ operation and introduced the notion of γ-open sets. H. Z. Ibrahim [1] introduced and discussed an operation of a topology $\alpha O(X)$ into the power set $P(X)$ of a space X and also he introduced the concept of α_γ-open sets. In 1992, J. Umehara, H. Maki and T. Noir [6] defined and discussed the properties of (γ,γ')-open sets. A. B. Khalaf, S. Jafari and H. Z. Ibrahim [2] introduced the notion of $\alpha O(X,\tau)_{[\gamma,\gamma']}$, which is the collection of all $\alpha_{[\gamma,\gamma']}$-open sets in a topological space (X,τ). In this paper, the author introduce and study the notion of $\alpha O(X,\tau)_{(\gamma,\gamma')}$ which is the collection of all $\alpha_{(\gamma,\gamma')}$-open by using operations γ and γ' on a topological space $\alpha O(X,\tau)$.

2 Preliminaries

Throughout this paper, (X,τ) represent nonempty topological space on which no separation axioms are assumed, unless otherwise mentioned. The closure and the interior of a subset A of X are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X,τ) is said to be α-open [4] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open (resp. α-closed) sets in a topological space (X,τ) is denoted by $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$). An operation γ [3] on a topology τ is a mapping from τ into power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. An operation $\gamma: \alpha O(X,\tau) \to P(X)$ [1] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X,\tau)$. We call the mapping γ an operation on $\alpha O(X,\tau)$. A subset A of X is called an α_γ-open set [1] if for each point $x \in A$, there exists an α-open set U of X containing x such that $U^\gamma \subseteq A$. The complement of an α_γ-open set is called α_γ-closed. The set of all α_γ-open sets of X is denote by $\alpha O(X,\tau)_\gamma$. The intersection of all α_γ-closed sets containing A is called the α_γ-closure of A and denoted by $\alpha_\gamma Cl(A)$. A point $x \in \alpha Cl_\gamma(A)$ iff $U^\gamma \cap A \neq \phi$ for each α-open set U containing x. An operation γ on $\alpha O(X,\tau)$ is said to be α-regular [1] if for every α-open sets U and V containing $x \in X$, there exists an α-open set W containing x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. A space X is said to be α_γ-regular

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[2] if for each $x \in X$ and for each α-open set V in X containing x, there exists an α-open set U in X containing x such that $U^{\gamma} \subseteq V$. A subset A of X is said to be $\alpha_{[\gamma,\gamma']}$-open [2] if for each $x \in A$, there exist α-open sets U and V of X containing x such that $U^{\gamma} \cap V^{\gamma'} \subseteq A$. A non-empty subset A of X with an operation γ on τ is called (γ, γ')-open [6] if for each $x \in A$, there exist open sets U and V of X containing x such that $U^{\gamma} \cup V^{\gamma'} \subseteq A$.

3 $\alpha_{(\gamma,\gamma')}$-Open Sets

In this section, we define and discuss the properties of $\alpha_{(\gamma,\gamma')}$-open sets.

Definition 1. *A subset A of (X, τ) is said to be $\alpha_{(\gamma,\gamma')}$-open if for each $x \in A$, there exist α-open sets U and V of X containing x such that $U^{\gamma} \cup V^{\gamma'} \subseteq A$. The set of all $\alpha_{(\gamma,\gamma')}$-open sets of (X, τ) is denoted by $\alpha O(X, \tau)_{(\gamma,\gamma')}$.*

Proposition 1. *If A_i is $\alpha_{(\gamma,\gamma')}$-open for every $i \in I$, then $\cup\{A_i : i \in I\}$ is $\alpha_{(\gamma,\gamma')}$-open.*

Proof. Let $x \in \cup_{i\in I} A_i$, then $x \in A_i$ for some $i \in I$. Since A_i is an $\alpha_{(\gamma,\gamma')}$-open set, so there exist α-open sets U and V of X containing x such that $U^{\gamma} \cup V^{\gamma'} \subseteq A_i \subseteq \cup_{i\in I} A_i$. Therefore, $\cup_{i\in I} A_i$ is an $\alpha_{(\gamma,\gamma')}$-open set of (X, τ).

*Remark.*If A and B are two $\alpha_{(\gamma,\gamma')}$-open sets in (X, τ), then the following example shows that $A \cap B$ need not be $\alpha_{(\gamma,\gamma')}$-open.

Example 1. Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = \begin{cases} A\cup, \{2\} & \text{if } 3 \in A, \\ A, & \text{if } 3 \notin A, \end{cases}$$

and

$$A^{\gamma'} = \begin{cases} A, & \text{if } A \neq \{2\}, \\ X, & \text{if } A = \{2\}. \end{cases}$$

Then, it is obvious that the sets $\{1, 2\}$ and $\{2, 3\}$ are $\alpha_{(\gamma,\gamma')}$-open, however their intersection $\{2\}$ is not $\alpha_{(\gamma,\gamma')}$-open.

*Remark.*From the above example, we notice that the family of all $\alpha_{(\gamma,\gamma')}$-open subsets of a space X is a supratopology and need not be a topology in general.

