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Suborbital graphs of a power subgroup of the modular group

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Abstract: In this paper, we define an invariant equivalence relation by using the group $\Gamma(2)$. Then we investigate some combinatorial properties of subgraphs of Γ^2.

Keywords: Modular group, imprimitive action, suborbital graphs.

1 Introduction

1.1 Motivation

Using the notion of the imprimitive action for an invariant equivalence relation on $\hat{\mathbb{Q}}$ by the congruence subgroup $\Gamma_0(n)$, Jones, Singerman and Wicks obtained suborbital graphs of the modular group Γ and showed that these graphs are the generalization of the well-known Farey graph[8]. Then Akbas found certain relationship between the lenghts of circuits in these graphs and periods of elliptic elements of the group $\Gamma_0(n)$[1]. This is important taking into account that the elliptic elements are one of the invariants of the group. Hence, suborbital graphs can be viewed as a tool to investigate permutation groups in terms of combinatorics[5].

Actually, the suborbital graphs of the group Γ^2 were studied in[7] for the relation $\Gamma^2_\infty \lneq \Gamma^2_0(n) \lneq \Gamma^2$ with $n \in \mathbb{N}$. In here, taking $\Gamma(2)$ instead of $\Gamma^2_0(n)$, we investigate some combinatorial properties of the newly constructed subgraphs of $\Gamma(2)$ different from[7]. We can summarize the cause of this choice as follows.

Congruence subgroups of Γ are very important in number theory; they all have finite index in Γ, but not every subgroup of finite index is a congruence subgroup. Some of them have a special interest. In[16], Singerman showed that $\Gamma_0(2)$ is isomorphic to the universal tessellation $\Gamma(2,\infty,\infty)$. He pointed that this is a chance taking into account the difficulties of construction of universal n-gonal tessellations. It is known that the groups $\Gamma(2,\infty,n)$ are Hecke groups and more complicated than the modular group $\Gamma(2,3,\infty)$.

Furthermore, the plane trees, the maps of genus 0 with a single face, can be probably seen as the simplest class of bipartite maps. In[9], $\Gamma(2)$ is given the automorphism group of the universal bipartite map $\hat{\mathfrak{B}}$ on $\mathbb{H}$. It is used as an illustration to emphasized the connections between maps on surfaces, permutations, Riemann surfaces.

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From this point of view, to collect new results about on $\Gamma(2)$, we used the relation $\Gamma_\infty^2 \lneq \Gamma(2) \lneq \Gamma^2$ for the imprimitive action in this paper.

1.2 Preliminaries

Define Γ^m as the subgroup of Γ generated by the m^{th} powers of all elements of Γ. Especially, Γ^2 and Γ^3 have been studied extensively by [11][12][13]. It turns out that,

$$\Gamma^2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : ab + bc + cd \equiv 0(mod2) \right\},$$

by Rankin [14]. From the equation $ab+bc+cd \equiv 0(mod2)$, we see that at least one of the letters a,b,c,d must be even. Suppose first that $a = 2a_0$. Then using the determinant, we have that b and c are odd. So, d must be odd as well. Hence, we get the element of Γ^2 as the matrices $\begin{pmatrix} 2a & b \\ c & d \end{pmatrix}$. Similarly, supposing $d = 2d_0$, we can get the elements of the form $\begin{pmatrix} a & b \\ c & 2d \end{pmatrix}$. Lastly, if a or d is not even, then both b and c will be even. To sum up, Γ^2 has three types of elements

$$\begin{pmatrix} 2a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & 2b \\ 2c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & 2d \end{pmatrix}.$$

where b,c and d of the first, a and d of the second and a,b,c of the third matrix are odd.

In this study, we also use congruance subgroup $\Gamma(2)$ of the modular group, so we give some information about this group. For any positive integer n, the group showed $\Gamma(n)$ is defined as follow:

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1(modn), b \equiv c \equiv 0(modn) \right\}.$$

For $n = 2$, the group $\Gamma(2)$ is generated by three elements

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and its cusps are $0, 1, \infty$.

