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Some results for nonexpansive mappings on metric spaces

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Abstract: Our work in this paper deals with some fixed point results for a class of nonexpansive single-valued and multi-valued mappings using Picard sequences in a complete metric space. Our results constitute generalizations of the corresponding results obtained by Demma et al. [6], Khojasteh et al. [11] and Vetro [15].

Keywords: Nonexpansive mappings, fixed point, metric space.

1 Introduction and preliminaries

Let (X,d) be a metric space and $f : X \to X$ be a single-valued mapping on X. A point $x \in X$ is a fixed point of f if $fx = x$. Let $F(f) = \{x \in X : fx = x\}$ denote the set of all fixed points of f on X. The mapping f is said to be

(i) *contraction* if there exists $k \in [0,1)$ such that $d(fx, fy) \leq kd(x,y)$ for all $x,y \in X$;
(ii) *nonexpansive* if $d(fx, fy) \leq d(x,y)$ for all $x,y \in X$;
(iii) *k-Lipschitz* if there exists a constant $k > 0$ such that $d(fx, fy) \leq kd(x,y)$ for all $x,y \in X$.

Banach's contraction principle [1] is a fundamental result in the fixed point theory, which has been used and extended in many different directions. These generalizations are made either by using contractive conditions or by imposing some additional conditions on the ambient spaces. The notion of nonexpansive mapping has an important role in fixed point theory. In fact, many researchers investigated the theory of nonexpansive mappings for establishing the existence of fixed points [7,8,10,14,15].

The aim of this paper is to give some results for the existence of fixed points for single-valued and multi-valued nonexpansive mappings in complete metric space endowed with a binary relation. We prove some theorems on distance between fixed points of single-valued mappings by using Picard sequence of any initial point x_0, say $\{x_n\}$ with $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. Also, we give results for fixed point of multi-valued nonexpansive mappings.

Now, we recall some notions for multi-valued mappings.

Let (X,d) be a metric space and let $CB(X)$ be the collection of all nonempty bounded and closed subsets of X. Let H be a Hausdorff metric induced by the metric d of X, that is

$$H(A,B) = max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

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for every $A,B \in CB(X)$ with

$$d(a,B) = inf\{d(a,x) : x \in B\}.$$

We recall the following definitions.

Definition 1. *A multivalued mapping $T : X \to CB(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that for any $x,y \in X$,*

$$H(Tx,Ty) \leq kd(x,y)$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \leq d(x,y)$$

for all $x,y \in X$. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

Definition 2. *Let M be a subset of $X \times X$ and let $f : X \to X$ be a mapping. Then, M is Banach f-invariant if $(fx, f^2x) \in M$ whenever $(x, fx) \in M$. Also, a subset Y of X is well ordered with respect to M if for all $x,y \in Y$ we have $(x,y) \in M$ or $(y,x) \in M$.*

The study of fixed points for multi-valued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [12] (see also [13]). The following result due Nadler [13] is a generalization of Banach contraction principle, to the case of a multi-valued mapping.

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued contraction mapping. Then T has a fixed point.*

Lemma 1. *If $\{x_n\}$ is a nonincreasing sequence of nonnegative real numbers, then for $r < m$, the sequence*

$$\left\{ \frac{x_n + 2x_{n+1} + r}{x_n + 2x_{n+1} + m} \right\}$$

is nonincreasing too.

Proof. We note that

$$\frac{x_n + 2x_{n+1} + r}{x_n + 2x_{n+1} + m} \geq \frac{x_{n+1} + 2x_{n+2} + r}{x_{n+1} + 2x_{n+2} + m}$$

if and only if

$$(x_n + 2x_{n+1} + r)(x_{n+1} + 2x_{n+2} + m) \geq (x_{n+1} + 2x_{n+2} + r)(x_n + 2x_{n+1} + m)$$

Since $\{x_n\}$ is a nonincreasing sequence, this inequality holds.

Corollary 1. *Let (X,d) be a metric space, $f : X \to X$ be a nonexpansive mapping and $x_0 \in X$. If $\{x_n\}$ is a Picard sequence, then the sequence*

$$\left\{ \frac{d(x_{n-1},x_n) + 2d(x_n,x_{n+1}) + r}{d(x_{n-1},x_n) + 2d(x_n,x_{n+1}) + m} \right\}$$

is nonincreasing for $r < m$.

Proof. From the nonexpansiveness of f and Lemma 1, the result holds.