In the following proposition, the intersection of two $\alpha_{(\gamma,\gamma')}$-open sets is also $\alpha_{(\gamma,\gamma')}$-open, under a certain condition.

Proposition 2. *Let γ and γ' be α-regular operations. If A and B are $\alpha_{(\gamma,\gamma')}$-open, then $A \cap B$ is $\alpha_{(\gamma,\gamma')}$-open.*

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since A and B are $\alpha_{(\gamma,\gamma')}$-open sets, there exist α-open sets U, V, W and S containing x such that $U^{\gamma} \cup V^{\gamma'} \subseteq A$ and $W^{\gamma} \cup S^{\gamma'} \subseteq B$. Since γ and γ' are α-regular operations, then there exist α-open sets K and L containing x such that $K^{\gamma} \cup L^{\gamma'} \subseteq (U^{\gamma} \cap W^{\gamma}) \cup (V^{\gamma'} \cap S^{\gamma'}) \subseteq (U^{\gamma} \cup V^{\gamma'}) \cap (W^{\gamma} \cup S^{\gamma'}) \subseteq A \cap B$. This implies that $A \cap B$ is an $\alpha_{(\gamma,\gamma')}$-open set.

Remark. By the above propositon, if γ and γ' are α-regular operations, then $\alpha O(X, \tau)_{(\gamma,\gamma')}$ forms a topology on X.

Proposition 3. *The set A is $\alpha_{(\gamma,\gamma')}$-open in X if and only if for each $x \in A$, there exists an $\alpha_{(\gamma,\gamma')}$-open set B such that $x \in B \subseteq A$.*

Proof. Obvious.

Remark. A subset A is an $\alpha_{(id,id')}$-open set of (X,τ) if and only if A is α-open in (X,τ). The operation $id = id' : \alpha O(X,\tau) \to P(X)$ is defined by $V^{id} = V$ for any set $V \in \alpha O(X,\tau)$. This operation is called the identity operation on $\alpha O(X,\tau)$. Therefore $\alpha O(X,\tau)_{(id,id')} = \alpha O(X,\tau)$.

Remark.[1] A subset A is an α_{id}-open set of (X,τ) if and only if A is α-open in (X,τ). Therefore, we have that $\alpha O(X,\tau)_{id} = \alpha O(X,\tau)$.

Remark. From Remarks 3 and 3, we have $\alpha O(X,\tau)_{(id,id')} = \alpha O(X,\tau) = \alpha O(X,\tau)_{id} = \alpha O(X,\tau)_{id'}$.

Remark. The following example shows that the concept of $\alpha_{(\gamma,\gamma')}$-open and open are independent.

Example 2. Let $X = \{1,2,3\}$ and $\tau = \{\phi, X, \{1\}\}$ be a topology on X. For each $A \in \alpha O(X,\tau)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A, & \text{if } A = \{1,2\}, \\ X, & \text{if } A \neq \{1,2\}, \\ \phi, & \text{if } A = \phi. \end{cases}$$

Then, $\alpha_{(\gamma,\gamma')}$-open sets are ϕ, X, and $\{1,2\}$.

Proposition 4. *Let γ and γ' be operations on $\alpha O(X)$. If A is (γ,γ')-open, then A is $\alpha_{(\gamma,\gamma')}$-open.*

Proof. Obvious.

The converse of the above proposition need not be true in general as it is shown in the following example.

Example 3. Let $X = \{1,2,3\}$ and $\tau = \{\phi, X, \{2\}\}$ be a topology on X. For each $A \in \alpha O(X,\tau)$, we define two operations γ and γ', respectively, by $A^{\gamma} = A^{\gamma'} = A$. Then, $\{1,2\}$ is $\alpha_{(\gamma,\gamma')}$-open but not (γ,γ')-open.

Proposition 5. *If A is $\alpha_{(\gamma,\gamma')}$-open, then A is $\alpha_{[\gamma,\gamma']}$-open.*

Proof. Obvious.

The converse of the above proposition need not be true in general as it is shown in the following example.

Example 4. Let $X = \{1,2,3\}$ and $\tau = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$ be a topology on X. For each $A \in \alpha O(X,\tau)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = \begin{cases} A \cup \{1\}, & \text{if } 2 \in A, \\ A, & \text{if } 2 \notin A, \end{cases}$$

and

$$A^{\gamma'} = \begin{cases} X, & \text{if } A \neq \{1\}, \\ A, & \text{if } A = \{1\}, \\ \phi, & \text{if } A = \phi. \end{cases}$$

Then, $\{1,3\}$ is $\alpha_{[\gamma,\gamma']}$-open but not $\alpha_{(\gamma,\gamma')}$-open.

Proposition 6. *If A is $\alpha_{(\gamma,\gamma')}$-open, then A is α_{γ}-open for any operation γ'.*

Proof. Obvious.

The converse of the above proposition need not be true in general as it is shown in the following example.

