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A comment on the bianchi groups

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Abstract: In this paper, we aim to discuss several the basic arithmetic structure of Bianchi groups. In particularly, we study fundamental domain and directed orbital graphs for the group $PSL(2,O_{-1})$.

Keywords: Quadratic number field, bianchi groups, suborbital graphs, circuits.

1 Introduction

It is known that the study of the group $PSL(2,\mathbb{C}) := \{T|T : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty,\ T(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{C}$ and $ad-bc=1\}$ consisting of all fractional linear transformations of the projective plane of points at infinity, with complex coefficients, was one of the major topics of mathematics in the 19th century and played an important role in the development of hyperbolic geometry. The group $PSL(2,\mathbb{C})$ has a natural action on 3-dimensional hyperbolic space $\mathbb{H}^3$ which may be described in geometric terms. An element $T \in PSL(2,\mathbb{C})$ induces a biholomorphic map of the Riemann sphere $\mathbb{C}\cup\{\infty\}$. Jules Henri Poincare (1854-1912) observed that $PSL(2,\mathbb{C})$ can be identified with the group of orientation preserving isometries of $\mathbb{H}^3$. As in the study of the hyperbolic plane, the importance of finding discrete groups of hyperbolic isometries of $\mathbb{H}^3$ was emphasized. Starting with an example by Emile Picard (1856-1941) and pursued by Luigi Bianchi (1856-1928) one particularly interesting class of discrete subgroups of $PSL(2,\mathbb{C})$ was considered. It is constructed in the following arithmetic way: Denote by O_d the ring of integers of an imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$ with $d<0$ and square free. Then O_d gives rise to a discrete group $\Gamma_d := PSL(2,O_d) < PSL(2,\mathbb{C})$ consisting of fractional linear transformations with coefficients in O_d. From [1,2] we can say that each subgroup τ of finite index in Γ_d operates properly discontinuously on $\mathbb{H}^3$ and a fundamental domain for this τ action on hyperbolic 3-space in noncompact but of finite volume. From various reasons, this class of arithmetically defined discrete groups of hyperbolic motions has gained new vivid interest in recent years. Beside that the Bianchi groups Γ_d are also of interest in their own group theoretical right. In [3,4], the structures of directed and undirected graphs were investigated. The relation between graphs and permutation groups was examined. In [5], authors mentioned permutation groups, primitivity and their applications to graph theory. In these papers [7,8,9,10] authors investigated some properties of suborbital graphs for the modular group, Picard group, the simple groups and $SL(3,\mathbb{Z})$ group. And also in [6], the author discussed suborbital graphs for the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$ group.

This paper is devoted to a computer aided analysis of subgoups of small index in Γ_d, in particular, we will deal as properties with subgroups of small index in Γ_d for $d=-1,-2,-3,-5,-7$. There is a fruitful interaction between geometric-topological, group theoretical and arithmetic questions and methods.

2 The action of $PSL(2,\mathbb{C})$ on $\mathbb{H}^3$

Consider the hyperbolic 3-space $\mathbb{H}^3 = \{(z,t)\in\mathbb{C}\times\mathbb{R} \mid t>0\}$, that is, the unique connected, simply connected Riemann manifold of dimension 3 with constant sectional curvature -1. A standard model for $\mathbb{H}^3$ is the upper half space model $\mathbb{H}^3$ with the metric coming from the line element $ds = \dfrac{\sqrt{dx^2+dy^2+dt^2}}{t}$ with $z=x+iy$. Every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $PSL(2,\mathbb{C})$

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acts on $\mathbb{H}^3$ as an orientation preserving isometry via the formula,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,t) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}y^2}{|cz+d|^2 + |c|^2y^2}, \frac{y}{|cz+d|^2 + |c|^2y^2} \right)$$

where $(z,t) \in \mathbb{H}^3$. It is well-known that every orientation preserving isometry of $\mathbb{H}^3$ arises this way.The group of all isometries of $\mathbb{H}^3$ is generated by $PSL(2,\mathbb{C})$ and complex conjugation $(z,t) \mapsto (\bar{z},t)$, which is an orientation reversing involution.
Indeed, the group $PSL(2,\mathbb{C})$ acts transitively on the points of $\mathbb{H}^3$ so that the stabiliser of any point in $\mathbb{H}^3$ is conjugate to the stabiliser of $(0,0,1)$. Therefore $\mathbb{H}^3$ and its geometry can be obtained from $SL(2,\mathbb{C})$ as its symmetric space as $SU(2,\mathbb{C})$ is a maximal compact subgroup. Likewise, the action on the sphere at infinity is transitive so that point stabilisers are conjugate to $\Upsilon = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \neq 0 \,\text{and}\, a,b \in \mathbb{C} \right\}$. Any finite subgroup of $PSL(2,\mathbb{C})$ must have a fixed point in $\mathbb{H}^3$ and so be conjugate to a subgroup of $SO(3,\mathbb{R})$. As such, it will either be cyclic, dihedral or conjugate to one of the regular solids groups and isomorphic to A_4, A_5 or S_4. The action of $PSL(2,\mathbb{C})$ on $\mathbb{H}^3$ leads to an action of $SL(2,\mathbb{C})$.

