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Uniqueness of weakly weighted-sharing a small function by a meromorphic function and its differential polynomial

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Abstract: In this paper we study the uniqueness of weakly weighted-sharing a small function by a meromorphic function and its differential polynomial. The result of the paper improve some recent results due to Hong-Yan Xu and Yi Hu [5].

Keywords: Meromorphic function, shared value, small function, weakly-sharing.

1 Introduction

Let f be a meromorphic function in the open complex plane $\mathbb{C}$. We use the standard notations of Nevanlinna theory, which can be found in [7]. We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

A meromorphic function $a = a(z)$ is called a small function of f if $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions of f. Clearly $\mathbb{C} \subset S(f)$.

Let f and g be two meromorphic functions in $\mathbb{C}$ and $a \in S(f) \cap S(g)$. We say that f and g share the function $a = a(z)$ CM (counting multiplicities) or IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same set of zeros counting multiplicities or ignoring multiplicities respectively.

Definition 1. [5] *Let k be a positive integer, and let f be a meromorphic function and $a \in S(f)$.*

(i) $\overline{N}(r,a;f| \geq k)$ *denotes the counting function of zeros of $f-a$ whose multiplicities are not less than k, where each zero is counted only once.*

(ii) $\overline{N}(r,a;f| \leq k)$ *denotes the counting function of zeros of $f-a$ whose multiplicities are not greater than k, where each zero is counted only once.*

(iii) $N_p(r,a;f) = \overline{N}(r,a;f) + \sum_{k=2}^{p} \overline{N}(r,a;f| \geq k)$.

Definition 2. [2] *For any complex number $c \in \mathbb{C} \cup \{\infty\}$, we denote by $\delta_p(c,f)$ the quantity*

$$\delta_p(c,f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,c;f)}{T(r,f)},$$

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where p is a positive integer. Clearly $\delta_p(c,f) \geq \delta(c,f)$.

Let $N_E(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ with the same multiplicities, and $N_0(r,a)$ be the counting functions of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. Denotes by $\overline{N}_E(r,a)$ and $\overline{N}_0(r,a)$ the reduced counting functions of f and g corresponding to the counting functions $N_E(r,a)$ and $N_0(r,a)$, respectively. If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_E(r,a)=S(r,f)+S(r,g),$$

then we say that f and g share a "CM". If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a)=S(r,f)+S(r,g),$$

then we say that f and g share a "IM".

Definition 3. [5] *Let f and g be two nonconstant meromorphic functions sharing a "IM", for $a\in S(f)\cap S(g)$, and a positive integer k or ∞.*

(i) *$\overline{N}_E^{k)}(r,a)$ denotes the counting function of zeros of $f-a$ whose multiplicities are equal to the corresponding zeros of $g-a$, both of their mutiplicities are not greater than k, where each zero is counted only once.*

(ii) *$\overline{N}_0^{(k}(r,a)$ denotes the reduced counting function of zeros of $f-a$ which are zeros of $g-a$, both of their mutiplicities are not less than k, where each zero is counted only once.*

(iii) *Let z_0 be the zeros of $f-a$ with multiplicity p and zeros of $g-a$ with multiplicity q. Denote by $\overline{N}_{f>k}(r,a;g)$ the reduced counting function of those zeros of $f-a$ and $g-a$ such that $p>q=k$.$\overline{N}_{g>k}(r,a;g)$ is defined analogously.*

(iv) *$\overline{N}_*(r,a;f,g)$ denotes the reduce counting function of zeros of $f-a$ whose multiplicities differ from the multiplicities of the corresponding zeros of $g-a$.*

Clearly,
$\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 4. [5] *For $a\in S(f)\cap S(g)$, if k is a positive integer or ∞, and*

$$\overline{N}(r,a;f|\leq k)-\overline{N}_E^{k)}(r,a)=S(r,f),\overline{N}(r,a;f|\geq k+1)-\overline{N}_0^{(k+1}(r,a)=S(r,f);$$

$$\overline{N}(r,a;g|\leq k)-\overline{N}_E^{k)}(r,a)=S(r,g),\overline{N}(r,a;g|\geq k+1)-\overline{N}_0^{(k+1}(r,a)=S(r,g),$$

or if $k=0$ and

$$\overline{N}(r,a;f)-\overline{N}_0(r,a)=S(r,f),\ \ \overline{N}(r,a;g)-\overline{N}_0(r,a)=S(r,g),$$

where $\overline{N}_0(r,a)$ is the reduce counting functions of all common zeros $f-a$ and $g-a$ ignoring multiplicities, then we say f and g weakly share a with weight k. Here, we write f,g share "(a,k)" to mean that f,g weakly share a with weight k.