Example 5. Let $X = \{1,2,3\}$ and τ be a discrete topology on X. For each $A \in \alpha O(X,\tau)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = \begin{cases} A, & \text{if } A = \{3\}, \\ X, & A \neq \{3\}, \\ \phi, & \text{if } A = \phi. \end{cases}$$

and

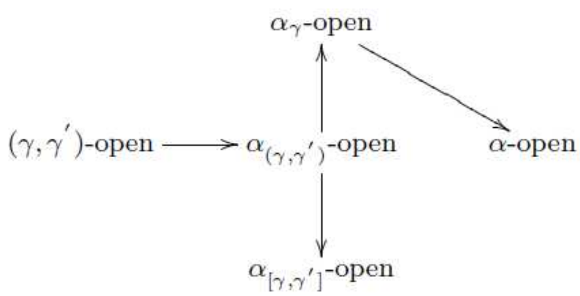
$$A^{\gamma'} = \begin{cases} X, & \text{if } A \neq \phi, \\ \phi, & \text{if } A = \phi. \end{cases}$$

Then, $\{3\}$ is α_{γ}-open but not $\alpha_{(\gamma,\gamma')}$-open.

Remark. A is α_{γ}-open if and only if A is $\alpha_{(\gamma,id)}$-open.

Remark.[1] Every α_{γ}-open subset of a space X is α-open.

*Remark.*We have the following implications but none of this implications are reversible.



Definition 2. *A topological space (X,τ) is said to be $\alpha_{(\gamma,\gamma')}$-regular if for each point x in X and every α-open set U in X containing x, there exist α-open sets W and S in X containing x such that $W^{\gamma} \cup S^{\gamma'} \subseteq U$.*

Proposition 7. *A topological space (X,τ) with operations γ and γ' on $\alpha O(X,\tau)$ is $\alpha_{(\gamma,\gamma')}$-regular if and only if $\alpha O(X,\tau) = \alpha O(X,\tau)_{(\gamma,\gamma')}$.*

Proof. Let (X,τ) be $\alpha_{(\gamma,\gamma')}$-regular and $A \in \alpha O(X,\tau)$. Since (X,τ) is $\alpha_{(\gamma,\gamma')}$-regular, then for each $x \in A$, there exist α-open sets W and S in X containing x such that $W^{\gamma} \cup S^{\gamma'} \subseteq A$. This implies that $A \in \alpha O(X,\tau)_{(\gamma,\gamma')}$. But we have $\alpha O(X,\tau)_{(\gamma,\gamma')} \subseteq \alpha O(X,\tau)$. Therefore $\alpha O(X,\tau) = \alpha O(X,\tau)_{(\gamma,\gamma')}$.
Conversely, let $\alpha O(X,\tau) = \alpha O(X,\tau)_{(\gamma,\gamma')}$, $x \in X$ and V be α-open in X containing x. Then, by assumption V is $\alpha_{(\gamma,\gamma')}$-open set. This implies that there exist α-open sets W and S in X containing x such that $W^{\gamma} \cup S^{\gamma'} \subseteq V$. Therefore, (X,τ) is $\alpha_{(\gamma,\gamma')}$-regular.

Remark. If a space X is $\alpha_{(\gamma,\gamma')}$-regular, then $\tau \subseteq \alpha O(X,\tau)_{(\gamma,\gamma')}$.

Remark. (X,τ) is $\alpha_{(\gamma,\gamma')}$-regular if and only if it is both α_{γ}-regular and $\alpha_{\gamma'}$-regular.

Definition 3. *A subset F of (X,τ) is said to be $\alpha_{(\gamma,\gamma')}$-closed if its complement $X \setminus F$ is $\alpha_{(\gamma,\gamma')}$-open. We denote the set of all $\alpha_{(\gamma,\gamma')}$-closed sets of (X,τ) by $\alpha C(X,\tau)_{(\gamma,\gamma')}$.*

Definition 4. *Let A be a subset of a topological space (X,τ). The intersection of all $\alpha_{(\gamma,\gamma')}$-closed sets containing A is called the $\alpha_{(\gamma,\gamma')}$-closure of A and denoted by $\alpha_{(\gamma,\gamma')}$-$Cl(A)$.*

Proposition 8. *For a point $x \in X$, $x \in \alpha_{(\gamma,\gamma')}$-$Cl(A)$ if and only if $V \cap A \neq \phi$ for every $\alpha_{(\gamma,\gamma')}$-open set V containing x.*

Proof. Obvious.

Proposition 9. *Let A and B be subsets of (X,τ). Then the following hold:*

(1) $A \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(A)$.
(2) If $A \subseteq B$, then $\alpha_{(\gamma,\gamma')}$-$Cl(A) \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(B)$.
(3) $A \in \alpha C(X,\tau)_{(\gamma,\gamma')}$ if and only if $\alpha_{(\gamma,\gamma')}$-$Cl(A) = A$.
(4) $\alpha_{(\gamma,\gamma')}$-$Cl(A) \in \alpha C(X,\tau)_{(\gamma,\gamma')}$.
(5) $\alpha_{(\gamma,\gamma')}$-$Cl(A\cap B) \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(A) \cap \alpha_{(\gamma,\gamma')}$-$Cl(B)$.
(6) If γ and γ' are α-regular, then $\alpha_{(\gamma,\gamma')}$-$Cl(A\cup B) = \alpha_{(\gamma,\gamma')}$-$Cl(A) \cup \alpha_{(\gamma,\gamma')}$-$Cl(B)$.

