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An unnoticed way of obtaining the binet form for fibonacci numbers

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Abstract: In this article, we drive the Binet form to Fibonacci and generalized Fibonacci numbers by applying the Laplace transform method that has not got enough credit for solutions of difference equations compare to other avaiable methods so far.

Keywords: Fibonacci numbers, difference equations, Laplace transform.

1 Introduction

The sequence F_n of the Fibonacci numbers is defined by the recurrence relation

$$F_n = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2. \end{cases}$$

A compact formula, known the Binet form, for the Fibonacci numbers is given by

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n}),$$

where $\phi = \frac{1+\sqrt{5}}{2}$, for the name of ϕ see [1]. There are several analytic proofs obtaining the Binet form using the method of matrices [2,3,4,5], the method of generating functions [6], the method of complex residues [7], for a discussion of more traditional methods [8] and the method of difference equation [9,10,11].

It is important to note that Binet (1786–1856) probably was not the first to figure out this. Leonhard Euler (1707–1783), Daniel Bernoulli (1700–1782), Abraham de Moivre (1667–1754) and also Gabriel Lamé (1795–1870) after whom the sequence is sometimes called, worked the same formula out more than a century earlier. For further reading, see [13,14,15,16,17,18,19,20].

Let $Y(t)$ be a real or complex function for $t > 0$ and s is a real or complex parameter. Then the Laplace transform of $Y(t)$ is defined by

$$\mathscr{L}\{Y(t)\} = y(s) = \int_0^\infty Y(t)e^{-st}dt,$$

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we assume that this integral exists.

$$\text{If} \quad \mathscr{L}\{Y(t)\} = y(s) \quad \text{and} \quad G(t) = \begin{cases} Y(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases} \quad \text{then} \quad \mathscr{L}\{G(t)\} = e^{-as}y(s). \tag{1}$$

For the Laplace transform of $Y(t)$ and its properties, see [21,22].

2 Main results

Lemma 1. *Let $Y(t) = a_n$ for $n \leq t < n+1$ where $n = 0,1,2,3,\cdots$. Then*

$$\mathscr{L}\{Y(t+1)\} = e^s y(s) - \frac{a_0 e^s(1-e^{-s})}{s}.$$

Proof. Applying $Y(t) = a_0$ for $0 \leq t < 1$. Then

$$\begin{aligned}
\mathscr{L}\{Y(t+1)\} &= \int_0^\infty e^{-st} Y(t+1)dt \\
&= e^s \int_1^\infty e^{-su} Y(u)du \\
&= e^s \left[\int_0^\infty e^{-su} Y(u)du - \int_0^1 e^{-su} Y(u)du\right] \\
&= e^s y(s) - e^s \int_0^1 e^{-su} a_0 du \\
&= e^s y(s) - \frac{a_0 e^s(1-e^{-s})}{s}.
\end{aligned}$$

Lemma 2. *Let $Y(t) = a_n$ for $n \leq t < n+1$ where $n = 0,1,2,3,\cdots$. Then*

$$\mathscr{L}\{Y(t+2)\} = e^{2s} y(s) - \frac{e^s(1-e^{-s})(a_0 e^s + a_1)}{s}.$$

Proof. Applying $Y(t) = a_0$ for $0 \leq t < 1$ and $Y(t) = a_1$ for $1 \leq t < 2$. Then

$$\begin{aligned}
\mathscr{L}\{Y(t+2)\} &= \int_0^\infty e^{-st} Y(t+2)dt \\
&= e^{2s} \int_2^\infty e^{-su} Y(u)du \\
&= e^{2s} \int_0^\infty e^{-su} Y(u)du - e^{2s} \int_0^1 e^{-su} Y(u)du - e^{2s} \int_1^2 e^{-su} Y(u)du \\
&= e^{2s} y(s) - e^{2s} \int_0^1 e^{-su} a_0 du - e^{2s} \int_1^2 e^{-su} a_1 du \\
&= e^{2s} y(s) - \frac{a_0 e^{2s}(1-e^{-s})}{s} - \frac{a_1 e^{2s}(e^{-s}-e^{-2s})}{s} \\
&= e^{2s} y(s) - \frac{e^s(1-e^{-s})(a_0 e^s + a_1)}{s}.
\end{aligned}$$

For general version of Lemma 1 and Lemma 2, see [12].

