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Examination of dependency on initial conditions of solution of a mixed problem with periodic boundary condition for a class of quasi-linear Euler-Bernoulli equation

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Abstract: In this paper, the dependency on initial conditions of the weak generalized solution which existence and uniqueness are proved by us [5] of a mixed problem with periodic boundary condition for a quasi-linear Euler-Bernoulli equation is examined. In order to this examination, we consider a second mixed problem with another initial conditions in addition to the problem and as in [5] the weak generalized solutions of both of problems are expressed as a Fourier series with undetermined variable coefficients, and a system of non-linear infinite integral equations for these coefficients are obtained. Then estimating the difference between the solutions of these systems on the Banach space B_T, we determine the dependency of the solution of mentioned problem on its initial conditions.

Keywords: Quasi-linear partial differential equation, Mixed problem, Weak generalized solution, Dependency on initial conditions, Fourier method, Periodic boundary condition.

1 Introduction

Vibration problems of beams which are composed of various materials and has different shaped in different environments reduced to Euler-Bernoulli equations. In fact, there is a Euler-Bernoulli beam theory in literature. This theory is based on Euler-Bernoulli equation. Using these equations, the different problems of construction, machinery, aircraft and defense industries, can be solved. In addition, conservation of momentum and continuity of the fluid flow are defined by Euler-Bernoulli equations. These equations are frequently used in fluid mechanics. At the same time, Euler-Bernoulli equations are used to define the coefficients of rise, drag and thrust on the wings of the wind vane machines and ejection angle. Hence, in the aforementioned applications depending on the examined object, homogeneous and non-homogeneous, quasi-linear of the Euler-Bernoulli equation with different initial and boundary conditions are studied for various problems. Generally, since initial data are obtained by experimentally in these problems, one encountered several errors. Therefore, one can say that the question of '*how the errors influence the solution*' is the basic problem. With this motivation, we examine the effects of small changes of initial conditions on the solution of a mixed problem with periodic boundary condition for a class of quasi-linear Euler-Bernoulli equation.

2 Establishing the problem

After summarizing the technical aspects of the problem as mentioned above, we suppose that the initial conditions are usually determined by experiment. In present study, the dependency on initial conditions of the weak generalized solution

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$u(t,x,\varepsilon)$ of the following mixed problem which has been examined in [5] with periodic boundary condition:

$$\frac{\partial^2 u}{\partial t^2} - \varepsilon b^4 \frac{\partial^4 u}{\partial t^2 \partial x^2} + a^2 \frac{\partial^4 u}{\partial x^4} = f(t,x,u), \ \ (t,x) \in D\{0 < t \leq T, \ 0 < x < \pi\}, \tag{1}$$

$$u(0,x,\varepsilon) = \varphi(x,\varepsilon), \ u_t(0,x,\varepsilon) = \psi(x,\varepsilon), \ (0 \leq x \leq \pi, \ 0 \leq \varepsilon \leq \varepsilon_0), \tag{2}$$

$$u(t,0,\varepsilon) = u(t,\pi,\varepsilon), \ u_x(t,0,\varepsilon) = u_x(t,\pi,\varepsilon),$$
$$u_{x^2}(t,0,\varepsilon) = u_{x^2}(t,\pi,\varepsilon), \ u_{x^3}(t,0,\varepsilon) = u_{x^3}(t,\pi,\varepsilon), \ (0 \leq t \leq T, \ 0 \leq \varepsilon \leq \varepsilon_0) \tag{3}$$

is examined, or other words we search the effects of small changes of initial conditions on the solution. For this aim, we also consider the following mixed problem:

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \varepsilon b^4 \frac{\partial^4 \tilde{u}}{\partial t^2 \partial x^2} + a^2 \frac{\partial^4 \tilde{u}}{\partial x^4} = f(t,x,\tilde{u}) + f_0(t,x), \ (t,x) \in D\{0 < t < T, \ 0 < x < \pi\}, \tag{4}$$

$$\tilde{u}(0,x,\varepsilon) = \tilde{\varphi}(x,\varepsilon), \ \tilde{u}_t(0,x,\varepsilon) = \tilde{\psi}(x,\varepsilon), \ (0 \leq x \leq \pi, \ 0 \leq \varepsilon \leq \varepsilon_0), \tag{5}$$