Proof. They are obvious.

Remark. From Remark 3 and Definition 3, the following hold for any subset A of X.

(1) $\alpha C(X,\tau)_{(\gamma,id)} = \{F : F \text{ is } \alpha_\gamma\text{-closed}\}$.
(2) $\alpha_{(\gamma,id)}$-$Cl(A) = \alpha_\gamma Cl(A)$.

Definition 5. *For a subset A of (X,τ), we define $\alpha Cl_{(\gamma,\gamma')}(A)$ as follows: $\alpha Cl_{(\gamma,\gamma')}(A) = \{x \in X : (U^\gamma \cup W^{\gamma'}) \cap A \neq \phi$ holds for every α-open sets U and W containing $x\}$.*

Proposition 10. *For a subset A of (X,τ), we have*

(1) $A \subseteq \alpha Cl(A) \subseteq \alpha Cl_\gamma(A) \subseteq \alpha Cl_{(\gamma,\gamma')}(A) \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(A)$.
(2) $\alpha_\gamma Cl(A) \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(A)$.

Proof. Obvious.

Theorem 1. *Let A and B be subsets of a topological space (X,τ). Then, we have the following properties:*

(1) $A \subseteq \alpha Cl_{(\gamma,\gamma')}(A)$.
(2) $\alpha Cl_{(\gamma,\gamma')}(\phi) = \phi$ and $\alpha Cl_{(\gamma,\gamma')}(X) = X$.
(3) $A \in \alpha C(X,\tau)_{(\gamma,\gamma')}$ if and only if $\alpha Cl_{(\gamma,\gamma')}(A) = A$.
(4) If $A \subseteq B$, then $\alpha Cl_{(\gamma,\gamma')}(A) \subseteq \alpha Cl_{(\gamma,\gamma')}(B)$.
(5) If γ and γ' are α-regular, then $\alpha Cl_{(\gamma,\gamma')}(A\cup B) = \alpha Cl_{(\gamma,\gamma')}(A) \cup \alpha Cl_{(\gamma,\gamma')}(B)$.
(6) $\alpha Cl_{(\gamma,\gamma')}(A\cap B) \subseteq \alpha Cl_{(\gamma,\gamma')}(A) \cap \alpha Cl_{(\gamma,\gamma')}(B)$.

Proof. (1), (2) and (4) They are obtained from Definition 5.
(3) Suppose A is $\alpha_{(\gamma,\gamma')}$-closed, so $X\setminus A$ is $\alpha_{(\gamma,\gamma')}$-open in (X,τ). We claim that $\alpha Cl_{(\gamma,\gamma')}(A) \subseteq A$. Let $x \notin A$. There exist α-open sets U and V of (X,τ) containing x such that $U^\gamma \cup V^{\gamma'} \subseteq X\setminus A$, that is, $(U^\gamma \cup V^{\gamma'}) \cap A = \phi$. Hence by Definition 5, we have that $x \notin \alpha Cl_{(\gamma,\gamma')}(A)$ and so $\alpha Cl_{(\gamma,\gamma')}(A) \subseteq A$. By (1), it is proved that $\alpha Cl_{(\gamma,\gamma')}(A) = A$.
Conversely, suppose that $\alpha Cl_{(\gamma,\gamma')}(A) = A$. Let $x \in X\setminus A$. Since $x \notin \alpha Cl_{(\gamma,\gamma')}(A)$, there exist α-open sets U and V containing x such that $(U^\gamma \cup V^{\gamma'}) \cap A = \phi$, that is, $U^\gamma \cup V^{\gamma'} \subseteq X\setminus A$. Therefore, A is $\alpha_{(\gamma,\gamma')}$-closed.
(5) Let $x \notin \alpha Cl_{(\gamma,\gamma')}(A) \cup \alpha Cl_{(\gamma,\gamma')}(B)$. Then, there exist α-open sets U, V, W and S of (X,τ) containing x such that $(U^\gamma \cup V^{\gamma'}) \cap A = \phi$ and $(W^\gamma \cup S^{\gamma'}) \cap B = \phi$. Since γ and γ' are α-regular, by definition of α-regular, there exist α-open sets K and L of (X,τ) containing x such that $k^\gamma \subseteq U^\gamma \cap W^\gamma$ and $L^{\gamma'} \subseteq V^{\gamma'} \cap S^{\gamma'}$. Thus, we have $(k^\gamma \cup L^{\gamma'}) \cap (A\cup B) \subseteq ((U^\gamma \cap$

$W^{\gamma})\cup(V^{\gamma'}\cap S^{\gamma'}))\cap(A\cup B)\subseteq((U^{\gamma}\cup V^{\gamma'})\cap(W^{\gamma}\cup S^{\gamma'}))\cap(A\cup B)=[((U^{\gamma}\cup V^{\gamma'})\cap(W^{\gamma}\cup S^{\gamma'}))\cap A]\cup[((U^{\gamma}\cup V^{\gamma'})\cap(W^{\gamma}\cup S^{\gamma'}))\cap B]=\phi$, that is, $(k^{\gamma}\cup L^{\gamma'})\cap(A\cup B)=\phi$. Hence, $x\notin\alpha Cl_{(\gamma,\gamma')}(A\cup B)$. This shows that $\alpha Cl_{(\gamma,\gamma')}(A)\cup\alpha Cl_{(\gamma,\gamma')}(B)\supseteq\alpha Cl_{(\gamma,\gamma')}(A\cup B)$.
(6) This obtained from (4).