Lemma 1. *The group $SL(2,\mathbb{C})$ is generated by the elements $\zeta_1 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\zeta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where $\alpha \in \mathbb{C}$. These generators operate on $\mathbb{H}^3$ as follows*

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} (z,r) = (z+\alpha, r), \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (z,r) = \left(\frac{-\bar{z}}{|z|^2+r^2}, \frac{r}{|z|^2+r^2} \right).$$

Proof. Suppose $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$. If $c \neq 0$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix}$$

and for $c = 0$ we obtain the factorization

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}.$$

We thus conclude that the matrices ζ_1, ζ_2 together with the matrices $W_\eta := \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$, $\eta \in \mathbb{C} \setminus \{0\}$, generate $SL(2,\mathbb{C})$.
But the elements W_η may be represented as products of the matrices ζ_1, ζ_2 as is evident from the following formulas

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \eta^2 - \eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \eta^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-\eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The first mathematical works on non-Euclidean geometry were published by N. Lobachevski (1829) and J. Bolyai (1832). E. Beltrami (1868) introduced the upper space model, the unit ball model and the projective disc model of n dimensional hyerbolic geometry, Milnor (1982). These models were well-known to the classical writers who used them in their investigations on discontinuous groups, e.g. Bianchi (1952), Fricke (1914), Klein (1897), Humbert (1919), Picard (1978), Poincare (1916). The upper-space model of three dimensional hyperbolic geometry is presented by Beardon

(1983) and Magnus (1974).
The formulas of elementary hyperbolic geometry become particularly nice if we represent the points of hyperbolic 3-space by matrices in $SL(2,\mathbb{C})$. It is know that the following formalism is due to H. Helling.

3 Bianchi groups and fundamental domain

Definition 1. *A quadratic number field is a field $F \subset \mathbb{C}$ such that F has dimensional 2 as a vector space over $\mathbb{Q}$. Such a field takes the form $F = \mathbb{Q}(\sqrt{n}) = \{p + q\sqrt{n} : p,q \in \mathbb{Q}$ and $n > 1$ square-free$\}$. If n is positive then F is a real quadratic number field, and if n is negative then F is an imaginary quadratic number field.*

The trace function of F is the additive homomorphism

$$tr : F \longrightarrow \mathbb{Q},\ \ tr(\alpha) = \alpha + \overline{\alpha}.$$

The norm function of F is the multiplicative homomorphism

$$N : F^* \longrightarrow \mathbb{Q},\ \ N(\alpha) = \alpha\overline{\alpha}.$$

Specifically, $N(p+q\sqrt{n}) = p^2 - q^2 n$. The integers of the quadratic field $F = \mathbb{Q}(\sqrt{n})$ are

$$O_F = \mathbb{Z}[g],\ \ g = \begin{cases} \frac{1+\sqrt{n}}{2} & \text{if } \ n \equiv 1\,(mod 4), \\ \sqrt{n} & \text{if } \ n \equiv 2,3\,(mod 4). \end{cases}$$

The discriminant of F is $\Delta_F = \begin{cases} n & \text{if } \ n \equiv 1\,(mod 4), \\ 4n & \text{if } \ n \equiv 2,3\,(mod 4). \end{cases}$

Thus we see that every quadratic number field is of the form $\mathbb{Q}(\sqrt{d})$, where d is any square free integer. For any natural number m, define $O_{d,m} := \mathbb{Z} + m\omega\mathbb{Z}$ where

$$\omega := \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } \ d \equiv 3\,(mod 4), \\ \sqrt{d} & \text{if } \ d \not\equiv 3\,(mod 4). \end{cases}$$

For simplicity, denote $O_{d,1}$ by O_d. We also define $O_1 := \mathbb{Z}$.