$$\tilde{u}(t,0,\varepsilon) = \tilde{u}(t,\pi,\varepsilon), \ \tilde{u}_x(t,0,\varepsilon) = \tilde{u}_x(t,\pi,\varepsilon),$$
$$\tilde{u}_{x^2}(t,0,\varepsilon) = \tilde{u}_{x^2}(t,\pi,\varepsilon), \ \tilde{u}_{x^3}(t,0,\varepsilon) = \tilde{u}_{x^3}(t,\pi,\varepsilon), \ (0 \leq t \leq T, \ 0 \leq \varepsilon \leq \varepsilon_0), \tag{6}$$

where $\varphi(x,\varepsilon), \psi(x,\varepsilon), \tilde{\varphi}(x,\varepsilon), \tilde{\psi}(x,\varepsilon), f(t,x,u)$ and $f_0(t,x)$ are given functions are known functions which have the required properties in their domain.

Definition 1. *The function $v(t,x) \in C(\bar{D})$ is called* test function *if it has continuous partial derivatives of order contained in equation (1) and satisfies both following conditions*

$$v(T,x) = v_t(T,x) = v_{x^2}(T,x) = v_{x^2t}(T,x) = 0$$

and the boundary condition (3).

We can give the following definition as in G. I. Chandirov's [1].

Definition 2. *The function $u(t,x,\varepsilon) \in C\{\bar{D} \times [0,\varepsilon_0]\}$ satisfying the integral identity*

$$\int_0^T \int_0^\pi \left\{ u \left[\frac{\partial^2 v}{\partial t^2} - \varepsilon b^4 \frac{\partial^4 v}{\partial t^2 x^2} + a^2 \frac{\partial^4 v}{\partial x^4} \right] - f(t,x,u)v \right\} dx\,dt + \int_0^\pi \varphi(x)[v_t(0,x) - \varepsilon b^2 v_{x^2t}(0,x)]\,dx - \int_0^\pi \psi(x)[v(0,x) - \varepsilon b^2 v_{x^2}(0,x)]\,dx = 0 \tag{7}$$

for an arbitrary test function $v(t,x)$ is called weak generalized solution *of problem (1)-(3).*

The set

$$\{\bar{u}(t,\varepsilon)\} = \{\frac{1}{2}u_0(t,\varepsilon), u_{c1}(t,\varepsilon), u_{s1}(t,\varepsilon), \ldots, u_{ck}(t,\varepsilon), u_{sk}(t,\varepsilon), \ldots\}$$

of continuous functions on $[0,T]$ for all $\varepsilon \in [0,\varepsilon_0]$ satisfying the condition

$$\frac{1}{2} \max_{t\in[0,T]} |u_0(t,\varepsilon)| + \sum_{k=1}^{\infty} \left[\max_{t\in[0,T]} |u_{ck}(t,\varepsilon)| + \max_{t\in[0,T]} |u_{sk}(t,\varepsilon)| \right] < \infty.$$

denote by B_T.

Let

$$\|\bar{u}(t,\varepsilon)\|_{B_T} = \frac{1}{2} \max_{t\in[0,T]} |u_0(t,\varepsilon)| + \sum_{k=1}^{\infty} \left[\max_{t\in[0,T]} |u_{ck}(t,\varepsilon)| + \max_{t\in[0,T]} |u_{sk}(t,\varepsilon)| \right]$$

be the norm in B_T. It can be shown that B_T is Banach space.

The following theorem concerning the existence and uniqueness of the weak generalized solution of the problem (1)-(3) is true [5].

Theorem 1. *Suppose the following conditions satisfy*

(a) $f(t,x,u)$ *is continuous respect to all arguments on* $D\times(-\infty,\infty)$,
(b) $|f(t,x,u)-f(t,x,v)|\le b(t,x)|u-v|$ *where* $b(t,x)\in L_2(D),\ b(t,x)>0$,
(c) $f(t,x,0)\in C(D)$,
(d) *The functions* $\varphi(x,\varepsilon),\ \psi(x,\varepsilon)$ *with* $\varphi(x,\varepsilon)\in C^1\{[0,\pi]\times[0,\varepsilon_0]\},\psi(x)\in C\{[0,\pi]\times[0,\varepsilon_0]\}$ *satisfy the following conditions for all* $\varepsilon\in[0,\varepsilon_0]$;

$$\varphi(0,\varepsilon)=\varphi(\pi,\varepsilon),\ \varphi'(0,\varepsilon)=\varphi'(\pi,\varepsilon),\ \psi(0,\varepsilon)=\psi(\pi,\varepsilon).$$

In this case, there is an unique weak generalized solution for the problem (1)-(3) in $D\times[0,\varepsilon_0]$.