Definition 6. *Let A be a subset of a topological space (X,τ). The union of all $\alpha_{(\gamma,\gamma')}$-open sets contained in A is called the $\alpha_{(\gamma,\gamma')}$-interior of A and is denoted by $\alpha_{(\gamma,\gamma')}$-$Int(A)$.*

Proposition 11. *For any subsets A,B of X, we have the following:*

(1) $\alpha_{(\gamma,\gamma')}$-$Int(A)$ *is an* $\alpha_{(\gamma,\gamma')}$*-open set in* X.
(2) A *is* $\alpha_{(\gamma,\gamma')}$*-open if and only if* $A=\alpha_{(\gamma,\gamma')}$-$Int(A)$.
(3) $\alpha_{(\gamma,\gamma')}$-$Int(\alpha_{(\gamma,\gamma')}$-$Int(A))=\alpha_{(\gamma,\gamma')}$-$Int(A)$.
(4) $\alpha_{(\gamma,\gamma')}$-$Int(A)\subseteq A$.
(5) *If* $A\subseteq B$, *then* $\alpha_{(\gamma,\gamma')}$-$Int(A)\subseteq\alpha_{(\gamma,\gamma')}$-$Int(B)$.
(6) $\alpha_{(\gamma,\gamma')}$-$Int(A\cup B)\supseteq\alpha_{(\gamma,\gamma')}$-$Int(A)\cup\alpha_{(\gamma,\gamma')}$-$Int(B)$.
(7) $\alpha_{(\gamma,\gamma')}$-$Int(A\cap B)\subseteq\alpha_{(\gamma,\gamma')}$-$Int(A)\cap\alpha_{(\gamma,\gamma')}$-$Int(B)$.

Proof. Obvious.

Proposition 12. *Let A be any subset of a topological space (X,τ). Then, the following statements are true:*

(1) $X\setminus\alpha_{(\gamma,\gamma')}$-$Int(A)=\alpha_{(\gamma,\gamma')}$-$Cl(X\setminus A)$.
(2) $X\setminus\alpha_{(\gamma,\gamma')}$-$Cl(A)=\alpha_{(\gamma,\gamma')}$-$Int(X\setminus A)$.
(3) $\alpha_{(\gamma,\gamma')}$-$Int(A)=X\setminus\alpha_{(\gamma,\gamma')}$-$Cl(X\setminus A)$.
(4) $\alpha_{(\gamma,\gamma')}$-$Cl(A)=X\setminus\alpha_{(\gamma,\gamma')}$-$Int(X\setminus A)$.

Proof. Obvious.

4 $\alpha_{(\gamma,\gamma')}$-g.closed sets

In this section, we define and study some properties of $\alpha_{(\gamma,\gamma')}$-g.closed sets.

Definition 7. *A subset A of (X,τ) is said to be an $\alpha_{(\gamma,\gamma')}$- generalized closed (briefly, $\alpha_{(\gamma,\gamma')}$- g.closed) set, if $\alpha_{(\gamma,\gamma')}$-$Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is an $\alpha_{(\gamma,\gamma')}$-open set in (X,τ).*

Remark. It is clear that every $\alpha_{(\gamma,\gamma')}$-closed set is $\alpha_{(\gamma,\gamma')}$-g.closed. But the converse is not true in general as it is shown in the following example.

Example 6. Let $X=\{1,2,3\}$ and $\tau=\{\phi,\{1\},\{2\},\{1,2\},\{1,3\},X\}$. For each $A\in\alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^{\gamma}=A^{\gamma'}=\begin{cases}A, & \text{if } A\in\{\phi,\{2\},\{1,3\}\},\\ X, & \text{otherwise.}\end{cases}$$

Now, if we let $A=\{1\}$, since the only $\alpha_{(\gamma,\gamma')}$-open supersets of A are $\{1,3\}$ and X, then A is $\alpha_{(\gamma,\gamma')}$-g.closed. But, it is easy to see that A is not $\alpha_{(\gamma,\gamma')}$-closed.

Proposition 13. *If A is $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-g.closed, then A is $\alpha_{(\gamma,\gamma')}$-closed.*

Proof. Since A is $\alpha_{(\gamma,\gamma')}$-open and $A\subseteq A$, we have $\alpha_{(\gamma,\gamma')}$-$Cl(A)\subseteq A$, also $A\subseteq\alpha_{(\gamma,\gamma')}$-$Cl(A)$, therefore $\alpha_{(\gamma,\gamma')}$-$Cl(A)=A$. That is, A is $\alpha_{(\gamma,\gamma')}$-closed.