d	ω	$N(p+q\omega)$	$trace$	$Elements\ of\ norm\ 1$
-1	i	p^2+q^2	$2p$	$\pm 1, \pm i$
-2	$\sqrt{-2}$	p^2+2q^2	$2p$	± 1
-3	$\frac{1+\sqrt{-3}}{2}$	p^2+pq+q^2	$2p$	$\pm 1, \pm\omega, \pm\omega^2$
-5	$\sqrt{-5}$	p^2+5q^2	$2p$	± 1
-7	$\frac{1+\sqrt{-7}}{2}$	$p^2+pq+2q^2$	$2p$	± 1

Let d is rational number. We examine $\mathbb{Q}(\sqrt{d}) = \{p + q\sqrt{d} : p,q \in \mathbb{Q}\} \subset \mathbb{C}$. Clearly if $\sqrt{d} \in \mathbb{Q}$ or $d = 1$, then $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}$. For $d = -1$, $\mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(i) = \{p + q\sqrt{-1} : p,q \in \mathbb{Q}\}$ is called Gaussian numbers. Also we may say that rational numbers within the root can be arranged. For instance, $\mathbb{Q}(\sqrt{\frac{1}{12}}) = \mathbb{Q}(\frac{1}{2}\sqrt{\frac{1}{3}}) = \mathbb{Q}(\sqrt{\frac{3}{3^2}}) = \mathbb{Q}(\sqrt{3})$.

Definition 2. *Let $d < 0$, square free integer and let O_d denote the ring of integers in $\mathbb{Q}(\sqrt{d})$. The groups $\Gamma_d = PSL(2,O_d)$ are called as the Bianchi groups. When $d = -1,-2,-3,-7,-11$ the rings O_d have an Euclidean algorithm and the Γ_d groups are known as the Euclidean Bianchi groups.*

Theorem 1. *Let K be an imaginary quadratic field of discriminant $\Delta_K < 0$, and let O_d be its ring of integers. Then the group $PSL(2,O_d)$ has following properties:*
(i)$PSL(2,O_d)$ *is a discrete subgroup of* $PSL(2,\mathbb{C})$.
(ii)$PSL(2,O_d)$ *has finite covolume, but is not cocompact.*
(iii)$PSL(2,O_d)$ *is a geometrically finite group.*
(iv)$PSL(2,O_d)$ *is finitely presented.*

We fix an imaginary quadratic number field K may construct now a fundamental domain $F_K \subset \mathbb{H}^3$ for group $PSL(2,O_d)$ where O_d is the ring of integers in K.

Lemma 2. *The fundamental domain $F_{\mathbb{Q}(i)}$ for $PSL(2,O_{-1})$ is described by*

$$F_{\mathbb{Q}(i)} = \{z+tj \in \mathbb{H}^3 \mid 0 \leq |Rez| \leq \frac{1}{2}, 0 \leq Imz \leq \frac{1}{2}, z\bar{z}+t^2 \geq 1\}.$$

$F_{\mathbb{Q}(i)}$ is a hyperbolic pyramid with one vertex at ∞ and the other four vertices in the points $P_1 = -\frac{1}{2}+\frac{\sqrt{3}}{2}j$, $P_2 = \frac{1}{2}+\frac{\sqrt{3}}{2}j$, $P_3 = \frac{1}{2}+\frac{1}{2}i+\frac{\sqrt{2}}{2}j$ and $P_4 = -\frac{1}{2}+\frac{1}{2}i+\frac{\sqrt{2}}{2}j$.

For $PSL(2,O_{-1})$, the Picard group, the region exterior to all isometric spheres, is the region exterior to all unit spheres whose centres lie on the integral lattice in $\mathbb{C}$. The stabiliser $PSL(2,O_{-1})_\infty$ is an extension of the translation subgroup by a rotation of order 2 about the origin. We thus obtain the fundamental region shown in figure.

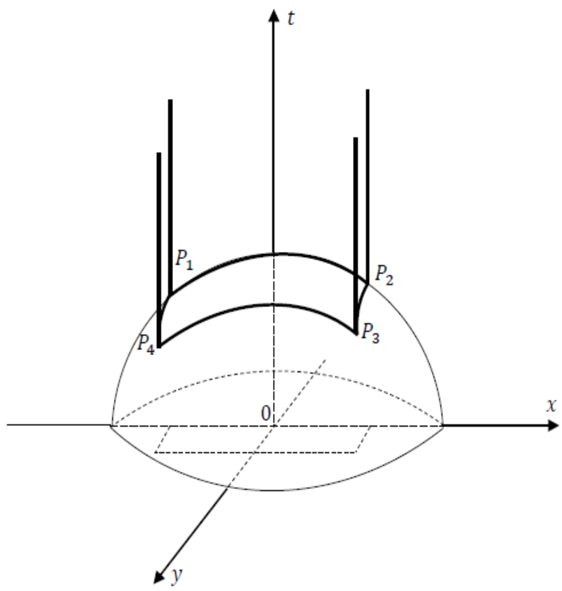

Fig. 1: Fundamental domain for Γ_{-1}

The Bianchi group Γ_d defined by $\Gamma_d := PSL(2,O_d) = SL(2,O_d)/\{\pm I\}$ is a discrete subgroup of $PSL(2,\mathbb{C})$ viewed as the group of orientation preserving isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$. Hence Γ_d acts properly discontinuously on $\mathbb{H}^3$. Fundamental domains for this action can analyze for small values of d.

