New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.254

Categorical properties of racks

Hatice Gulsun Akay and Ibrahim Ilker Akca

Eskisehir Osmangazi University, Department of Mathematics and Computer Science, Eskisehir, Turkey

Received: 13 November 2017, Accepted: 11 January 2018
Published online: 21 February 2018.

Abstract: In this paper, we give some categorical objects of racks such as product, pullback and equalizer objects.

Keywords: Rack, pullback, equalizer.

1 Introduction

A rack [3] is a set with a non-associative binary operation satisfying two rack conditions. The theory of racks is connected to the group theory. This relation leads to the functor $\mathbf{Conj}\colon \mathbf{Grp} \to \mathbf{Rack}$ between the categories of racks and of groups which admits a left adjoint functor $\mathbf{As}\colon \mathbf{Rack} \to \mathbf{Grp}$; see [4], [7] for more details.

The earliest work on racks is due to Conway and Wraith [3] which is inspired by the conjugacy operation in a group and focuses in the special case of racks, called quandles; but they also were aware of the generalization. In the literature, racks are also called "automorphic sets" [2], "crystals" [8] and "(left) distributive quasigroups" [10].

In this study, we firstly recall the definitions and some examples for racks. Most of them appear in [7]. Afterwards, we give some categorical properties of racks which are the constructions of product, pullback and equalizer objects. These categorical objects are defined by the universal property diagrams in [1], [9] and examined for more specific categories such as category of crossed modules of racks and (modified) categories of interest in [5], [6].

2 Racks

We recall some notions from [7] which will be used in sequel.

Definition 1. *A rack R is a set with a binary operation satisfying:*

(R1) *For all $a,b \in R$, there exists a unique $c \in R$ such that:*

$$c \lhd a = b,$$

(R2) *For all $a,b,c \in R$, we have:*

$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c).$$

A rack which aditionally satisfies the idempotency condition:

$$r \lhd r = r$$

is called a "quandle" (for all $r \in R$).

* Corresponding author e-mail: gulsunhatice@gmail.com

Definition 2. *A "pointed" rack R is a rack equipped with a fixed element $1 \in R$ such that (for all $r \in R$):*

$$1 \lhd r = 1 \quad \text{and} \quad r \lhd 1 = r.$$

Remark. We only work with the pointed racks in the rest.

Definition 3. *Let R and S be two (pointed) racks. A rack morphism is a map:*

$$f : R \to S$$

such that:

$$f\left(r \lhd r'\right) = f\left(r\right) \lhd f\left(r'\right) \qquad (\text{and } f\left(1\right) = 1)$$

for all $r, r' \in R$.

Thus we get the category of (pointed) racks denoted by **Rack.**

Some examples of racks are:

(1) The trivial rack T_n of order n is the set $\{0, 1, 2, ..., n-1\}$ with the rack operation (for all $x, y \in T_n$):

$$x \lhd y = x.$$

The infinitive trivial rack T_∞ is the set $\mathbb{Z}$ equipped with the same operation.

(2) The dihedral rack D_n is the set $\{0, 1, 2, ..., n-1\}$ with the rack operation:

$$x \lhd y = 2y - x \mod n$$

for all $x, y \in D_n$ and the infinitive dihedral rack D_∞ is the set $\mathbb{Z}$ equipped with:

$$x \lhd y = 2y - x$$

for all $x, y \in \mathbb{Z}$.

(3) The cyclic rack C_n of order n is the set $\{0, 1, 2, ..., n-1\}$ with the rack operation:

$$x \lhd y = x + 1 \mod n$$

for all $x, y \in C_n$, while the infinitive cyclic rack is the set $\mathbb{Z}$ equipped with:

$$x \lhd y = x + 1$$

for all $x, y \in \mathbb{Z}$.