Obviously, if f and g share "(a,k)", then f and g share "(a,p)" for any p $(0\leq p\leq k)$. Also, we note that f and g share a "IM" or "CM" if and only if f and g share "$(a,0)$" or "(a,∞)", respectively.

Definition 5. [5] *Let*

$$L(f)=f^{(n)}+a_{n-1}f^{(n-1)}+\ldots+a_0f, \tag{$*$}$$

be a differential polynomial of f, where a_j $(j=0,1,\ldots,n-1)\in S(f)$.

In 2003, Yu [8] considered the uniqueness problem of an entire function or meromorphic function when it shares one small function with its derivative and proved the following results.

Theorem 1. *Let $n \geq 1$, let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(n)}$ share a CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(n)}$.*

Theorem 2. *Let $n \geq 1$, let f be a non-constant non-entire meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$, f and a do not have any common pole. If f, $f^{(n)}$ share a CM and $4\delta(0, f) + 2(8+n)\Theta(\infty, f) > 19 + 2n$, then $f \equiv f^{(n)}$.*

In 2004, Liu and Gu [3] applied a different method and obtained the following results.

Theorem 3. *Let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(n)}$ share a CM, f and a do not have any common pole of same multiplicity and $2\delta(0, f) + 4\Theta(\infty, f) > 5$, then $f \equiv f^{(n)}$.*

Theorem 4. *Let $n \geq 1$, let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(n)}$ share a CM and $\delta(0, f) > \frac{1}{2}$, then $f \equiv f^{(n)}$.*

In 2011, Hong-Yan Xu and Yi Hu [5] obtained the following result which improve the results of [15, 8].

Theorem 5. *Let $n \geq 1$, let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose that $L(f)$ is defined by $(*)$, If $f, L(f)$ share "(a,k)". Then $f \equiv L(f)$ if one of the following assumptions holds,*

1. $2 \leq k \leq \infty$ *and*
$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5,$$
2. $k = 1$ *and*
$$\left(\frac{7}{2} + n\right)\Theta(\infty, f) + \frac{3}{2}\delta_2(0, f) + \delta_{2+n}(0, f) > n + 5,$$
3. $k = 0$ *and*
$$(6+2n)\Theta(\infty, f) + \delta_2(0, f) + 2\Theta(0, f) + 2\delta_{2+n}(0, f) > (2n+10).$$

We define a monomial $M[f]$ and differential polynomial $H[f]$ as follows,
Let $p_0, p_1, ..., p_k$ be non-negative integers. We call

$$M[f] = f^{p_0}(f')^{p_1}...(f^{(k)})^{p_k}$$

a differential monomial in f with degree $d_M = p_0 + p_1 + ... + p_k$ and weight $\Gamma_M = p_0 + 2p_1 + ... + (k+1)p_k$, and

$$H[f] = \sum_{j=1}^{n} a_j M_j[f], \tag{1}$$

where a_j are small functions of f, is called a differential polynomial in f of degree $d = \max\{d_{M_j}, 1 \leq j \leq n\}$ and weight $\Gamma = \max\{\Gamma_{M_j}, 1 \leq j \leq n\}$, furthermore if $deg(M_j) = d\,(j = 1, 2, ..., n)$, then $H[f]$ is a homogeneous differential polynomial in f of degree d.

In this paper, we improve the above Theorems and obtain the following results.