Lemma 3. *The fundamental domain $F_{\mathbb{Q}(\sqrt{-3})}$ for $PSL(2,O_{-3})$ is described by*

$$F_{\mathbb{Q}(\sqrt{-3})} = \{z+tj \in \mathbb{H}^3 \mid 0 \leq |Rez|, \frac{\sqrt{3}}{3}Rez \leq Imz, Imz \leq \frac{\sqrt{3}}{3}(1-Rez)\}$$

$$\cup\{z \in \mathbb{C} \mid 0 \leq Imz \leq \frac{1}{2}, -\frac{\sqrt{3}}{3}Rez \leq Imz \leq \frac{\sqrt{3}}{3}Rez\}.$$

The picture of $F_{\mathbb{Q}(\sqrt{-3})}$ is closed quadrangle with corners in the points $Q_1 = 0$, $Q_2 = \frac{1}{2}-\frac{\sqrt{-3}}{6}$, $Q_3 = \frac{1}{2}+\frac{\sqrt{-3}}{6}$ and $Q_4 = \frac{\sqrt{-3}}{3}$.

Now, we give examples about generator of Bianchi groups. We note that in here Γ_d^{ab} commutator factor group of Γ_d.

Example 1. **(I)** The group Γ_{-1} is generated by the matrices

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ B=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\ C=\begin{pmatrix}1&i\\0&1\end{pmatrix},\ E=\begin{pmatrix}-i&0\\0&i\end{pmatrix}$$

where $i=\sqrt{-1}$. A presentation of Γ_{-1} is given by the following relations

$$B^2=(AB)^3=ACA^{-1}C^{-1}=(AE)^2=I,$$

$$E^2=(CE)^2=(BE)^2=(CBE)^3=I$$

and we have $\Gamma_{-1}^{ab}=\mathbb{Z}_2^2$. **(II)** The group Γ_{-2} is generated by the matrices

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ B=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\ C=\begin{pmatrix}1&\omega\\0&1\end{pmatrix}.$$

where $\omega=\sqrt{-2}$. A presentation of Γ_{-2} is given by the following relations

$$B^2=(AB)^3=ACA^{-1}C^{-1}=(BC^{-1}BC)^2=I$$

and again we can compute $\Gamma_{-2}^{ab}=\mathbb{Z}\times\mathbb{Z}_2$. **(III)** The group Γ_{-3} is generated by the matrices

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ B=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\ C=\begin{pmatrix}1&\omega\\0&1\end{pmatrix}.$$

Where $\omega=-\frac{1}{2}+\frac{\sqrt{-3}}{2}$. A presentation of Γ_{-3} is given by the following relations

$$B^2=(AB)^3=ACA^{-1}C^{-1}=(ACBC^{-2}B)^2=I,$$

$$(ACBC^{-1}B)^3=A^{-2}C^{-1}BCBC^{-1}BC^{-1}BCB=I$$

and then one may finds $\Gamma_{-3}^{ab}=\mathbb{Z}_3$. **(IV)** The group Γ_{-7} is generated by the matrices

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ B=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\ C=\begin{pmatrix}1&\omega\\0&1\end{pmatrix}$$

Where $\omega=\frac{1+\sqrt{-7}}{2}$. A presentation of Γ_{-7} is given by the the following relations

$$B^2=(AB)^3=ACA^{-1}C^{-1}=(BAC^{-1}BC)^2=I$$

and one computes $\Gamma_{-7}^{ab}=\mathbb{Z}\oplus\mathbb{Z}_2$.