(4) Given a group G, we may define a rack structure on G by setting (for all $g, h \in G$):

$$g \lhd h = h^{-1} g h.$$

This rack is called the "conjugation" rack of G and denoted by **ConjG**. This construction provides a functor:

$$\mathbf{Conj} : \mathbf{Grp} \to \mathbf{Rack}.$$

(5) We may define a different rack structure on G by setting (for all $g, h \in G$):

$$g \lhd h = h g^{-1} h.$$

that is called "core" rack. However this construction is not functorial.

(6) Let P and R be two racks, then the cartesian product:

$$P \times R = \{(p,r) \mid p \in P, r \in R\}$$

has a rack structure with:

$$(p,r) \lhd (p',r') = (p \lhd p', r \lhd r')$$

for all $(p,r), (p',r') \in P \times R$.

Definition 4. *Let R be a rack and X be a set. We say that X is an R-set when there are bijections $(\cdot r) : X \to X$ for all $r \in R$ such that:*

$$(x \cdot r) \cdot r' = (x \cdot r') \cdot (r \lhd r'),$$

for all $x \in X$ and $r, r' \in R$.

Definition 5. *Let R, S be two racks. We say that S acts on R by automorphisms when there is a (right) rack action of S on R and:*

$$(r \lhd r') \cdot s = (r \cdot s) \lhd (r' \cdot s)$$

for all $s \in S$ and $r, r' \in R$.

The following notion is likely to be semi-direct product of groups:

Definition 6. *If there exists a (right) rack action of R on S, the "hemi-semi-direct product" $S \rtimes R \subset S \times R$ is the rack defined by the rack operation:*

$$(s,r) \lhd (s',r') = (s \cdot r', r \lhd r')$$

for all $(s,r), (s',r') \in S \rtimes R$.

Definition 7. *For a given rack R, a non empty subset $S \subseteq R$ is called a subrack if $s \lhd s' \in S$ for all $s, s' \in S$.*

3 Categorical properties of racks

In this section we give the constructions of product, pullback and equalizer objects for the category of racks.

Theorem 1. *The category of racks has products.*

Proof. Let P and R be two racks. Define:

$$P \times R = \{(p,r) \mid p \in P, r \in R\}.$$

We already know that $P \times R$ is a rack. Also it is easy to verify that the projection maps $p_1 : P \times R \to P$ and $p_2 : P \times R \to R$ are rack morphisms.

Now we will check the universal property. Let T be any rack and $\alpha : T \to P$, $\beta : T \to R$ be two rack morphisms. Then we need to prove that there exists a unique rack morphism:

$$\varphi : T \to P \times R$$

such that makes following diagram commutes:

$$\begin{tikzcd} & T \arrow[dl, "\alpha"'] \arrow[d, dotted, "\exists!\varphi"] \arrow[dr, "\beta"] & \\ P & P\times R \arrow[l, "p_1"] \arrow[r, "p_2"'] & R \end{tikzcd} \tag{1}$$

Define:

$$\begin{aligned} \varphi : T &\to P\times R \\ t &\mapsto \varphi(t) = (\alpha(t), \beta(t)). \end{aligned}$$

φ is a rack morphism since:

$$\begin{aligned} \varphi(t \lhd t') &= (\alpha(t \lhd t'), \beta(t \lhd t')) \\ &= (\alpha(t) \lhd \alpha(t'), \beta(t) \lhd \beta(t')) \\ &= (\alpha(t), \beta(t)) \lhd (\alpha(t'), \beta(t')) \\ &= \varphi(t) \lhd \varphi(t') \end{aligned}$$

for all $t, t' \in T$. Furthermore we get:

$$\begin{aligned} p_1\varphi(t) &= p_1(\alpha(t), \beta(t)) \\ &= \alpha(t) \end{aligned}$$

and

$$\begin{aligned} p_2\varphi(t) &= p_2(\alpha(t), \beta(t)) \\ &= \beta(t) \end{aligned}$$

for all $t \in T$ that proves the commutativity of (1).