Proposition 14. *The intersection of an $\alpha_{(\gamma,\gamma')}$-g.closed set and an $\alpha_{(\gamma,\gamma')}$-closed set is always $\alpha_{(\gamma,\gamma')}$-g.closed.*

Proof. Let A be an $\alpha_{(\gamma,\gamma')}$-g.closed set and F be an $\alpha_{(\gamma,\gamma')}$-closed set. Assume that U is an $\alpha_{(\gamma,\gamma')}$-open set such that $A\cap F\subseteq U$. Set $G=X\setminus F$. Then we have $A\subseteq U\cup G$, since G is $\alpha_{(\gamma,\gamma')}$-open, then $U\cup G$ is $\alpha_{(\gamma,\gamma')}$-open and since A is $\alpha_{(\gamma,\gamma')}$-g.closed, then $\alpha_{(\gamma,\gamma')}$-$Cl(A)\subseteq U\cup G$. Now, $\alpha_{(\gamma,\gamma')}$-$Cl(A\cap F)\subseteq\alpha_{(\gamma,\gamma')}$-$Cl(A)\cap\alpha_{(\gamma,\gamma')}$-$Cl(F)=\alpha_{(\gamma,\gamma')}$-$Cl(A)\cap F\subseteq (U\cup G)\cap F=(U\cap F)\cup\phi\subseteq U$.

The intersection of two $\alpha_{(\gamma,\gamma')}$-g.closed sets need not be $\alpha_{(\gamma,\gamma')}$-g.closed in general. It is shown by the following example.

Example 7. Let $X=\{1,2,3\}$ and τ be a discrete topology on X. For each $A\in\alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^{\gamma}=A^{\gamma'}=\begin{cases}A, & \text{if } A\in\{\phi,\{1\}\},\\ X, & \text{otherwise.}\end{cases}$$

Set $A=\{1,2\}$ and $B=\{1,3\}$. Clearly, A and B are $\alpha_{(\gamma,\gamma')}$-g.closed sets, since X is their only $\alpha_{(\gamma,\gamma')}$-open superset. But, $C=\{1\}=A\cap B$ is not $\alpha_{(\gamma,\gamma')}$-g.closed, since $C\subseteq\{1\}\in\alpha O(X,\tau)_{(\gamma,\gamma')}$ and $\alpha_{(\gamma,\gamma')}$-$Cl(C)=X\nsubseteq\{1\}$.

Proposition 15. *If γ ana γ' are α-regular operations on $\alpha O(X,\tau)$. Then the finite union of $\alpha_{(\gamma,\gamma')}$-g.closed sets is always an $\alpha_{(\gamma,\gamma')}$-g.closed set.*

Proof. Let A and B be two $\alpha_{(\gamma,\gamma')}$-g.closed sets, and $A\cup B\subseteq U$, where U is $\alpha_{(\gamma,\gamma')}$-open. Since A and B are $\alpha_{(\gamma,\gamma')}$-g.closed sets, we have $\alpha_{(\gamma,\gamma')}$-$Cl(A)\subseteq U$ and $\alpha_{(\gamma,\gamma')}$-$Cl(B)\subseteq U$ and so $\alpha_{(\gamma,\gamma')}$-$Cl(A)\cup\alpha_{(\gamma,\gamma')}$-$Cl(B)\subseteq U$. But, we have $\alpha_{(\gamma,\gamma')}$-$Cl(A)\cup\alpha_{(\gamma,\gamma')}$-$Cl(B)=\alpha_{(\gamma,\gamma')}$-$Cl(A\cup B)$ by Proposition 9 (6). Therefore, $\alpha_{(\gamma,\gamma')}$-$Cl(A\cup B)\subseteq U$ and so $A\cup B$ is an $\alpha_{(\gamma,\gamma')}$-g.closed set.

The union of two $\alpha_{(\gamma,\gamma')}$-g.closed sets need not be $\alpha_{(\gamma,\gamma')}$-g.closed in general. It is shown by the following example.

Example 8. Let $X=\{1,2,3\}$ and τ be a discrete topology on X. For each $A\in\alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^{\gamma}=A^{\gamma'}=\begin{cases}A, & \text{if } A\in\{\phi,\{1,2\},\{1,3\},\{2,3\}\},\\ X, & \text{otherwise,}\end{cases}$$

Let $A=\{1\}$ and $B=\{2\}$. Here A and B are $\alpha_{(\gamma,\gamma')}$-g.closed but $A\cup B=\{1,2\}$ is not $\alpha_{(\gamma,\gamma')}$-g.closed, since $\{1,2\}$ is $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-$Cl(\{1,2\})=X$.