4 Congruence subgroups and digraphs

It is known that number theoretic interest in the Bianchi groups has centered primarily on the congruence subgroups and the congruence subgroup property. There has also been a considerable amount of work on quadratic forms with entries in O_d and their relation to Γ_d. If σ is an ideal in O_d then the principal congruence subgroup $mod\sigma$, $\Gamma_d(\sigma)$ consists of those transformations in Γ_d corresponding to matrices in $SL(2,O_d)$ congruent to $\pm Imod\sigma$.

$$\Gamma_d(\sigma)=\{\pm T: T\in SL(2,O_d), T\equiv Imod\sigma\}.$$

$\Gamma_d(\sigma)$ can also be described as the kernel of the natural this map $\psi: SL(2,O_d)\longrightarrow SL(2,O_d/\sigma)$ modulo $\pm I$. Thus each principal congruence subgroup is normal and of finite index. A congruence subgroup is a subgroup which contains a principal congruence subgroup. Notice that in the Euclidean cases each ideal is principal and a formula in M. Newman allows us to compute the index of a principal congruence subgroup. Namely if $\alpha\in O_d$, $d\in\{-1,-2,-3,-7,-11\}$ then

$|\Gamma_d:\Gamma_d(\alpha)|=\rho|\alpha|^3\prod_{p|\alpha}(1-\frac{1}{p^2})$ where p runs over the primes dividing α and $\rho=1$ if $\alpha|2$ or $\rho=\frac{1}{2}$ otherwise.
Now we say some subgroup

$$W_0:=\left\{\begin{pmatrix}a & b\\ c & -\overline{a}\end{pmatrix}\in SL(2,\mathbb{C})\;\middle|\; a\in\mathbb{C}, b,c\in\mathbb{R}\right\},$$

$$W_1:=\left\{\begin{pmatrix}a & -b\\ -c & -\overline{a}\end{pmatrix}\in SL(2,\mathbb{C})\;\middle|\; a\in\mathbb{C}, b,c\in\mathbb{R}\right\},$$

$$W_2:=\left\{\begin{pmatrix}a & bi\\ ci & -\overline{a}\end{pmatrix}\in SL(2,\mathbb{C})\;\middle|\; a\in\mathbb{C}, b,c\in\mathbb{R}\right\}.$$

The group $SL(2,\mathbb{C})$ acts on W_0 by $\omega\mapsto M\omega\overline{M}^{-1}$ for $\omega\in W_0$, $M\in SL(2,\mathbb{C})$. Similarly, the group $SL(2,\mathbb{C})$ acts on W_1 and W_2. The full congruence subgroup is

$$\Gamma(\sigma):=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,O_d)\;\middle|\;\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} mod\,\sigma\right\}$$

where σ is a non zero ideal in the ring of integers O_d. Note that ∞ is a cusp of $\Gamma(\sigma)$.
Another subgroup is

$$\Gamma_0(\sigma):=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,O_d)\;\middle|\;\gamma\equiv 0\;\; mod\,\sigma\right\}.$$

Some of the groups Bianchi are subgroups of discrete groups which can be generated by hyperbolic reflections. The relationship of the Bianchi to reflection groups is studied in Shvartsman (1987), Shaikheev (1987), Vinberg (1987) and Ruzmanov (1990).

We can give applications for $PSL(2,O_{-1})$, hence we will obtain an extension of suborbital graphs.

Theorem 2. *$PSL(2,O_{-1})$ acts transitively on $\Sigma:=\mathbb{Q}(\sqrt{-1})\cup\{\infty\}$. We represent ∞ as $\pm\frac{\varepsilon}{0}$ where ε is 1 or i.*

Proof. We can show that the orbit containing ∞ is Σ. If $\frac{x}{y}\in\Sigma$, then as $(x,y)=1$, there exist $\alpha,\beta\in\mathbb{Z}[i]$ with $\alpha y-\beta x=1$. Then the element $\begin{pmatrix}\alpha & x\\ \beta & y\end{pmatrix}$ of $PSL(2,O_{-1})$ sends 0 to $\frac{x}{y}$.

Lemma 4. *The stabilizer of 0 in $PSL(2,O_{-1})$ is the set*

$$PSL(2,O_{-1})_0:=\Gamma_{-1,0}=\left\{\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}\;\middle|\;\lambda\in\mathbb{Z}[i]\right\}.$$

Proof. The stabilizer of a point in Σ is a infinite cyclic group. As the action is transitive, stabilizer of any two points are conjugate. Hence it is enough to examine the stabilizer of 0 in $PSL(2,O_{-1})$. $T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$ and so $\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$. Then $b=0$, $d=1$ and as $detT=1$, $a=1$. Therefore $c=\lambda\in\mathbb{Z}[i]$. Thus $T=\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}$. Indeed this shows that the set $PSL(2,O_{-1})_0$ is equal to $\left\langle\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\rangle$.