Theorem 6. *Let f be a non-constant meromorphic function and $H[f]$ be a non-constant homogeneous differential polynomial of degree d and weight Γ satisfying $\Gamma \geq (k+2)d - 2$. Let $a(z) \in S(f)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $H[f] - a$ share $(0, k)$. Then $\frac{H[f]-a}{f-a} = C$ for some non-zero constant C if one of the following asumptions holds,*

(i) $2 \leq k \leq \infty$ *and*
$$4\Theta(\infty, f) + \delta_2(0, f) + d\delta_{2+\Gamma-d}(0, f^d) > 5, \tag{2}$$

(ii) $k = 1$ *and*
$$\left(\frac{7}{2} + \Gamma - d\right)\Theta(\infty, f) + \frac{3}{2}\delta_2(0, f) + d\delta_{2+\Gamma-d}(0, f^d) > \Gamma + 4, \tag{3}$$

(iii) $k = 0$ *and*
$$(6 + 2\Gamma - 2d)\Theta(\infty, f) + \delta_2(0, f) + 2\Theta(0, f) + d\delta_{1+\Gamma-d}(0, f^d) + d\delta_{2+\Gamma-d}(0, f^d) > 2\Gamma + 9. \tag{4}$$

Especially, when $k=0$, i.e., f and H share a IM, if (4) holds, then $f\equiv H[f]$.

From Theorem 6 we have the following corollary.

Corollary 1. *Let f be a non-constant entire function and $a\equiv a(z)\,(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,a)=S(r,f)$. If $f,H[f]$ share "(a,k)", $k\geq 2$ and $\delta_{2+\Gamma-d}(0,f^d)>\frac{1}{d+1}$, or if $f,H[f]$ share "$(a,1)$" and $\delta_{2+\Gamma-d}(0,f^d)>\frac{2d+1}{3+2d}$, or if $f,H[f]$ share "$(a,0)$" and $\delta_{2+\Gamma-d}(0,f^d)>\frac{2d+2}{d}-\frac{1}{d}\left(\delta_2(0,f)+2\Theta(0,f)+d\delta_{1+\Gamma-d}(0,f^d)\right)$, then $\frac{H[f]-a}{f-a}=C$ for some non-zero constant C and $f\equiv H[f]$ for $k=0$, where $H[f]$ is defined by (1).*

2 Some lemmas

For the proof of our main results, we need the following lemmas.

Lemma 1. [4] *Let $H[f]$ be a non-constant differential polynomial. Let z_0 be a pole of f order p and neither a zero nor a pole of coefficients of $H[f]$. Then z_0 is a pole of $H[f]$ with order at most $pd+(\Gamma-d)$.*

Lemma 2. [4] *Let f be a non-constant meromorphic function, $H[f]$ is a homogeneous differential polynomial in f of degree d and weight Γ, and let p be a positive integer. If $H[f]\not\equiv 0$ and $\Gamma\geq(k+2)d-(p+1)$, we have*

$$N_p\left(r,\frac{1}{H}\right)\leq T(r,H)-dT(r,f)+N_{p+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f), \tag{5}$$

$$N_p\left(r,\frac{1}{H}\right)\leq(\Gamma-d)\overline{N}(r,f)+N_{p+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f). \tag{6}$$

Lemma 3. [6] *Let k be a nonnegative integer or ∞, F and G be two nonconstant meromorphic functions, F and G share "$(1,k)$". Let*

$$\Delta=\left(\frac{F''}{F'}-2\frac{F'}{F-1}\right)-\left(\frac{G''}{G'}-2\frac{G'}{G-1}\right). \tag{7}$$

If $\Delta\not\equiv 0$, $2\leq k\leq\infty$, then

$$T(r,F)\leq N_2(r,\infty;F)+N_2(r,0;F)+N_2(r,\infty;G)+N_2(r,0;G)+S(r,F)+S(r,G).$$

The same inequalities holds for $T(r,G)$.

When f and g share 1 "IM", $\overline{N}_L(r,1;f)$ denotes the counting function of the 1-points of f whose multiplicities are greater than 1-points of g, where each zero is counted only once. Similarly, we denote $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;f)$ denotes the counting function of those simple 1-points of f and g, and $\overline{N}_E^{(2}(r,1;f)$ denotes the counting function of those multiplicity 1-points of f and g, each point in these counting functions is counted only once. In the same way, one can define $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

Lemma 4. [5] *If f,g be two nonconstant meromorphic functions such that they share "$(1,1)$", then*

$$2\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>2}(r,1;g)\leq N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 5. [5] *Let f,g share "$(1,1)$". Then*

$$\overline{N}_{f>2}(r,1;g)\leq\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}\overline{N}_0(r,0,f')+S(r,f).$$

Lemma 6. [5] *Let f and g be two nonconstant meromorphic functions sharing* "$(1,0)$". *Then*

$$\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)\leq N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 7. [5] *Let f,g share* "$(1,0)$". *Then*

$$\overline{N}_L(r,1;f)\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 8. [5] *Let f,g share* "$(1,0)$". *Then*

(i) $\overline{N}_{f>1}(r,1;g)\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0,f')+S(r,f)$;
(ii) $\overline{N}_{g>1}(r,1;f)\leq\overline{N}(r,0;g)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0,f')+S(r,g)$.