Definition 3. *[**Gronwall Inequality**] In* $[0,T]$, *let* $a(t)$ *be a non-negative, continues function,* $b(t)$ *and* $c(t)$ *be non-negative, integrable functions,* $f(t)$ *be a bounded function and if the following inequality;*

$$a(t)\le\int_0^t[a(\tau)b(\tau)+c(\tau)]\ d\tau+f(t)$$

then

$$\max_{0\le t\le T}a(t)\le\Big[\int_0^\tau c(\tau)\,d\tau+\sup|f(\tau)|\Big]\exp\int_0^\tau b(\tau)\,d\tau.$$

Definition 4. *[**Bessel Inequality**] Let the function* $f(x)$ *satisfy the conditions of Dirichlet theorem on* $[0,\pi]$. *For the coefficients of its Fourier series which can be written by the functions* $1,\cos 2kx,\sin 2kx\ (k=\overline{1,\infty})$, *the following inequality*

$$\frac{f_0^2}{2}+\sum_{k=1}^{\infty}(f_{ck}^2+f_{sk}^2)\le\frac{2}{\pi}\int_0^\pi f^2(x)\,dx$$

is true. Here Fourier coefficients are determined as follows

$$f_0=\frac{2}{\pi}\int_0^\pi f(x)\,dx,\quad f_{ck}=\frac{2}{\pi}\int_0^\pi f(x)\cos 2kx\,dx,\quad f_{sk}=\frac{2}{\pi}\int_0^\pi f(x)\sin 2kx\,dx,\ (k=\overline{1,\infty}).$$

3 Solution

To examine the dependency of the weak generalized solution on initial conditions, let us look for the generalized solution of (1)-(3) as formally in the following form

$$u(t,x,\varepsilon)=\frac{1}{2}u_0(t,\varepsilon)+\sum_{k=1}^{\infty}[u_{ck}(t,\varepsilon)\cos 2kx+u_{sk}(t,\varepsilon)\sin 2kx] \tag{8}$$

for following unknown functions $u_0(t,\varepsilon),u_{ck}(t,\varepsilon),u_{sk}(t,\varepsilon),\ (k=\overline{1,\infty})$. In order to determinate unknowns using equation (8) formally in problem (1)-(3), we get the infinite system of integral equations;

$$\begin{aligned}
u_0(t,\varepsilon)&=\varphi_0+\psi_0 t+\frac{2}{\pi}\int_0^t\int_0^\pi(t-\tau)f\{W_p\}\ d\xi\,d\tau,\\
u_{ck}(t,\varepsilon)&=\varphi_{ck}\cos\alpha_k t+\frac{\psi_{ck}}{\alpha_k}\sin\alpha_k t+\frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\{W_p\}\times\cos 2k\xi\sin\alpha_k(t-\tau)\ d\xi\,d\tau,\\
u_{sk}(t,\varepsilon)&=\varphi_{sk}\cos\alpha_k t+\frac{\psi_{sk}}{\alpha_k}\sin\alpha_k t+\frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\{W_p\}\times\sin 2k\xi\sin\alpha_k(t-\tau)\ d\xi\,d\tau,\\
\alpha_k&=\frac{a(2k)^2}{\sqrt{1+\varepsilon(2kb)^2}},\ k=\overline{1,\infty}.
\end{aligned} \tag{9}$$

where $W_p = \tau,\xi,\frac{1}{2}u_0(\tau,\varepsilon)+\sum_{n=1}^{\infty}[u_{cn}(\tau,\varepsilon)\cos 2n\xi + u_{sn}(\tau,\varepsilon)\sin 2n\xi]$. In similar way let look for the generalized solution of (4)-(6) in the form