Consider φ' with the same property as φ, i.e. the following conditions hold:

$$\begin{aligned} p_1\varphi' &= \alpha \\ p_2\varphi' &= \beta. \end{aligned}$$

Define $(p, r) \in P\times R$ by $\varphi'(t) = (p, r)$. We get:

$$\begin{aligned} p_1\varphi'(t) = \alpha(t) &\Rightarrow p_1(p, r) = \alpha(t) \\ &\Rightarrow p = \alpha(t) \end{aligned}$$

and

$$\begin{aligned} p_2\varphi'(t) = \beta(t) &\Rightarrow p_2(p, r) = \beta(t) \\ &\Rightarrow r = \beta(t) \end{aligned}$$

for all $t \in T$ which yields:

$$\begin{aligned}\varphi'(t) &= (p,r) \\ &= (\alpha(t), \beta(t)) \\ &= \varphi(t)\end{aligned}$$

and proves that φ is unique.

Theorem 2. *The category of racks has pullbacks.*

Proof. Let $f : P \to T$ and $g : R \to T$ be two rack morphisms. Define:

$$P \times_T R = \{(p,r) \mid f(p) = g(r)\}$$

which is a subrack of $P \times R$; see [5]. Then we get the following commutative diagram:

$$\begin{tikzcd} P \times_T R \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "g"] \\ P \arrow[r, "f"'] & T \end{tikzcd}$$

Let Q be any rack with two rack morphisms $\alpha : Q \to P$ and $\beta : Q \to R$ where the following diagram commutes:

$$\begin{tikzcd} Q \arrow[r, "\beta"] \arrow[d, "\alpha"'] & R \arrow[d, "g"] \\ P \arrow[r, "f"'] & T \end{tikzcd}$$

Then there must be a unique rack morphism:

$$\varphi : Q \to P \times_T R$$

that makes the following diagram commutative:

$$\begin{tikzcd} Q \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & P \times_T R \arrow[r, "p_2"'] \arrow[d, "p_1"] & R \arrow[d, "g"] \\ & P \arrow[r, "f"'] & T \end{tikzcd} \tag{2}$$

namely:

$$p_1\varphi = \alpha$$
$$p_2\varphi = \beta.$$

For this aim, define:

$$\begin{aligned}\varphi : Q &\to P\times_T R\\ q &\mapsto \varphi(q) = (\alpha(q),\beta(q)).\end{aligned}$$

Then φ is a rack morphism since:

$$\begin{aligned}\varphi\left(q\lhd q'\right) &= \left(\alpha\left(q\lhd q'\right),\beta\left(q\lhd q'\right)\right)\\ &= \left(\alpha\left(q\right)\lhd\alpha\left(q'\right),\beta\left(q\right)\lhd\beta\left(q'\right)\right)\\ &= (\alpha(q),\beta(q))\lhd(\alpha(q'),\beta(q'))\\ &= \varphi(q)\lhd\varphi(q')\end{aligned}$$

for all $q,q'\in Q$. Furthermore we get:

$$\begin{aligned}p_1\varphi\left(q\right) &= p_1\left(\alpha(q),\beta(q)\right)\\ &= \alpha(q)\\ p_2\varphi\left(q\right) &= p_2\left(\alpha(q),\beta(q)\right)\\ &= \beta(q)\end{aligned}$$

for all $q\in Q$ that proves the commutativity of (2).

Consider φ' with the same property as φ, i.e. the following conditions hold:

$$p_1\varphi' = \alpha$$
$$p_2\varphi' = \beta.$$

Define $(p,r)\in P\times_T R$ by $\varphi'(q) = (p,r)$. We get:

$$\begin{aligned}p_1\varphi'\left(q\right) = \alpha\left(q\right) &\Rightarrow p_1(p,r) = \alpha\left(q\right)\\ &\Rightarrow p = \alpha\left(q\right)\\ p_2\varphi'\left(q\right) = \beta\left(q\right) &\Rightarrow p_2\left(p,r\right) = \beta\left(q\right)\\ &\Rightarrow r = \beta\left(q\right)\end{aligned}$$

for all $q\in Q$ which yields:

$$\begin{aligned}\varphi'(q) &= (p,r)\\ &= \left(\alpha\left(q\right),\beta\left(q\right)\right)\\ &= \varphi(q)\end{aligned}$$

and proves that φ is unique.