2 Main results

Firstly, we prove some results for nonexpansive mappings defined on a metric space endowed with an arbitrary binary relation.

Theorem 2. *Let (X,d) be a complete metric space,$M \subset X \times X$ and $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx,fy) \le \left(\frac{d(x,fy)+d(y,fx)+d(y,f^2x)+r}{d(x,fx)+d(x,f^2x)+d(y,fy)+m} + k \right) d(x,y) \tag{1}$$

for all $(x,y) \in M$, where $k \in [0,1)$, $m,r \in \mathbb{R}^+$ such that $r < m$. Also assume that

(a) *M is Banach f-invariant;*
(b) *if $\{x_n\}$ is a sequence in X such that $(x_{n-1},x_n) \in M$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to \infty$, then $(x_{n-1},z) \in M$ for all $n \in \mathbb{N}$;*
(c) *$F(f)$ is well ordered with respect to M.*
If there exists $x_0 \in X$ such that $(x_0, fx_0) \in M$ and

$$\frac{d(x_0,fx_0)+2d(fx_0,f^2x_0)+r}{d(x_0,fx_0)+2d(fx_0,f^2x_0)+m} + k < 1 \tag{2}$$

then
(i) *f has a fixed point $z \in X$;*
(ii) *for any $x_0 \in X$, the Picard sequence converges to a fixed point of f ;*
(iii) *if $z,w \in X$ are two different fixed points of f, then $d(z,w) \ge \max\left\{\frac{m(1-k)-r}{3}, 0\right\}$.*

Proof. Let $x_0 \in X$ be such that $(x_0, fx_0) \in M$ and $\{x_n\}$ be a Picard sequence of initial point $x_0 \in X$. Suppose that (2) holds. Taking $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, x_{n-1} is a fixed point of f. That is, the existence of a fixed point is obvious. Now we suppose that $x_n \ne x_{n-1}$ for all $n \in \mathbb{N}$. Since M is Banach f-invariant, we obtain that $(x_1,x_2) = \left(fx_0, f^2x_0\right) \in M$ for $(x_0,x_1) = (x_0, fx_0) \in M$. From (1) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$\begin{aligned} d(x_n,x_{n+1}) = d(fx_{n-1},fx_n) &\le \left[\frac{d(x_{n-1},x_{n+1})+d(x_n,x_{n+1})+r}{d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_n,x_{n+1})+m} + k \right] d(x_{n-1},x_n) \\ &\le \left[\frac{d(x_{n-1},x_n)+2d(x_n,x_{n+1})+r}{d(x_{n-1},x_n)+2d(x_n,x_{n+1})+m} + k \right] d(x_{n-1},x_n) \end{aligned} \tag{3}$$

for all $n \in \mathbb{N}$. From the inequality (3) and Corollary 1, we get

$$\begin{aligned} d(x_n,x_{n+1}) &\le \left[\frac{d(x_{n-1},x_n)+2d(x_n,x_{n+1})+r}{d(x_{n-1},x_n)+2d(x_n,x_{n+1})+m} + k \right] d(x_{n-1},x_n) \\ &\le \left[\frac{d(x_0,x_1)+2d(x_1,x_2)+r}{d(x_0,x_1)+2d(x_1,x_2)+m} + k \right] d(x_{n-1},x_n) \\ &= \left[\frac{d(x_0,fx_0)+2d(fx_0,f^2x_0)+r}{d(x_0,fx_0)+2d(fx_0,f^2x_0)+m} + k \right] d(x_{n-1},x_n) \end{aligned} \tag{4}$$

where

$$\gamma = \frac{d(x_0,x_1)+2d(x_1,x_2)+r}{d(x_0,x_1)+2d(x_1,x_2)+m} + k < 1.$$

From (4), $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, the sequence $\{x_n\}$ converges to $z \in X$. Now, we will show that z is a fixed point of f. Using hypothesis (b), we get that $(x_n,z) \in M$. Therefore, from the condition (1) with

$x = x_n$ and $y = z$, we obtain

$$d(x_{n+1}, fz) = d(fx_n, fz) \leq \left[\frac{\begin{array}{c} d(x_n, fz) + d(z, fx_n) \\ +d(z, f^2x_n) + r \end{array}}{\begin{array}{c} d(x_n, fx_n) + d(x_n, f^2x_n) \\ +d(z, fz) + m \end{array}} + k\right] d(x_n, z) \tag{5}$$
$$= \left[\frac{d(x_n, fz) + d(z, x_{n+1}) + d(z, x_{n+2}) + r}{d(x_n, x_{n+1}) + d(x_n, x_{n+2}) + d(z, fz) + m} + k\right] d(x_n, z).$$

If we take limit as $n \to +\infty$ on both sides of (5), we get $d(z, fz) \leq 0$. This implies that $d(z, fz) = 0$, that is, $z = fz$ and hence z is a fixed point of f. Thus (i) and (ii) hold.