Lemma 3. *Let* $Y(t) = r^n$ *for* $n \leq t < n+1$ *where* $n = 0,1,2,3,\cdots$. *Then*

$$\mathscr{L}\{Y(t)\} = \frac{1-e^{-s}}{s(1-re^{-s})}.$$

Proof. Geometric series is employed when necessary.

$$\begin{aligned}\mathscr{L}\{r^n\} &= \int_0^1 e^{-st} r^0 dt + \int_1^2 e^{-st} r^1 dt + \int_2^3 e^{-st} r^2 dt + \cdots \\ &= \frac{1-e^{-s}}{s} + r\frac{e^{-s}-e^{-2s}}{s} + r^2\frac{e^{-2s}-e^{-3s}}{s} + \cdots \\ &= \frac{1-e^{-s}}{s}\left(1+re^{-s}+r^2e^{-2s}+\cdots\right) \\ &= \frac{1-e^{-s}}{s}\frac{1}{1-re^{-s}} \\ &= \frac{1-e^{-s}}{s(1-re^{-s})}.\end{aligned}$$

Theorem 1. *Let* $F_0 = 0$ *and* $F_1 = 1$. *If*

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0, \tag{2}$$

then

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right). \tag{3}$$

Proof. By applying Lemma 1, Lemma 2 and taking the Laplace transform of both sides of (2) one obtains.

$$\left(e^{2s}-e^s-1\right)y(s) = \frac{e^s(1-e^{-s})}{s}.$$

Then

$$\begin{aligned}y(s) &= \frac{e^s(1-e^{-s})}{\sqrt{5}s}\left(\frac{1}{e^s-\frac{1+\sqrt{5}}{2}} - \frac{1}{e^s-\frac{1-\sqrt{5}}{2}}\right) \\ &= \frac{(1-e^{-s})}{\sqrt{5}s}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}e^{-s}} - \frac{1}{1-\frac{1-\sqrt{5}}{2}e^{-s}}\right).\end{aligned}$$

Applying Lemma 3 and the inverse Laplace transform one obtains

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Theorem 2. *Let* $F_0 = p$ *and* $F_1 = p+q$ *if*

$$F_{n+2} = F_n + F_{n+1}, \quad n \geq 0, \tag{4}$$

then either

$$F_n=\frac{F_1-F_0}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)+\frac{F_0}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

or

$$F_n=\frac{F_1+F_0(\sqrt{5}-1)}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

For generalized Fibonacci numbers, see [23].

Proof. By applying Lemma 1, Lemma 2 and taking the Laplace transform of both sides of (4) one obtains

$$\left(e^{2s}-e^s-1\right)y(s)=\frac{e^s(1-e^{-s})(F_0e^s+F_1)}{s}-\frac{F_0e^s(1-e^{-s})}{s}.$$

Then

$$\begin{aligned}y(s)&=(F_1-F_0)\frac{e^s(1-e^{-s})}{\sqrt{5}s}\left(\frac{1}{e^s-\frac{1+\sqrt{5}}{2}}-\frac{1}{e^s-\frac{1-\sqrt{5}}{2}}\right)+F_0\frac{e^{2s}(1-e^{-s})}{\sqrt{5}s}\left(\frac{1}{e^s-\frac{1+\sqrt{5}}{2}}-\frac{1}{e^s-\frac{1-\sqrt{5}}{2}}\right)\\&=(F_1-F_0)\frac{1-e^{-s}}{\sqrt{5}s}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}e^{-s}}-\frac{1}{1-\frac{1-\sqrt{5}}{2}e^{-s}}\right)+F_0\frac{e^s(1-e^{-s})}{\sqrt{5}s}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}e^{-s}}-\frac{1}{1-\frac{1-\sqrt{5}}{2}e^{-s}}\right).\end{aligned}$$

Now using Lemma 3, (1) and taking the inverse Laplace transform one obtains either

$$F_n=\frac{F_1-F_0}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)+\frac{F_0}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

or

$$F_n=\frac{F_1+F_0(\sqrt{5}-1)}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Corollary 1. *If $F_0=0$ and $F_1=1$ then one obtains Theorem 1.*

Corollary 2. *If $F_0=0$ and $F_1=1$ then one obtains Lucas numbers, denoted by L_n, $L_n=2F_{n+1}-F_n$, where F_n is in the form (3).*

For Lucas numbers and its properties, see [13, 15, 16, 17, 18].

3 Conclusion

The Laplace transform method are applied to generalized Fibonacci sequence to obtain the Binet form. The result obtained in [12] is applied to Pell and Tribonacci numbers. With this article, we aim to bring up the importance of Laplace transform method for difference equations to interested readers.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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