Definition 3.$PSL(2,O_{-1})^0(N):=\Gamma^0_{-1}(N)=\{T\in PSL(2,O_{-1})\;\;|b\equiv 0\,(modN), N\in\mathbb{Z}[i]\}$

It is clear that $\Gamma_{-1,0} < \Gamma_{-1}^{0}(N) < \Gamma_{-1}$. We may use an equivalence relation $\approx$ induced on Σ by Γ_{-1}.
Now let $\frac{r}{s}, \frac{x}{y} \in \Sigma$. Corresponding to these, there are two matrices

$$S_1 := \begin{pmatrix} \star & r \\ \star & s \end{pmatrix}, \quad S_2 := \begin{pmatrix} \star & x \\ \star & y \end{pmatrix}$$

in Γ_{-1} for which $S_1(0) = \frac{r}{s}$, $S_2(0) = \frac{x}{y}$. Therefore $\frac{r}{s} \approx \frac{x}{y}$ if and only if

$$S_1^{-1}S_2 = \begin{pmatrix} s & -r \\ \star & \star \end{pmatrix} \begin{pmatrix} \star & x \\ \star & y \end{pmatrix} = \begin{pmatrix} \star & sx-ry \\ \star & \star \end{pmatrix} \in \Gamma_{-1}^{0}(N).$$

So $sx - ry \equiv 0\,(modN)$.

Definition 4. *Let $\mathbb{G}$ be a graph and a sequence $v_1, v_2, \ldots, v_k$ of different vertices. Then form $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_k \longrightarrow v_1$, where $k > 2$ and k positive integer, is called a directed circuit in $\mathbb{G}$.*

Definition 5. *Let (Γ_{-1}, Σ) be transitive permutation group. Then Γ_{-1} acts on $\Sigma \times \Sigma$ by $\Theta : \Gamma_{-1} \times (\Sigma \times \Sigma) \longrightarrow \Sigma \times \Sigma$, $\Theta(T, (\alpha_1, \alpha_2)) = (T(\alpha_1), T(\alpha_2))$, where $T \in \Gamma_{-1}$ and $\alpha_1, \alpha_2 \in \Sigma$. The orbits of this action are called suborbitals of Γ_{-1}.*

Now we investigate the suborbital directed graphs or digraphs for the action Γ_{-1} on Σ. We say that the subgraph of vertices form the block

$$[\infty] := \left[\frac{1}{0}\right] = \left\{ \frac{x}{y} \in \Sigma \;\middle|\; x \equiv 1\,(modN),\, y \equiv 0\,(modN) \right\}$$

is denoted by $F_{u,N} := F(\frac{1}{0}, \frac{u}{N})$ where $(u,N) = 1$.

Theorem 3. *There is an edge $\frac{r}{s} \longrightarrow \frac{x}{y}$ in $F_{u,N}$ if and only if there exists a unit $\varepsilon \in \mathbb{Z}[i]$ such that $x \equiv \pm\varepsilon ur\,(modN)$, $y \equiv \pm\varepsilon us\,(modN)$ and $ry - sx = \varepsilon N$.*

Proof. Suppose that there exists an edge $\frac{r}{s} \longrightarrow \frac{x}{y} \in F_{u,N}$. Hence there exist some $T \in \Gamma_{-1}$ such that sends the pair $(\infty, \frac{u}{N})$ to the pair $(\frac{r}{s}, \frac{x}{y})$. Clearly $T(\infty) = \frac{r}{s}$ and $T(\frac{u}{N}) = \frac{x}{y}$. For $T(z) = \frac{az+b}{cz+d}$ we have that $\frac{a}{c} = \frac{r}{s}$ and $\frac{au+bN}{cu+dN} = \frac{x}{y}$. Then there exist the units $\varepsilon_0, \varepsilon_1 \in \mathbb{Z}[i]$ such that $a = \varepsilon_0 r$, $c = \varepsilon_0 s$ and $au + bN = \varepsilon_1 x$, $cu + dN = \varepsilon_1 y$. So, we can write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} \varepsilon_0 r & \varepsilon_1 x \\ \varepsilon_0 s & \varepsilon_1 y \end{pmatrix}$. Finally, taking with $\varepsilon = \varepsilon_0\varepsilon_1$, we get that $x \equiv \pm\varepsilon ur\,(modN)$, $y \equiv \pm\varepsilon us\,(modN)$. And also from the determinant $ry - sx = \varepsilon N$ is achieved.