Theorem 3. *The category of racks has equalizers.*

Proof. Let $f,g : P\to R$ be two rack morphisms. Define the set:

$$Q = \{p\in P \mid f\left(p\right) = g\left(p\right)\}.$$

Q is a subrack of P since:

$$\begin{aligned} f\left(p \lhd p'\right) &= f\left(p\right) \lhd f\left(p'\right) \\ &= g\left(p\right) \lhd g\left(p'\right) \\ &= g\left(p \lhd p'\right) \end{aligned}$$

for all $p, p' \in P$.

Also the inclusion morphism $u : Q \to P$ is a rack morphism since:

$$\begin{aligned} u\left(p \lhd p'\right) &= p \lhd p' \\ &= u\left(p\right) \lhd u\left(p'\right) \end{aligned}$$

for all $p, p' \in Q$. Furthermore for all $p \in Q$, we have:

$$\begin{aligned} \left(fu\right)\left(p\right) &= f\left(p\right) \\ &= g\left(p\right) \\ &= \left(gu\right)\left(p\right) \end{aligned}$$

and get:

$$fu = gu.$$

Let T be any rack with a rack morphism $v : T \to P$ where:

$$fv = gv.$$

Then there must be a unique rack morphism:

$$\phi : T \to Q$$

such that the following diagram commutes:

$$\begin{tikzcd} Q \arrow[r, hook, "u"] & P \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & R \\ T \arrow[u, dotted, "\exists!\phi"] \arrow[ur, "v"'] & & \end{tikzcd} \tag{3}$$

We can say that $v\left(t\right) \in Q$ since:

$$f(v\left(t\right)) = g(v\left(t\right))$$

for all $t \in T$. Define ϕ by $\phi\left(t\right) = v\left(t\right)$ for all $t \in T$. Then we get:

$$\begin{aligned} u\phi(t) &= uv(t) \\ &= v(t) \end{aligned}$$

for all $t \in T$ that satisfies $u\phi = v$ and proves the commutativity of (3).

Consider ϕ' with the same property as ϕ, i.e. $u\phi' = v$. Define $q \in Q$ by $\phi'(t) = q$. We get:

$$\begin{aligned} u\phi'(t) = v(t) &\Rightarrow u(q) = v(t) \\ &\Rightarrow q = v(t) \end{aligned}$$

for all $t \in T$ which yields:

$$\begin{aligned} \phi'(t) &= q \\ &= v(t) \\ &= \phi(t) \end{aligned}$$

and proves that φ is unique.

Acknowledgment

The first author is partially supported by TÜBİTAK.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Barr, M. and Wells, C., Category Theory, Lecture Notes for ESSLLI, (1999).
[2] Brieskorn, E., Automorphic Sets and Singularities, In ”Braids (Santa Cruz, CA, 1986)”, Contemporary Mathematics, (1988), 78, p.45–115.
[3] Conway, J.H., Wraith, G., Unpublished Correspondence, (1959).
[4] Crans , A., Wagemann, F., Crossed modules of racks, Homology, Homotopy and its Appl., (2014), 16(2), p.85–106.
[5] Emir, K., Gülsün Akay, H., Pullback Crossed Modules in the Category of Racks, Doi: 10.15672/HJMS.2017.532.
[6] Emir, K., Çetin, S., Limits in Modified Categories of Interest, to appear in Bulletin of the Iranian Mathematical Society.
[7] Fenn, R., Rourke, C., Racks and Links in Codimension Two, J. Knot Theory Ramifications 1, (1992), p.343–406.
[8] Kauffman, L.H., Knot-Crystals Classical Knot Theory in Modern Guise, Knots and Physics, World Sci.
[9] Mac Lane, S., Categories for the working mathematician, Second edition, Graduate Texts in Mathematics, 5. Springer-Verlag, New York, (1998).
[10] Stanovský, D., Left distributive left quasigroup, Ph.D. Thesis, Charles University in prague, (2004).