2 The Action of Γ^2 on $\hat{\mathbb{Q}}$

Every element of $\hat{\mathbb{Q}}$ can be represented as a reduced fraction $\frac{x}{y}$, with $x,y \in \mathbb{Z}$ and $(x,y) = 1$; since $\frac{x}{y} = \frac{-x}{-y}$, this representation is not unique. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action $z \to \frac{az+b}{cz+d}$ of Γ^2 on $\hat{\mathbb{Q}}$ now becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax+by}{cx+dy}.$$

Lemma 1.*[7]*

(i) *The action of Γ^2 on $\hat{\mathbb{Q}}$ is transitive.*

(ii) *The stabilizer of a point is in infinite cyclic group.*

Proposition 1.[4] *Let (G,Ω) be transitive. Then (G,Ω) is primitive if and only if G_α, the stabilizer of a point $\alpha \in \Omega$, is a maximal subgroup of G for each $\alpha \in \Omega$.*

Indeed, suppose that $G_\alpha < H < G$. Since G acts transitively, every element of Ω has the form $g(\alpha)$ for some $g \in G$. One easily checks that there is a well-defined $G-$ invariant equivalence relation $\approx$ on Ω, given by $g(\alpha) \approx g'(\alpha)$ if and only if $g' \in gH$.

We now apply these ideas to the case where G is Γ^2, and Ω is $\hat{\mathbb{Q}}$. Here Γ^2_∞, the stabilizer of ∞, is the subgroup of Γ^2 generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, so by finding subgroups H of Γ^2 containing Γ^2_∞ $\left(\text{or equivalently, containing } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right)$, we can produce Γ^2- invariant equivalence relations on $\hat{\mathbb{Q}}$. From the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, some obvious choices for H are the congruence subgroups

$$\Gamma(2) = \{A \in \Gamma : A \equiv I \ (mod 2)\}.$$

Clearly, $\Gamma^2_\infty < \Gamma(2) < \Gamma^2$, so Γ^2 acts imprimitively on $\hat{\mathbb{Q}}$. Let $\approx$ denote Γ^2- invariant equivalence relation induced on $\hat{\mathbb{Q}}$ by $\Gamma(2)$. If $v = \frac{r}{s}$ and $w = \frac{x}{y}$ are elements of $\hat{\mathbb{Q}}$, then $v = g(\infty)$ and $w = g'(\infty)$ for elements $g, g' \in \Gamma^2$ of the form

$$g = \begin{pmatrix} r & k \\ s & l \end{pmatrix}, \begin{pmatrix} x & z \\ y & t \end{pmatrix};$$

now $v \approx w$ if and only if $g^{-1}g' \in H = \Gamma(2)$, and since $g^{-1} = \begin{pmatrix} l & -k \\ -s & r \end{pmatrix}$ we see that $v \approx w$ if and only if

$$lx - ky \equiv rt - sz \equiv \pm 1 \ (mod 2)$$

$$lz - kt \equiv ry - sx \equiv \pm 1 \ (mod 2)$$

To put this another way, $v = \frac{r}{s}$ and $w = \frac{x}{y}$ are equivalent if and only if they "have the same reduction mod2", that is,

$$x \equiv \ ur \text{ and } y \equiv \ us \ (mod 2)$$

for some unit $u \in U_2$.

By our general discussion of imprimitivity, the number $\Psi(2)$ of equivalence classes under $\approx$ is given by

$$\Psi(2) = |\Gamma^2 : \Gamma(2)| = 3.$$

3 Suborbital Graphs for Γ^2 on $\hat{\mathbb{Q}}$

Let (G,Ω) be a transitive permutation group. Then G acts on $\Omega \times \Omega$ by

$$g : (\alpha, \beta) \rightarrow (g(\alpha), g(\beta))$$

$(g \in G, \ \alpha, \beta \in \Omega)$. The orbits of this action are called suborbitals of G, that containing (α,β) being denoted by $O(\alpha,\beta)$ From $O(\alpha,\beta)$ we can form a suborbital graph $G(\alpha,\beta)$: Its vertices are the elements of Ω, and there is a directed edge from γ to the δ if $(\gamma,\delta) \in O(\alpha,\beta)$.

Clearly $O(\alpha,\beta)$ is also a suborbital graph, and it is either equal to our disjoint from $O(\alpha,\beta)$. In the latter case, $G(\alpha,\beta)$ is just $G(\alpha,\beta)$ with the arrows reversed, and we call $G(\alpha,\beta)$ and $G(\beta,\alpha)$ paired suborbital graphs. In the former case, $G(\alpha,\beta)=G(\beta,\alpha)$ and the graphs consists of pairs of oppositely directed edges; it is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call self-paired. These ideas were first introduced by Sims[15] and are also described in a paper by Newmann[10] and in books Tsuzuku[17], Biggs and White[4], the emphasis being on applications to finite groups. The reader is also refereed to [2][3][6][7] for some relevant previous work on suborbital graphs.