Conversely, we can take the plus sign. Therefore there exist $b, d \in \mathbb{Z}[i]$ such that $x = \varepsilon ur + bN$, $y = \varepsilon us + dN$. If we choose $a = \varepsilon r$ and $c = \varepsilon s$, then we find $x = au + bN$ and $y = cu + dN$. So $T(\infty, \frac{u}{N}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} \varepsilon r & x \\ \varepsilon s & y \end{pmatrix}$. As $\varepsilon(ry - sx) = N$ we have $ad - bc = 1$ and $T \in \Gamma_{-1}$. Hence $\frac{r}{s} \longrightarrow \frac{x}{y}$ in $F_{u,N}$. Similarly, minus sign case may shown.

Theorem 4. *$F_{u,N}$ contains directed hyperbolic triangles if and only if there exists a unit $\varepsilon \in \mathbb{Z}[i]$ such that $\varepsilon^2u^2 - \varepsilon u + 1 \equiv 0\,(modN)$ and $\varepsilon^2u^2 + \varepsilon u + 1 \equiv 0\,(modN)$.*

Proof. We suppose that $F_{u,N}$ contains a directed hyperbolic triangle. Since the transitive action, the form of the directed hyperbolic triangle can written like this

$$\frac{1}{0} \longrightarrow \frac{u}{N} \longrightarrow \frac{r}{N} \longrightarrow \frac{1}{0}.$$

As the edge condition in above the theorem, we have to be provided for the second edge $\frac{u}{N} \longrightarrow \frac{r}{N}$, that is $u - r = \varepsilon$ and $r \equiv \pm\varepsilon u^2\,(modN)$. Then we have $\varepsilon r \equiv \pm\varepsilon^2u^2\,(modN)$. Therefore $\mp\varepsilon^2u^2 + \varepsilon r \equiv 0\,(modN)$ and $\varepsilon u - \varepsilon r = \varepsilon^2 = \pm 1$ are obtained. Hence there are two cases. The first case is $-\varepsilon^2u^2 + \varepsilon r \equiv 0\,(modN)$ and $\varepsilon r = \varepsilon u - 1$. The second case is $\varepsilon^2u^2 + \varepsilon r \equiv 0\,(modN)$ and $\varepsilon r = \varepsilon u + 1$. Finally we may say that for $\varepsilon \in \mathbb{Z}[i]$ these equations $\varepsilon^2u^2 - \varepsilon u + 1 \equiv 0(modN)$ and

$\varepsilon^2u^2+\varepsilon u+1\equiv 0(modN)$ are satisfied. Conversely, we can solve these equations only with special conditions. Let $\varepsilon\in\mathbb{Z}[i]$ be a unit such that $\varepsilon^2u^2-\varepsilon u+1\equiv 0(modN)$. Above the theorem implies that there is a directed hyperbolic triangle

$$\frac{1}{0}\longrightarrow\frac{u}{N}\longrightarrow\frac{u-\frac{1}{\varepsilon}}{N}\longrightarrow\frac{1}{0}$$

in $F_{u,N}$. Similarly, we can find another directed hyperbolic triangle for $\varepsilon^2u^2+\varepsilon u+1\equiv 0(modN)$, in this case

$$\frac{1}{0}\longrightarrow\frac{u}{N}\longrightarrow\frac{u+\frac{1}{\varepsilon}}{N}\longrightarrow\frac{1}{0}$$

in $F_{u,N}$.

Now we get example for first equation.

Example 2. Let $N=1+6i$. If we take $u=2+i$, then this equation $(3+4i)\varepsilon^2-(2+i)\varepsilon+1\equiv 0\,mod(1+6i)$ is achieved for $\varepsilon=i$. Hence we get this directed hyperbolic triangle as

$$\frac{1}{0}\longrightarrow\frac{2+i}{1+6i}\longrightarrow\frac{2}{1+6i}\longrightarrow\frac{1}{0}$$

in $F_{2+i,1+6i}$. And also, it is clear that $(2+i,1+6i)=1$ and $ry-sx=\varepsilon N$.
Similarly, let $N=i$. Again we choose $u=-1$, then $\varepsilon^2+\varepsilon+1\equiv 0\,mod(i)$ is held if and only if $\varepsilon=-i$. So, directed triangle is

$$\infty\longrightarrow i\longrightarrow 1+i\longrightarrow\infty$$

in $F_{-1,i}$.