$$\tilde{u}(t,x,\varepsilon)=\frac{1}{2}\tilde{u}_0(t,\varepsilon)+\sum_{k=1}^{\infty}[\tilde{u}_{ck}(t,\varepsilon)\cos 2kx+\tilde{u}_{sk}(t,\varepsilon)\sin 2kx] \tag{10}$$

and we get the following infinite system of integral equations for the unknown functions

$$\begin{aligned}
\tilde{u}_0(t,\varepsilon) &= \tilde{\varphi}_0+\tilde{\psi}_0 t+\frac{2}{\pi}\int_0^t\int_0^{\pi}(t-\tau)f\{\tau,\xi,\frac{1}{2}\tilde{u}_0(\tau,\varepsilon)+\sum_{n=1}^{\infty}[\tilde{u}_{cn}(\tau,\varepsilon)\cos 2n\xi+\tilde{u}_{sn}(\tau,\varepsilon)\sin 2n\xi]\}\ d\xi d\tau+\\
&\frac{2}{\pi}\int_0^t\int_0^{\pi}(t-\tau)f_0(\tau,\xi)\,d\tau d\xi,\\
\tilde{u}_{ck}(t,\varepsilon) &= \tilde{\varphi}_{ck}\cos\alpha_k t+\frac{\tilde{\psi}_{ck}}{\alpha_k}\sin\alpha_k t+\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}f\{\tau,\xi,\frac{1}{2}\tilde{u}_0(\tau,\varepsilon)+\sum_{n=1}^{\infty}[\tilde{u}_{cn}(\tau,\varepsilon)\cos 2n\xi+\tilde{u}_{sn}(\tau,\varepsilon)\sin 2n\xi]\}\times\\
&\cos 2k\xi\sin\alpha_k(t-\tau)\ d\xi\, d\tau+\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}f_0(\tau,\xi)\cos 2k\xi\sin\alpha_k(t-\tau)\,d\tau d\xi,\\
\tilde{u}_{sk}(t,\varepsilon) &= \tilde{\varphi}_{sk}\cos\alpha_k t+\frac{\tilde{\psi}_{sk}}{\alpha_k}\sin\alpha_k t+\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}f\{\tau,\xi,\frac{1}{2}\tilde{u}_0(\tau,\varepsilon)+\sum_{n=1}^{\infty}[\tilde{u}_{cn}(\tau,\varepsilon)\cos 2n\xi+\tilde{u}_{sn}(\tau,\varepsilon)\sin 2n\xi]\}\times\\
&\sin 2k\xi\sin\alpha_k(t-\tau)\ d\xi\, d\tau+\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}f_0(\tau,\xi)\sin 2k\xi\sin\alpha_k(t-\tau)\,d\tau d\xi.
\end{aligned} \tag{11}$$

By the conditions of theorem (1), for both of (9) and (11) the existence and uniqueness of the solutions of infinite system of integral equations are proved [5]. For simplicity take

$$Au(\tau,\xi,\varepsilon)=\frac{1}{2}u_0(\tau,\varepsilon)+\sum_{n=1}^{\infty}[u_{cn}(\tau,\varepsilon)\cos 2n\xi+u_{sn}(\tau,\varepsilon)\sin 2n\xi]$$

and

$$A\tilde{u}(\tau,\xi,\varepsilon)=\frac{1}{2}\tilde{u}_0(\tau,\varepsilon)+\sum_{n=1}^{\infty}[\tilde{u}_{cn}(\tau,\varepsilon)\cos 2n\xi+\tilde{u}_{sn}(\tau,\varepsilon)\sin 2n\xi].$$