Proof. (proof of Theorem 6.) Let

$$F=\frac{f}{a},\qquad G=\frac{H[f]}{a}. \tag{8}$$

From the conditions of Theorem 6, we know that F and G share "$(1,k)$", and from (8), we have

$$T(r,F)=T(r,f)+S(r,f),T(r,G)=O(T(r,f))+S(r,f). \tag{9}$$
$$\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)+S(r,f). \tag{10}$$

It is obvious that f is a transcendental meromorphic function. Let Δ be defined by (7). We distinguish two cases

Case 1. $\Delta\equiv 0$. integrating (7), yields

$$\frac{1}{F-1}=\frac{C}{G-1}+D, \tag{11}$$

where C and D are constants and $C\neq 0$. If there exists a pole z_0 of f with multiplicity p which is not zero or pole of a, then z_0 is a pole of G with multiplicity $pd+(\Gamma-d)$, a pole of F with multiplicity p. This contradicts (11) as H contains at least one derivative. Therefore, we have

$$\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)=\overline{N}(r,\infty;f)=S(r,f). \tag{12}$$

(11) also shows that F and G share the value 1 CM. Next, we will prove $D=0$. Suppose $D\neq 0$, then we have

$$\frac{1}{F-1}=\frac{D\left(G-1+\frac{C}{D}\right)}{G-1}. \tag{13}$$

So, we have

$$\overline{N}\left(r,0;D\left(G-1+\frac{C}{D}\right)\right)=\overline{N}\left(r,\infty;\frac{F-1}{G-1}\right)=S(r,f). \tag{14}$$

Subcase 1.1. If $\frac{C}{D}\neq 1$, then by using (12), (14) and the second fundamental theorem, we have

$$\begin{aligned}T(r,F)&\leq\overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\left(r,0;G-1+\frac{C}{D}\right)+S(r,F)\\&\leq\overline{N}(r,0;G)+S(r,F)\leq(1+o(1))T(r,G).\end{aligned}$$

This gives that

$$T(r,G)=\overline{N}(r,0;G)+S(r,F)=N_1(r,0;G)+S(r,F).$$

So we have

$$T(r,H)=N(r,0;H)+S(r,f)=N_1(r,0;H)+S(r,f).$$

Let $p = 1$, then from assumption we have

$$\Gamma \geq (k+2)d - 2 = (k+2)d - (p+1).$$

Thus from (5) in Lemma 2, we get

$$T(r,H) = N_1(r,0;H) + S(r,f) \leq T(r,H) - dT(r,f) + N_{1+\Gamma-d}(r,0;f^d) + S(r,f).$$

So we have

$$dT(r,f) \leq N_{1+\Gamma-d}(r,0;f^d) + S(r,f).$$

This gives that

$$dT(r,f) = N_{1+\Gamma-d}(r,0;f^d) + S(r,f).$$

So we have

$$\delta_{2+\Gamma-d}(r,0;f^d) = \delta_{1+\Gamma-d}(r,0;f^d) = 0.$$

Since (12), we get

$$\Theta(\infty, f) = 1. \tag{15}$$

Subcase 1.2. $k \geq 2$. By using (2) and the definition of deficiency, we get a contradiction.

Subcase 1.3. $k = 1$. By using (3) and the definition of deficiency, we get a contradiction.

Subcase 1.4. $k = 0$. By using (4) and the definition of deficiency, we get a contradiction.