If $w \in X$, with $z \neq w$, is another fixed point of f, then using (1) with $x = z$ and $y = w$, we get

$$d(z,w) = d(fz, fw) \leq [\frac{d(z, fw) + d(w, fz) + d(w, f^2z) + r}{d(z, fz) + d(z, f^2z) + d(w, fw) + m} + k]d(z,w) = \left[\frac{3d(z,w) + r}{m} + k\right] d(z,w)$$

and hence $d(z,w) \geq \frac{m(1-k)-r}{3}$, that is, (iii) holds.

We consider a weak contractive condition to prove the following theorem.

Theorem 3. *Let (X,d) be a complete metric space, $M \subset X \times X$ and $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx) + d(y, f^2x) + r}{d(x, fx) + d(x, f^2x) + d(y, fy) + m} + k\right) d(x,y) + Ld(y, fx) \tag{6}$$

for all $(x,y) \in M$, where $k \in [0,1)$, $m,r,L \in \mathbb{R}^+$ such that $r < m$. Also assume that

(a) *M is Banach f-invariant;*
(b) *if $\{x_n\}$ is a sequence in X such that $(x_{n-1}, x_n) \in M$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to \infty$, then $(x_{n-1}, z) \in M$ for all $n \in \mathbb{N}$;*
(c) *$F(f)$ is well ordered with respect to M. If there exists $x_0 \in X$ such that $(x_0, fx_0) \in M$ and (2) holds then*
  (i) *f has a fixed point $z \in X$;*
  (ii) *for any $x_0 \in X$, the Picard sequence converges to a fixed point of f;*
  (iii) *if $z,w \in X$ are two different fixed points of f, then $d(z,w) \geq \max\left\{\frac{m(1-k-L)-r}{3}, 0\right\}$.*

Proof. Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be a Picard sequence of initial point x_0. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of f. If $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, using the contractive condition (6) with $x = x_{n-1}$ and $y = x_n$, we get

$$d(x_n, x_{n+1}) \leq \left[\frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1}) + r}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + m} + k\right] d(x_{n-1}, x_n) + Ld(x_n, x_n),$$
$$\leq \left[\frac{d(x_{n-1}, x_n) + 2d(x_n, x_{n+1}) + r}{d(x_{n-1}, x_n) + 2d(x_n, x_{n+1}) + m} + k\right] d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Then, by Corollary 1, $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, the sequence $\{x_n\}$ converges to some $z \in X$. Now, we prove that z is a fixed point for f. Using (6) with $x = x_n$ and $y = z$, we obtain

$$d(x_{n+1}, fz) = d(fx_n, fz) \leq \left[\frac{d(x_n, fz) + d(z, fx_n) + d(z, f^2x_n) + r}{d(x_n, fx_n) + d(x_n, f^2x_n) + d(z, fz) + m} + k\right] d(x_n, z) + Ld(z, fx_n) \tag{7}$$
$$= \left[\frac{d(x_n, fz) + d(z, x_{n+1}) + d(z, x_{n+2}) + r}{d(x_n, x_{n+1}) + d(x_n, x_{n+2}) + d(z, fz) + m} + k\right] + Ld(z, x_{n+1})$$

On taking limit as $n \to +\infty$ on both sides of (7), we get $d(z, fz) \leq 0$. This implies that $d(z, fz) = 0$, that is, $z = fz$. Hence z is a fixed point of f. Thus (i) and (ii) hold.

If $w \in X$, with $z \neq w$, is another fixed point of f, then using (6) with $x = z$ and $y = w$, we get

$$d(z, w) = d(fz, fw) \leq [\frac{d(z, fw) + d(w, fz) + d(w, f^2z) + r}{d(z, fz) + d(z, f^2z) + d(w, fw) + m} + k]d(z, w) + Ld(w, fz)$$
$$= \left[\frac{3d(z, w) + r}{m} + k\right] d(z, w) + Ld(z, w)$$

and hence $d(z, w) \geq \max\left\{\frac{m(1-k-L)-r}{3}, 0\right\}$, that is, (iii) holds.