Let us write the difference between the systems (9) and (11) as following

$$\begin{aligned}
u_0(t,\varepsilon)-\tilde{u}_0(t,\varepsilon) =&(\varphi_0-\tilde{\varphi}_0)+(\psi_0-\tilde{\psi}_0)t+\frac{2}{\pi}\int_0^t\int_0^{\pi}(t-\tau)\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\ d\xi d\tau-\\
&\frac{2}{\pi}\int_0^t\int_0^{\pi}(t-\tau)f_0(\tau,\xi)\,d\tau d\xi,\\
u_{ck}(t,\varepsilon)-\tilde{u}_{ck}(t,\varepsilon) =&(\varphi-\tilde{\varphi}_{ck})\cos\alpha_k t+\frac{\psi-\tilde{\psi}_{ck}}{\alpha_k}\sin\alpha_k t+\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\times\\
&\cos 2k\xi\sin\alpha_k(t-\tau)\ d\xi\, d\tau-\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}f_0(\tau,\xi)\cos 2k\xi\sin\alpha_k(t-\tau)\,d\tau d\xi,\\
u_{sk}-\tilde{u}_{sk}(t,\varepsilon) =&(\varphi-\tilde{\varphi}_{sk})\cos\alpha_k t+\frac{\varphi-\tilde{\psi}_{sk}}{\alpha_k}\sin\alpha_k t+\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\times\\
&\sin 2k\xi\sin\alpha_k(t-\tau)\ d\xi\, d\tau-\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}f_0(\tau,\xi)\sin 2k\xi\sin\alpha_k(t-\tau)\,d\tau d\xi.
\end{aligned}$$

Grouping in obtained differences and we write the following sum

$$\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)=\frac{1}{2}[u_0(t,\varepsilon)-\tilde{u}_0(t,\varepsilon)]+\sum_{k=1}^{\infty}\{[u_{ck}(t,\varepsilon)-\tilde{u}_{ck}(t,\varepsilon)]+[u_{sk}(t,\varepsilon)-\tilde{u}_{sk}(t,\varepsilon)]\}=$$
$$\frac{1}{2}(\varphi_0-\tilde{\varphi}_0)+\sum_{k=1}^{\infty}[(\varphi_{ck}-\tilde{\varphi}_{ck})+(\varphi_{sk}-\tilde{\varphi}_{sk})]+\frac{1}{2}(\psi_0-\tilde{\psi}_0)t+\sum_{k=1}^{\infty}\frac{1}{\alpha_k}[(\psi_{ck}-\tilde{\psi}_{ck})+(\psi_{sk}-\tilde{\psi}_{sk})]+$$
$$\frac{1}{2}\frac{2}{\pi}\int_0^t\int_0^\pi(t-\tau)\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}d\xi\,d\tau+$$
$$\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\cos 2k\xi\,d\xi+$$
$$\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\sin 2k\xi\,d\xi\Big)\sin\alpha_k(t-\tau)\,d\tau-$$
$$\frac{2}{\pi}\int_0^t\int_0^\pi(t-\tau)f_0(\tau,\xi)\,d\xi\,d\tau-\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\cos 2k\xi\,d\xi+\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\sin 2k\xi\,d\xi\Big)\sin\alpha_k(t-\tau)\,d\tau.$$

Taking into account that the absolute values of both of sides and $0\le t-\tau\le T$, then applying the Cauchy inequality with respect to τ to integrals on the right hand side, we get

$$|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|\le\frac{1}{2}|\varphi_0-\tilde{\varphi}_0|+\sum_{k=1}^{\infty}[|\varphi_{ck}-\tilde{\varphi}_{ck}|+|\varphi_{sk}-\tilde{\varphi}_{sk}|]+\frac{T}{2}|\psi_0-\tilde{\psi}_0|+\sum_{k=1}^{\infty}\frac{1}{\alpha_k}[|\psi_{ck}-\tilde{\psi}_{ck}|+|\psi_{sk}-\tilde{\psi}_{sk}|]+$$
$$\frac{\sqrt{T}}{2}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}d\xi\right)^2d\tau\right]^{1/2}+$$
$$\sqrt{T}\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\Big[\int_0^t\Big(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\cos k\xi\,d\xi+$$
$$\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\sin 2k\xi\,d\xi\Big)^2d\tau\Big]^{1/2}+$$
$$\frac{\sqrt{T}}{2}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\,d\xi\right)^2d\tau\right]^{1/2}+\sqrt{T}\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\cos k\xi\,d\xi+\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\sin 2k\xi\,d\xi\right)^2d\tau\right]^{1/2}$$