Subcase 1.5. If $\frac{C}{D} = 1$, then from (13), we have

$$\frac{1}{F-1} \equiv C\frac{G}{G-1}.$$

This gives us that

$$\left(F - 1 - \frac{1}{C}\right)G \equiv -\frac{1}{C}.$$

Using that $F = \frac{f}{a}$ and $G = \frac{H}{a}$, we get

$$f - \left(a + \frac{1}{C}\right) \equiv -\frac{a^2}{C} \cdot \frac{1}{H}. \tag{16}$$

Using (12), (16), Lemma 1 and the first fundamental theorem, we get

$$\begin{aligned}
(d+1)T(r,f) &= T\left(r,0;f^d\left(f - \left(1+\frac{1}{C}\right)a\right)\right) + O(1)\\
&= T\left(r,\infty;-\frac{CH}{f^d a^2}\right) + O(1)\\
&= N\left(r,\infty;\frac{H}{f^d}\right) + S(r,f)\\
&\leq dN(r,0;f) + S(r,f)\\
&\leq (d+o(1))T(r,f),
\end{aligned}$$

which is a contradiction, hence $D = 0$. This gives from (11) that

$$\frac{G-1}{F-1} \equiv C.$$

So we get $\frac{H[f]-a}{f-a} = C\,(C \neq 0.)$ Next, we will prove $C = 1$ when $l = 0$. Suppose $C \neq 1$, then we have

$$F \equiv \frac{1}{C}(G-1+C)$$

and

$$N(r,0;F) = N(r,(1+C);G). \tag{17}$$

By the second fundamental theorem and (12) (17), we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,(1+C);G) + S(r,f) \\ &\leq \overline{N}(r,0;G) + \overline{N}(r,0;F) + S(r,f) \\ &= N_1(r,0;G) + \overline{N}(r,0;F). \end{aligned}$$

By Lemma 2 for $p = 1$, we have

$$dT(r,f) \leq N_{1+\Gamma-d}(r,0;f^d) + \overline{N}(r,0;f) + S(r,f).$$

From the above formula and the definition of deficiency, we have

$$d\delta_{1+\Gamma-d}(0,f^d) + \Theta(0,f) \leq 1. \tag{18}$$

So we have

$$d\delta_{2+\Gamma-d}(0,f^d) + \delta_2(0,f) \leq 1,\ d\delta_{1+\Gamma-d}(0,f^d) \leq 1. \tag{19}$$

Combining (18) (19) (15) with the assumptions of Theorem 6, we get a contradiction. So $C = 1$ and $F \equiv G$, i.e. $f \equiv H[f]$. This is just the conclusion of this theorem.

Case 2. $\Delta \not\equiv 0$.

Subcase 2.1. $k \geq 2$. It follows from Lemma 3 that

$$T(r,G) \leq N_2(r,\infty;F) + N_2(r,0;F) + N_2(r,\infty;G) + N_2(r,0;G) + S(r,F) + S(r,G). \tag{20}$$

Noting that

$$N_2(r,0;G) = N_2\left(r,0;\frac{H}{a}\right) \leq N_2(r,0;H) + S(r,f).$$

Let $p = 2$, then from assumption we have

$$\Gamma \geq (k+2)d - 2 > (k+2)d - (p+1).$$

Thus, from (5) in Lemma 2 we obtain that

$$T(r,H) \leq 4\overline{N}(r,\infty;f) + N_2(r,0;f) + T(r,H) - dT(r,f) + N_{2+\Gamma-d}(r,0;f^d) + S(r,f).$$

So we have

$$dT(r,f) \leq 4\overline{N}(r,\infty;f) + N_2(r,0;f) + N_{2+\Gamma-d}(r,0;f^d) + S(r,f).$$

This gives that

$$4\Theta(\infty,f) + \delta_2(0,f) + d\delta_{2+\Gamma-d}(0,f^d) \leq 5.$$

Which contradicts the assumption (2) of Theorem 6.

Subcase 2.2. $k = 1$. We know that F, G share "$(1,1)$", hence we have

$$N(r,\infty;H) \le \overline{N}(r,\infty;F) + \overline{N}(r,1;F| \ge 2) + \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f), \tag{21}$$

and

$$N(r,1;F| = 1) \le N(r,0;H) + S(r,f) \le N(r,\infty;H) + S(r,f), \tag{22}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is similarly defined. By the second fundamental theorem, we see that

$$\begin{aligned} T(r,F) + T(r,G) \le\ & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) \\ & + \overline{N}(r,1;F) + \overline{N}(r,1;G) - \overline{N}_0(r,0;F') \\ & - \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{23}$$