Remark. If in two above theorems let $r = 0, m = 1$, we have the following results, which generalize Theorems 3.1 and 3.2 in [15], respectively.

Corollary 2. *Let (X, d) be a complete metric space, $M \subset X \times X$ and $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left[\frac{d(x, fy) + d(y, fx) + d(y, f^2x)}{d(x, fx) + d(x, f^2x) + d(y, fy) + 1} + k\right] d(x, y)$$

for all $(x, y) \in M$, where $k \in [0, 1)$, $r \in \mathbb{R}^+$ such that $r < 1$. Then

(a) *M is Banach f-invariant;*
(b) *if $\{x_n\}$ is a sequence in X such that $(x_{n-1}, x_n) \in M$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to \infty$, then $(x_{n-1}, z) \in M$ for all $n \in \mathbb{N}$;*
(c) *$F(f)$ is well ordered with respect to M. If there exists $x_0 \in X$ such that $(x_0, fx_0) \in M$ and*

$$\frac{d(x_0, fx_0) + 2d(fx_0, f^2x_0)}{d(x_0, fx_0) + 2d(fx_0, f^2x_0) + 1} + k < 1 \tag{8}$$

then
  (i) *f has a fixed point $z \in X$;*
  (ii) *for any $x_0 \in X$, the Picard sequence converges to a fixed point of f;*
  (iii) *if $z, w \in X$ are two different fixed points of f, then $d(z, w) \geq \max\left\{\frac{1-k}{3}, 0\right\}$.*

Corollary 3. *Let (X, d) be a complete metric space, $M \subset X \times X$ and $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx) + d(y, f^2x)}{d(x, fx) + d(x, f^2x) + d(y, fy) + 1} + k\right) d(x, y) + Ld(y, fx)$$

for all $(x, y) \in M$, where $k \in [0, 1)$, $r \in \mathbb{R}^+$ such that $r < 1$. Also assume that

(a) *M is Banach f-invariant;*

(b) *if* $\{x_n\}$ *is a sequence in* X *such that* $(x_{n-1}, x_n) \in M$ *for all* $n \in \mathbb{N}$ *and* $x_n \to z \in X$ *as* $n \to \infty$, *then* $(x_{n-1}, z) \in M$ *for all* $n \in \mathbb{N}$;

(c) $F(f)$ *is well ordered with respect to* M. *If there exists* $x_0 \in X$ *such that* $(x_0, fx_0) \in M$ *and* (8) *holds then*

  (i) f *has a fixed point* $z \in X$;

  (ii) *for any* $x_0 \in X$, *the Picard sequence converges to a fixed point of* f;

  (iii) *if* $z, w \in X$ *are two different fixed points of* f, *then* $d(z,w) \geq \max\left\{\frac{1-L-k}{3}, 0\right\}$.

Remark. The above the corollaries generalize the Theorems 3.1 and 3.2 of [15].

Taking $M = X \times X$ in Theorems 2 and 3, we have the following results.

Theorem 4. *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx) + d(y, f^2x) + r}{d(x, fx) + d(x, f^2x) + d(y, fy) + m} + k\right) d(x,y) \tag{9}$$

for all $x, y \in X$, *where* $k \in [0,1)$, $m, r \in \mathbb{R}^+$ *such that* $r < m$. *If there exists* $x_0 \in X$ *such that*

$$\frac{d(x_0, fx_0) + 2d(fx_0, f^2x_0) + r}{d(x_0, fx_0) + 2d(fx_0, f^2x_0) + m} + k < 1 \tag{10}$$

then

(i) f *has a fixed point* $z \in X$;

(ii) *for any* $x_0 \in X$, *the Picard sequence converges to a fixed point of* f;

(iii) *if* $z, w \in X$ *are two different fixed points of* f, *then* $d(z,w) \geq \max\left\{\frac{m(1-k)-r}{3}, 0\right\}$.

Theorem 5. *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx) + d(y, f^2x) + r}{d(x, fx) + d(x, f^2x) + d(y, fy) + m} + k\right) d(x,y) + Ld(y, fx)$$

for all $x, y \in X$, *where* $k \in [0,1)$, $m, r, L \in \mathbb{R}^+$ *such that* $r < m$. *If there exists* $x_0 \in X$ *and* (10) *holds then*

(i) f *has a fixed point* $z \in X$;

(ii) *for any* $x_0 \in X$, *the Picard sequence converges to a fixed point of* f;

(iii) *if* $z, w \in X$ *are two different fixed points of* f, *then* $d(z,w) \geq \max\left\{\frac{m(1-k-L)-r}{3}, 0\right\}$.