Applying Hölder's inequality to the second and third sums, we have that

$$|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|\le M_\varepsilon+\frac{\sqrt{T}}{2}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}d\xi\right)^2d\tau\right]^{1/2}+$$
$$\sqrt{T}\left(\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\right)^{1/2}\Big\{\sum_{k=1}^{\infty}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\cos 2k\xi\,d\xi+$$
$$\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\sin 2k\xi\,d\xi\Big)^2d\tau\Big\}^{1/2}+\frac{\sqrt{T}}{2}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\,d\xi\right)^2d\tau\right]^{1/2}+$$
$$\sqrt{T}\left(\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\right)^{1/2}\left\{\sum_{k=1}^{\infty}\int_0^t\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\cos k\xi\,d\xi+\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\sin 2k\xi\,d\xi\right)^2d\tau\right\}^{1/2}$$

where

$$M_\varepsilon(\varphi,\psi)=M(\|\varphi(x,\varepsilon)-\tilde{\varphi}(x,\varepsilon)\|,\|\psi(x,\varepsilon)-\tilde{\psi}(x,\varepsilon)\|)$$
$$=\frac{1}{2}|\varphi_0-\tilde{\varphi}_0|+\sum_{k=1}^{\infty}[|\varphi_{ck}-\tilde{\varphi}_{ck}|+|\varphi_{sk}-\tilde{\varphi}_{sk}|]+\frac{T}{2}|\psi_0-\tilde{\psi}_0|+\sum_{k=1}^{\infty}\frac{1}{\alpha_k}[|\psi_{ck}-\tilde{\psi}_{ck}|+|\psi_{sk}-\tilde{\psi}_{sk}|].$$

Taking square to both of sides and using following inequality

$$(\alpha_1+\alpha_2+\ldots+\alpha_n)^2 \le n(\alpha_1^2+\alpha_2^2+\ldots+\alpha_n^2)$$

we get

$$|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|^2 \le 5\Big\{M_s^2(\varphi,\psi)+\frac{T}{4}\int_0^t\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\,d\xi\right)^2 d\tau+$$
$$T\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\sum_{k=1}^{\infty}\int_0^t\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\cos k\xi\,d\xi+\right.$$
$$\left.\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\sin 2k\xi\,d\xi\right)^2 d\tau+\frac{T}{4}\int_0^t\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\,d\xi\right)^2 d\tau+$$
$$T\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\sum_{k=1}^{\infty}\int_0^t\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\cos k\xi\,d\xi+\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\sin 2k\xi\,d\xi\right)^2 d\tau\Big\}.$$

Applying the inequality $(a+b)^2\le 2(a^2+b^2)$ to third and fifth sums in parentheses on the right side and using integrable term by term within the required conditions, let us do following modifications:

$$|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|^2 \le 5\Big\{M_s^2(\varphi,\psi)+\frac{T}{4}\int_0^t\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\,d\xi\right)^2 d\tau+$$
$$2T\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\int_0^t\sum_{k=1}^{\infty}\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\cos k\xi\,d\xi\right)^2+$$
$$2T\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\int_0^t\sum_{k=1}^{\infty}\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\sin k\xi\,d\xi\right)^2+\frac{T}{4}\int_0^t\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\,d\xi\right)^2 d\tau+$$
$$2T\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\int_0^t\sum_{0}^{\infty}\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\cos k\xi\,d\xi\right)^2 d\tau+2T\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\int_0^t\sum_{0}^{\infty}\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\sin 2k\xi\,d\xi\right)^2 d\tau\Big\}.$$

Supposing

$$M_T=\max\left\{\frac{5T}{2},10T\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\right\},$$

the last inequality can be written as following

$$|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|^2 \le 5M_\varepsilon^2(\varphi,\psi)+M_T\Big\{\int_0^t\frac{1}{2}\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\,d\xi\right)^2+$$
$$\sum_{k=1}^{\infty}\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\cos k\xi\,d\xi\right)^2+$$
$$\sum_{k=1}^{\infty}\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}\sin k\xi\,d\xi\right)^2\Big\}d\tau+$$
$$M_T\int_0^t\Big\{\frac{1}{2}\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\,d\xi\right)^2+\sum_{k=1}^{\infty}\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\cos 2k\xi\,d\xi\right)^2+\sum_{k=1}^{\infty}\left(\frac{2}{\pi}\int_0^\pi f_0(\tau,\xi)\sin 2k\xi\,d\xi\right)^2\Big\}d\tau.$$