Using Lemmas (4)and (5), (21) and (22) we can get

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1;G) \le\ & N(r,1;F| = 1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}(r,1;G) \\ \le\ & N(r,1;F| = 1) + N(r,1;G) - \overline{N}_L(r,1;F) - \overline{N}_L(r,1;G) + \overline{N}_{F>2}(r,1;G) \\ \le\ & \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2) + \overline{N}(r,\infty;F) + \overline{N}_*(r,1;F,G) + T(r,G) \\ & - m(r,1;G) + O(1) + \frac{1}{2}\overline{N}(r,\infty;F) - \overline{N}_L(r,1;F) - \overline{N}_L(r,1;G) + \frac{1}{2}\overline{N}(r,0;F) \\ & + N_0(r,0;F') + N_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{24}$$

Combining (23) and (24), we can obtain

$$\begin{aligned} T(r,F) \le\ & \frac{7}{2}\overline{N}(r,\infty;F) + N_2(r,0;F) + N_2(r,0;G) + \frac{1}{2}\overline{N}(r,0;F) + S(r,f) \\ \le\ & \frac{7}{2}\overline{N}(r,\infty;F) + \frac{3}{2}N_2(r,0;F) + N_2(r,0;G) + S(r,f). \end{aligned}$$

By the definition of F, G and (6), we have

$$\begin{aligned} T(r,f) \le\ & \frac{7}{2}\overline{N}(r,\infty;F) + \frac{3}{2}N_2(r,0;F) + N_2(r,0;H) + S(r,f) \\ \le\ & \frac{7}{2}\overline{N}(r,\infty;f) + \frac{3}{2}N_2(r,0;f) + (\Gamma - d)\overline{N}(r,\infty;f) + N_{2+\Gamma-d}(r,0;f^d) + S(r,f). \end{aligned}$$

So

$$\left(\frac{7}{2} + \Gamma - d\right)\Theta(\infty,f) + \frac{3}{2}\delta_2(0,f) + d\delta_{2+\Gamma-d}(0,f^d) \le \Gamma + 4,$$

which contradicts the assumption (3) of Theorem 6.

Subcase 2.3. $k = 0$. We know that F, G share "$(1,0)$", hence we have

$$\begin{aligned} N(r,\infty;H) \le\ & \overline{N}(r,\infty;F) + \overline{N}(r,1;F| \ge 2) + \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2) \\ & + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f), \end{aligned} \tag{25}$$

and

$$N_E^{1)}(r,1;F) = N_E^{1)}(r,1;G) + S(r,f),\ N_E^{(2}(r,1;F) = N_E^{(2}(r,1;G) + S(r,f),$$

$$N_E^{1)}(r,1;F) \le N(r,\infty;H) + S(r,f). \tag{26}$$

Using Lemmas 6-8 and (25) and (26), we get

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\le \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}(r,1;G)\\ &\le N_E^{1)}(r,1;F)+N(r,1;G)-\overline{N}_L(r,1;G)+\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;G)\\ &\le \overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+\overline{N}(r,\infty;F)+\overline{N}_*(r,1;F,G)+T(r,G)\\ &-m(r,1;G)+O(1)-\overline{N}_L(r,1;G)+\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;G)\\ &+N_0(r,0;F^{'})+N_0(r,0;G^{'})+S(r,F)+S(r,G).\end{aligned} \tag{27}$$

Combining (23) and (27) and by Lemma 2, we can obtain

$$\begin{aligned}T(r,f) &\le 6\overline{N}(r,\infty;F)+N_2(r,0;F)+2\overline{N}(r,0;F)+N_2(r,0;G)+\overline{N}(r,0;G)+S(r,f)\\ &\le (6+2\Gamma-2d)\overline{N}(r,\infty;f)++N_2(r,0;f)+2\overline{N}(r,0;f)+N_{2+\Gamma-d}(r,0;f^d)\\ &+N_{1+\Gamma-d}(r,0;f^d)+S(r,f).\end{aligned}$$

So

$$(6+2\Gamma-2d)\Theta(\infty,f)+\delta_2(0,f)+2\Theta(0,f)+d\delta_{1+\Gamma-d}(0,f^d)+d\delta_{2+\Gamma-d}(0,f^d)\le 2\Gamma+9,$$

which contradicts the assumption (4) of Theorem 6. Now the proof has been completed.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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