Theorem 1. *$\frac{r}{s}\to\frac{x}{y}\in G_{u,2}$ if and only if*

$$x\equiv \pm ur\ (mod 4),\ x\equiv \pm us\ (mod 2)\ \text{and}\ ry-sx=\pm 2$$

Proof. By the transitivity of Γ^2, without loos of generality, we assume that $\frac{r}{s}<\frac{x}{y}$ where all letters are positive integers. Thus, we have that $ry-sx<0$. Since $\frac{r}{s}\to\frac{x}{y}\in G_{u,2}$, there exist some $T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma^2$ such that $T(\frac{1}{0},\frac{u}{2})=(\frac{r}{s},\frac{x}{y})$. As $ry-sx<0$, the multiplication of $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & u\\ 0 & 2\end{pmatrix}$ is equal to $\begin{pmatrix} -r & x\\ -s & y\end{pmatrix}$ or $\begin{pmatrix} r & -x\\ s & -y\end{pmatrix}$. If the first case is valid, we have that $a=-r,\ c=-s,\ au+2b=x,\ cu+2d=y$ and $ry-sx=-2$. That is, $x\equiv\ -ur\ (mod 2)$ and $y\equiv\ -us\ (mod 2)$. Since s is even we see that b and c must be even because $T(\frac{1}{0})=\frac{-r}{-s}=\frac{a}{c}$. Since b is even, we have that $x\equiv\ -ur\ (mod 4)$ and $y\equiv\ -us\ (mod 2)$.

In the opposite direction, we shall prove the theorem for minus sign. Suppose that $x\equiv\ -ur\ (mod 4)$, $y\equiv\ -us\ (mod 2)$ and $ry-sx=-2$. In this, there exist integers b,d such that $x=-ur-4b$, $y=-us-2d$. So, it is clear that $\begin{pmatrix} -r & -2b\\ -s & -d\end{pmatrix}\in\Gamma^2$ which means $\frac{r}{s}\to\frac{x}{y}\in G_{u,2}$. Because $-2=ry-sx=r(-us-2d)-s(-ur-4b)$. This implies $rd-2bs=1$. We can illustrate one example for this subgraph obtained from elements $A=\begin{pmatrix} 3 & -2\\ 2 & -1\end{pmatrix}$ and $B=\begin{pmatrix} 5 & -8\\ 2 & -3\end{pmatrix}$ with figure1.

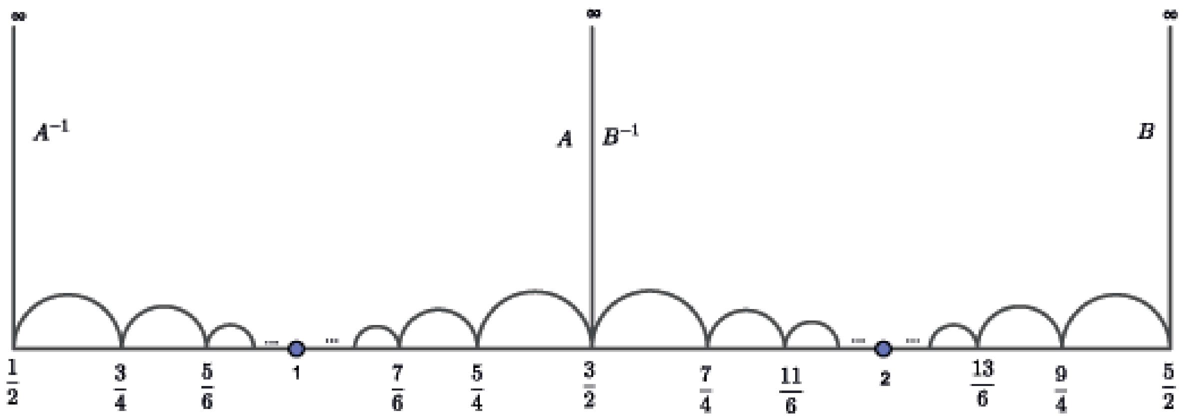

Fig. 1: The paths of Γ^2.