Applying Bessel's inequality to the integrals in second and third sums on the right side, we obtain that

$$|\bar{u}(t,\varepsilon)-\tilde{u}(t,\varepsilon)|^2 \le 5M_\varepsilon^2(\varphi,\psi)+M_T\int_0^t\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\}^2\,d\xi\,d\tau$$
$$+M_T\int_0^t\frac{2}{\pi}\int_0^\pi f_0^2(\tau,\xi)\,d\xi\,d\tau.$$

Here, supposing Lipschitz condition $|f(t,x,u)-f(t,x,\tilde{u})| \leq b(t,x)|u-\tilde{u}|$ is satisfied, and taking into account the inequality $|Au(\tau,\xi,\varepsilon)-A\tilde{u}(\tau,\xi,\varepsilon)| \leq |\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|$ we have

$$|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|^2 \leq 5M_\varepsilon^2(\varphi,\psi)+M_T\int_0^t \frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)|Au(\tau,\xi,\varepsilon)-A\tilde{u}(\tau,\xi,\varepsilon)|^2\,d\xi\,d\tau+M_T\int_0^t \frac{2}{\pi}\int_0^\pi f_0^2(\tau,\xi)\,d\xi\,d\tau$$

or equivalently

$$|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|^2 \leq 5M_\varepsilon^2(\varphi,\psi)+M_T\int_0^t \frac{2}{\pi}\int_0^\pi \left[b^2(\tau,\xi)|\bar{u}(\tau,\varepsilon)-\bar{\tilde{u}}(\tau,\varepsilon)|^2+f_0^2(\tau,\xi)\right]d\xi\,d\tau$$

Finally applying Gronwall inequality to the last inequality, we have

$$\max_{0\leq t\leq T}|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|^2 \leq \left[\int_0^T\left(\frac{2}{\pi}\int_0^\pi f_0^2(\tau,\xi)\,d\xi\right)d\tau+5\sup M_\varepsilon^2(\varphi,\psi)\right]\exp\int_0^T\frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)\,d\xi\,d\tau.$$

for $D\{[0,T]\times[0,\pi]\}$ where for all $\varepsilon\in[0,\varepsilon_0]$

$$\|\bar{u}(t,x,\varepsilon)-\bar{\tilde{u}}(t,x,\varepsilon)\|_{C(D)} \leq \max_{0\leq t\leq T}|\bar{u}(t,\varepsilon)-\bar{\tilde{u}}(t,\varepsilon)|$$

then it can be written as from the last inequality

$$\|\bar{u}(t,x,\varepsilon)-\bar{\tilde{u}}(t,x,\varepsilon)\|_{C(D)} \leq \left[\int_0^T\left(\frac{2}{\pi}\int_0^\pi f_0^2(\tau,\xi)\,d\xi\right)d\tau+5\sup M_\varepsilon^2(\varphi,\psi)\right]\exp\int_0^T\frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)\,d\xi\,d\tau.$$

To sum up, we can give the following theorem

Theorem 2. *Suppose the following conditions satisfy*

(a) *For all $\varepsilon\in[0,\varepsilon_0]$, in $D\{0<t\leq T, 0<x<\pi\}$, there is a weak generalized solution for the problems (1)-(3) and (4) - (6) separately.*
(b) *All conditions of Theorem (1) are satisfied for both of problems;*
(c) *With $f_0(t,x)\in C(\overline{D})$, conditions of Dirichlet Theorem are satisfied respect to x for all $t\in[0,T]$; then if $\sigma>0$, $\exists\delta(\sigma)>0$ so that for all $\varepsilon\in[0,\varepsilon_0]$, following inequalities $\|\varphi(t,x,\varepsilon)-\tilde{\varphi}(x,\varepsilon)\|_{C[0,\pi]}<\delta(\sigma)$, $|\psi(t,x,\varepsilon)-\tilde{\psi}(x,\varepsilon)\|_{C[0,\pi]}<\delta(\sigma)$ and $\|f_0(t,x)\|<\delta(\sigma)$ are satisfied,*

then $\|u(t,x,\varepsilon)-\tilde{u}(t,x,\varepsilon)\|<\delta$ is true.

In other words, satisfying the conditions of Theorem 2, the weak generalized solution of the problem (1)-(3) is continuous respect to initial conditions, means that the solution is determined.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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