Proposition 16. *If a subset A of (X,τ) is $\alpha_{(\gamma,\gamma')}$-g.closed and $A\subseteq B\subseteq\alpha_{(\gamma,\gamma')}$-$Cl(A)$, then B is an $\alpha_{(\gamma,\gamma')}$-g.closed set in (X,τ).*

Proof. Let U be an $\alpha_{(\gamma,\gamma')}$-open set of (X,τ) such that $B\subseteq U$. Since A is $\alpha_{(\gamma,\gamma')}$-g.closed, we have $\alpha_{(\gamma,\gamma')}$-$Cl(A)\subseteq U$. Now, by Proposition 9 and assumptions, it is shown that $\alpha_{(\gamma,\gamma')}$-$Cl(A)\subseteq\alpha_{(\gamma,\gamma')}$-$Cl(B)\subseteq\alpha_{(\gamma,\gamma')}$-$Cl[\alpha_{(\gamma,\gamma')}$-$Cl(A)]=\alpha_{(\gamma,\gamma')}$-$Cl(A)\subseteq U$ and so $\alpha_{(\gamma,\gamma')}$-$Cl(B)\subseteq U$. Therefore, B is an $\alpha_{(\gamma,\gamma')}$-g.closed set of (X,τ).

Proposition 17. *For each $x\in X$, $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed or $X\setminus\{x\}$ is $\alpha_{(\gamma,\gamma')}$-g.closed in (X,τ).*

Proof. Suppose that $\{x\}$ is not $\alpha_{(\gamma,\gamma')}$-closed. Then, $X\setminus\{x\}$ is not $\alpha_{(\gamma,\gamma')}$-open. Let U be any $\alpha_{(\gamma,\gamma')}$-open set such that $X\setminus\{x\}\subseteq U$. Then, this implies $U=X$ and so $\alpha_{(\gamma,\gamma')}$-$Cl(X\setminus\{x\})\subseteq U$. Hence, $X\setminus\{x\}$ is $\alpha_{(\gamma,\gamma')}$-g.closed.

Proposition 18. *The following statements $(1),(2)$ and (3) are equivalent for a subset A of (X,τ).*

(1) *A is $\alpha_{(\gamma,\gamma')}$-g.closed in (X,τ).*
(2) *$\alpha_{(\gamma,\gamma')}$-$Cl(\{x\})\cap A\neq\phi$ for every $x\in\alpha_{(\gamma,\gamma')}$-$Cl(A)$.*
(3) *$\alpha_{(\gamma,\gamma')}$-$Cl(A)\setminus A$ does not contain any non-empty $\alpha_{(\gamma,\gamma')}$-closed set.*

Proof. $(1) \Rightarrow (2)$. Suppose that there exists a point $x \in \alpha_{(\gamma,\gamma')}\text{-}Cl(A)$ such that $\alpha_{(\gamma,\gamma')}\text{-}Cl(\{x\}) \cap A = \phi$. Since $\alpha_{(\gamma,\gamma')}\text{-}Cl(\{x\})$ is $\alpha_{(\gamma,\gamma')}$-closed by Proposition 9, $X \setminus \alpha_{(\gamma,\gamma')}\text{-}Cl(\{x\})$ is an $\alpha_{(\gamma,\gamma')}$-open set of (X,τ). Since $A \subseteq X \setminus (\alpha_{(\gamma,\gamma')}\text{-}Cl(\{x\}))$ and A is $\alpha_{(\gamma,\gamma')}$-g.closed, this implies $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \subseteq X \setminus \alpha_{(\gamma,\gamma')}\text{-}Cl(\{x\})$ and hence $x \notin \alpha_{(\gamma,\gamma')}\text{-}Cl(A)$. This is a contradiction. Therefore, we conclude that $\alpha_{(\gamma,\gamma')}\text{-}Cl(\{x\}) \cap A \neq \phi$ holds for every $x \in \alpha_{(\gamma,\gamma')}\text{-}Cl(A)$.
$(2) \Rightarrow (3)$. Suppose that there exists a non-empty $\alpha_{(\gamma,\gamma')}$-closed set F such that $F \subseteq \alpha_{(\gamma,\gamma')}\text{-}Cl(A) \setminus A$ and so $A \cap F = \phi$. Let $y \in F$. Then, $y \in \alpha_{(\gamma,\gamma')}\text{-}Cl(A)$ and $y \notin A$. By (2), it is obtained that $\phi \neq \alpha_{(\gamma,\gamma')}\text{-}Cl(\{y\}) \cap A \subseteq \alpha_{(\gamma,\gamma')}\text{-}Cl(F) \cap A = F \cap A$ and so $F \cap A \neq \phi$. This is a contradiction and so (3) is claimed.
$(3) \Rightarrow (1)$. Let $A \subseteq U$, where U is $\alpha_{(\gamma,\gamma')}$-open in (X,τ). If $\alpha_{(\gamma,\gamma')}\text{-}Cl(A)$ is not contained in U, then $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \cap (X \setminus U) \neq \phi$. Now, since $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \cap (X \setminus U) \subseteq \alpha_{(\gamma,\gamma')}\text{-}Cl(A) \setminus A$ and $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \cap (X \setminus U)$ is a non-empty $\alpha_{(\gamma,\gamma')}$-closed set, we obtain a contradiction and therefore A is $\alpha_{(\gamma,\gamma')}$-g.closed.