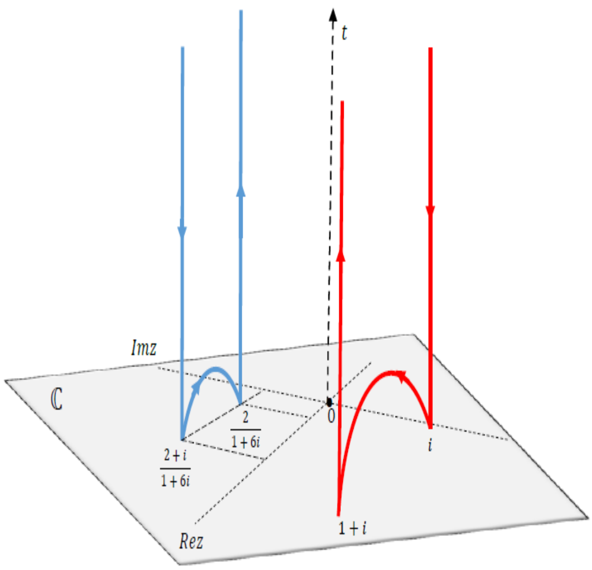

Fig. 2: Circuits in $F_{2+i,1+6i}$ and $F_{-1,i}$

Corollary 1. *The transformation* $\psi := \begin{pmatrix} -\varepsilon u & \frac{\varepsilon^2 u^2 \mp \varepsilon u + 1}{\varepsilon N} \\ -\varepsilon N & \varepsilon u \mp 1 \end{pmatrix}$ *which is defined by means of the congruence* $\varepsilon^2 u^2 \mp \varepsilon u + 1 \equiv 0\,(modN)$ *is an elliptic element of order 3. Obviously that* $det\psi = 1$, $\psi^3 = I$ *and* $tr(\psi) = \mp 1$*. Moreover, it is easily seen that* $\psi\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}u\\N\end{pmatrix}$, $\psi^2\begin{pmatrix}1\\0\end{pmatrix} := \psi\begin{pmatrix}u\\N\end{pmatrix} = \begin{pmatrix}u \mp \frac{1}{\varepsilon}\\N\end{pmatrix}$, $\psi^3\begin{pmatrix}1\\0\end{pmatrix} := \psi\begin{pmatrix}u \mp \frac{1}{\kappa}\\N\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$.

Corollary 2. *The transformation* $\eta := \begin{pmatrix} -\varepsilon u & \frac{\varepsilon^2 u^2 \pm \varepsilon u - 1}{\varepsilon N} \\ -\varepsilon N & \varepsilon u \pm 1 \end{pmatrix}$ *has* $det\eta = -1$ *and* $tr(\eta) = \pm 1$*. Furthermore,* $\eta\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}u\\N\end{pmatrix}$, $\eta\begin{pmatrix}u\\N\end{pmatrix} = \begin{pmatrix}u \mp \frac{1}{\varepsilon}\\N\end{pmatrix}$, $\eta^2\begin{pmatrix}u\\N\end{pmatrix} = \begin{pmatrix}u \mp \frac{1}{2\varepsilon}\\N\end{pmatrix}$, $\eta^3\begin{pmatrix}u\\N\end{pmatrix} = \begin{pmatrix}u \mp \frac{2}{3\varepsilon}\\N\end{pmatrix}$, $\eta^4\begin{pmatrix}u\\N\end{pmatrix} = \begin{pmatrix}u \mp \frac{3}{5\varepsilon}\\N\end{pmatrix}, \ldots, \eta^n\begin{pmatrix}u\\N\end{pmatrix} = \begin{pmatrix}u \mp \frac{F_n}{F_{n+1}\varepsilon}\\N\end{pmatrix}$, *where* $n \geq 0$ *positive integer and* F_n *Fibonacci numbers. It is known that* $F_0 = 0$, $F_1 = 1$ *and* $F_{n+2} = F_n + F_{n+1}$*. That is, two directed paths*

$$\frac{1}{0} \longrightarrow \frac{u}{N} \longrightarrow \frac{u \mp \frac{1}{\varepsilon}}{N} \longrightarrow \frac{u \mp \frac{1}{2\varepsilon}}{N} \longrightarrow \frac{u \mp \frac{2}{3\varepsilon}}{N} \longrightarrow \frac{u \mp \frac{3}{5\varepsilon}}{N} \ldots \longrightarrow \frac{u \mp \frac{F_n}{F_{n+1}\varepsilon}}{N}$$

are obtained.

Example 3. Let $N = 41$ and $u = 7$. We consider first case. If we take $\varepsilon = -1$ then we have this directed path

$$\frac{1}{0} \longrightarrow \frac{7}{41} \longrightarrow \frac{7+1}{41} \longrightarrow \frac{7+\frac{1}{2}}{41} \longrightarrow \frac{7+\frac{2}{3}}{41} \longrightarrow \frac{7+\frac{3}{5}}{41} \ldots \longrightarrow \frac{7+\frac{\sqrt{5}-1}{2}}{41}.$$

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