Theorem 2. *Let $T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma^2$. Then $\frac{1}{2}\to\frac{1}{4}\in F^2$ if and only if $T=\begin{pmatrix} 4n+1 & -2n\\ 16n+2 & 1-8n\end{pmatrix}$ with $n\in\mathbb{Z}$.*

Proof. Let $T(\frac{1}{2})=\frac{1}{4}=\frac{a+2b}{c+2d}$. Because $c+2d=a+2b+2$, T is $\begin{pmatrix} 2a-1 & 1-a \\ 2c & 2-c \end{pmatrix}$. Since $T\in\Gamma$, $4a-c=3$ and so $a=4k+1$, $k\in\mathbb{Z}$. Therefore, $T=\begin{pmatrix} 2k+1 & -k \\ 8k+2 & 1-4k \end{pmatrix}$. On the other hand, because $T\in\Gamma^2$ $(2k+1)(-k)+(-k)(8k+2)+(8k+2)(1-4k)\equiv\ 0\ (mod2)$.From the last congruence, $k\equiv\ 0\ (mod2)$. So, $k=2,\ n\in\mathbb{Z}$. As a result, T is in the form $\begin{pmatrix} 4n+1 & -2n \\ 16n+2 & 1-8n \end{pmatrix}$. The opposite is obvious.

Theorem 3. *Let $n\geq 2,\ n\in\mathbb{Z}$. The transformations of the matrices $T=\begin{pmatrix} 4n+1 & -2n \\ 16n+2 & 1-8n \end{pmatrix}$ are hyperbolic transformations and fixed points of these transformation are attracting fixed points.*

Proof. The trace of the transformations of the matrices $T=\begin{pmatrix} 4n+1 & -2n \\ 16n+2 & 1-8n \end{pmatrix}$ is equal to $|2-4n|$. So, when $n=0$ and 1, these transformations are parabolic, when $n\geq 2$, they are hyperbolic, because of $|2-4n|\geq 2$. The fixed points of these transformations,

$$z_{1,2}=\frac{3n\pm\sqrt{n^2-n}}{8n+1}.$$

Therefore, $T'(z)=\frac{1}{[(16n+2z+1-8n)]^2}$ and so $T'(z_{1,2})=\left(\frac{1}{1-2n\pm\sqrt{n^2-n}}\right)<1$, with $\forall n\geq 2$. Thus, these fixed points are attracting fixed points. ∎

Theorem 4. *Let $T_n:=\begin{pmatrix} an+1 & bn \\ cn & dn+1 \end{pmatrix}\in\Gamma(n),\ c\neq 0$ and $n\geq 2$. Then $|T_n(\infty)-T_n^2(\infty)|\leq\frac{1}{2n}$.*

Proof. For $T=\begin{pmatrix} an+1 & bn \\ cn & dn+1 \end{pmatrix}$, T^2 is equal to $\begin{pmatrix} (an+1)^2+bcn^2 & bn(an+dn+2) \\ cn(an+dn+2) & bcn^2+(dn+1)^2 \end{pmatrix}$. Therefore,

$$T(\infty)=\frac{an+1}{cn} \text{ and } T^2(\infty)=\frac{(an+1)^2+bcn^2}{cn(an+dn+2)}.$$

So,

$$|T(\infty)-T^2(\infty)|=\left|\frac{an+1}{cn}-\frac{(an+1)^2+bcn^2}{cn(an+dn+2)}\right|=\left|\frac{1}{cn^2(a+d)+2cn}\right|=\frac{1}{|c|n}\frac{1}{|n(a+d)+2|}.$$

Let M be $\frac{1}{|n(a+d+2)|}$. In this case, when $n\geq 2$, $|n(a+d)+2|\geq|2(a+d)+2|=2|a+d+1|\geq 2$. So, M becomes less than $\frac{1}{2}$, when $n\geq 2$ and $c\neq 0$. As a result $|T(\infty)-T^2(\infty)|\leq\frac{1}{2n}$. ∎

From this theorem, we can easily say that, the maximum value of distance of two vertices $T(\infty)$ and $T^2(\infty)$ is $\frac{1}{4}$ for all $T\in\Gamma(2)$. The following is a result of this theorem.

Corollary 1. *Let $T=\begin{pmatrix} 4n+1 & -2n \\ 16n+2 & 1-8n \end{pmatrix}$. Then $|T(\infty)-T^2(\infty)|\leq\frac{1}{4}$.*

Proof. Because $T(\infty)=\frac{4n+1}{16n+2}$ and $T^2(\infty)=\frac{-16n^2+4n+1}{-64n^2+24n+4}$, then

$$|T(\infty)-T^2(\infty)|=\left|\frac{4n+1}{16n+2}-\frac{-16n^2+4n+1}{-64n^2+24n+4}\right|=\left|\frac{1}{4(16n^2-6n-1)}\right|=\frac{1}{4}\frac{1}{|16n^2-6n-1|}$$

If M is taken as $\frac{1}{|16n^2-6n-1|}$, M becomes 1 , with $n=0$ and $M<1$, when $n\neq 0$. As a result $|T(\infty)-T^2(\infty)|\leq\frac{1}{4}$. ∎

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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