Proposition 19. *If A is an $\alpha_{(\gamma,\gamma')}$-g.closed set of a space X, then the following are equivalent:*

(1) *A is $\alpha_{(\gamma,\gamma')}$-closed.*
(2) *$\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \setminus A$ is $\alpha_{(\gamma,\gamma')}$-closed.*

Proof. $(1) \Rightarrow (2)$. Since A is $\alpha_{(\gamma,\gamma')}$-closed, then $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) = A$ holds by Proposition 9 (3) and so $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \setminus A = \phi$ and the set ϕ is $\alpha_{(\gamma,\gamma')}$-closed.
$(2) \Rightarrow (1)$. Since A is $\alpha_{(\gamma,\gamma')}$-g.closed, $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \setminus A$ does not contain any non-empty $\alpha_{(\gamma,\gamma')}$-closed subset and so $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \setminus A = \phi$. This shows that A is $\alpha_{(\gamma,\gamma')}$-closed.

Proposition 20. *For a space (X,τ), the following are equivalent:*

(1) *Every subset of X is $\alpha_{(\gamma,\gamma')}$-g.closed.*
(2) $\alpha O(X,\tau)_{(\gamma,\gamma')} = \alpha C(X,\tau)_{(\gamma,\gamma')}$.

Proof. $(1) \Rightarrow (2)$. Let $U \in \alpha O(X,\tau)_{(\gamma,\gamma')}$. Then, by hypothesis, U is $\alpha_{(\gamma,\gamma')}$-g.closed which implies that $\alpha_{(\gamma,\gamma')}\text{-}Cl(U) \subseteq U$, so, $\alpha_{(\gamma,\gamma')}\text{-}Cl(U) = U$. Thus, we have $U \in \alpha C(X,\tau)_{(\gamma,\gamma')}$; and so $\alpha O(X,\tau)_{(\gamma,\gamma')} \subseteq \alpha C(X,\tau)_{(\gamma,\gamma')}$.
Conversely, let $V \in \alpha C(X,\tau)_{(\gamma,\gamma')}$. Then, $X \setminus V \in \alpha O(X,\tau)_{(\gamma,\gamma')}$. By using the above techinque, it is shown that $V \in \alpha O(X,\tau)_{(\gamma,\gamma')}$; and so $\alpha C(X,\tau)_{(\gamma,\gamma')} \subseteq \alpha O(X,\tau)_{(\gamma,\gamma')}$. Therefore, we have the proof of (2).
$(2) \Rightarrow (1)$. If A is a subset of a space (X,τ) such that $A \subseteq U$ where $U \in \alpha O(X,\tau)_{(\gamma,\gamma')}$, then $U \in \alpha C(X,\tau)_{(\gamma,\gamma')}$. Therefore $\alpha_{(\gamma,\gamma')}\text{-}Cl(A) \subseteq \alpha_{(\gamma,\gamma')}\text{-}Cl(U) = U$ which shows that A is $\alpha_{(\gamma,\gamma')}$-g.closed.

Definition 8. *A subset A of X is $\alpha_{(\gamma,\gamma')}$- g.open if its complement $X \setminus A$ is $\alpha_{(\gamma,\gamma')}$-g.closed in (X,τ).*

It is clear that every $\alpha_{(\gamma,\gamma')}$-open set is $\alpha_{(\gamma,\gamma')}$-g.open, but the converse is not true in general as it is shown in the following example.

Example 9. Consider Example 6, if $A = \{2,3\}$, then A is $\alpha_{(\gamma,\gamma')}$-g.open but not $\alpha_{(\gamma,\gamma')}$-open.

Corollary 1. *A subset A of (X,τ) is $\alpha_{(\gamma,\gamma')}$-g.open if and only if $F \subseteq \alpha_{(\gamma,\gamma')}\text{-}Int(A)$ whenever $F \subseteq A$ and F is $\alpha_{(\gamma,\gamma')}$-closed in (X,τ).*

Proof. By Definition 8 and Proposition 12, the proof is obtained.

The union of two $\alpha_{(\gamma,\gamma')}$-g.open sets need not be $\alpha_{(\gamma,\gamma')}$-g.open in general. It is shown by the following example.

Example 10. Consider Example 7, if $A = \{2\}$ and $B = \{3\}$ then A and B are $\alpha_{(\gamma,\gamma')}$-g.open sets in X, but $A \cup B = \{2,3\}$ is not an $\alpha_{(\gamma,\gamma')}$-g.open set in X.

Corollary 2. *Let γ and γ' be an α-regular operations on $\alpha O(X,\tau)$, and let A and B be two $\alpha_{(\gamma,\gamma')}$-g.open sets in a space (X,τ). Then $A \cap B$ is also $\alpha_{(\gamma,\gamma')}$-g.open.*

Proof. By Definition 8 and Proposition 15, it is proved.

Corollary 3. *Every singleton point set in a space* (X,τ) *is either* $\alpha_{(\gamma,\gamma')}$*-g.open or* $\alpha_{(\gamma,\gamma')}$*-closed.*

Proof. By Definition 8 and Proposition 17, it is proved.

Corollary 4. *If* $\alpha_{(\gamma,\gamma')}$*-Int*$(A)\subseteq B\subseteq A$ *and A is* $\alpha_{(\gamma,\gamma')}$*-g.open, then B is* $\alpha_{(\gamma,\gamma')}$*-g.open.*

Proof. By Definition 8 and Propositions 12 and 16, the proof is obtained.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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