Remark. (i) If choosing $r = k = 0$ and $m = 1$ in Theorem 4, our result generalize Theorem 1 in [11].

(ii) If choosing $r = 0$ and $m = 1$ in Theorems 4 and 5, our results generalize Theorems 4.1 and 4.2 in [15].

In this part of the paper, we give some theorems for nonexpansive multi-valued mappings which can be proved in a similar way as above theorems. From now on, let $K(X)$ be the collection of all nonempty compact subsets of X.

Theorem 6. *Let* (X,d) *be a complete metric space and* $T : X \to K(X)$ *be a nonexpansive multi-valued mapping such that*

$$H(Tx, Ty) \leq \left(\frac{d(x, Ty) + d(y, Tx) + d(y, T^2x) + r}{d(x, Tx) + d(x, T^2x) + d(y, Ty) + m} + k\right) d(x,y) \tag{11}$$

for all $x, y \in X$, *where* $k \in [0,1)$, $m, r \in \mathbb{R}^+$ *such that* $r < m$. *If there exists* $x_0 \in X$ *such that*

$$\frac{d(x_0, Tx_0) + 2d(Tx_0, T^2x_0) + r}{d(x_0, Tx_0) + 2d(Tx_0, T^2x_0) + m} + k < 1 \tag{12}$$

holds, where $x_1 \in Tx_0$ *is such that* $d(x_0, x_1) = d(x_0, Tx_0)$, *then* T *has a fixed point.*

Proof. Let be $x_0 \in X$ an arbitrary point. Since Tx_0 is compact there exists $x_1 \in Tx_0$ such that $d(x_0,x_1) = d(x_0,Tx_0)$. If we suppose that $x_0 = x_1$ or $x_1 \in Tx_1$, then x_1 is a fixed point of T. So, the proof is complete. Now, we assume that $x_0 \neq x_1, x_1 \notin Tx_1$ and (12) holds. From $d(x_1,Tx_1) > 0$, we have $H(Tx_0,Tx_1) > 0$. Again by the compactness of Tx_1 we obtain $x_2 \in Tx_1$ such that $d(x_1,x_2) = d(x_1,Tx_1)$. Using (12), we have

$$\begin{aligned} d(x_1,x_2) = d(x_1,Tx_1) &\leq H(Tx_0,Tx_1) \\ &\leq \left(\frac{d(x_0,Tx_1) + d(x_1,Tx_0) + d\left(x_1,T^2x_0\right) + r}{d(x_0,Tx_0) + d\left(x_0,T^2x_0\right) + d(x_1,Tx_1) + m} + k \right) d(x_0,x_1) \\ &\leq \left(\frac{d(x_0,x_2) + d(x_1,x_2) + r}{d(x_0,x_1) + d(x_1,x_2) + d(x_1,x_2) + m} + k \right) d(x_0,x_1) \\ &\leq \left(\frac{d(x_0,x_1) + 2d(x_1,x_2) + r}{d(x_0,x_1) + 2d(x_1,x_2) + m} + k \right) d(x_0,x_1) \end{aligned}$$

Now, we assume that $x_1, x_2, ..., x_n \in X$ such that $x_{i+1} \in Tx_i$, $x_i \notin Tx_i$ and

$$d(x_i,x_{i+1}) = d(x_i,Tx_i) \leq \left(\frac{d(x_{i-1},x_i) + 2d(x_i,x_{i+1}) + r}{d(x_{i-1},x_i) + 2d(x_i,x_{i+1}) + m} + k \right) d(x_{i-1},x_i)$$

for all $\imath = 1,2,...,n-1$. Repeating the same procedure as above, we get Tx_n is compact. That is, there exists $x_{n+1} \in Tx_n$ such that $d(x_n,x_{n+1}) = d(x_n,Tx_n)$. Then

$$d(x_n,x_{n+1}) = d(x_n,Tx_n) \leq H(Tx_{n-1},Tx_n) \leq \left(\frac{d(x_{n-1},x_n) + 2d(x_n,x_{n+1}) + r}{d(x_{n-1},x_n) + 2d(x_n,x_{n+1}) + m} + k \right) d(x_{n-1},x_n). \tag{13}$$

Let be $x_n = x_{n+1}$ for all $n \in \mathbb{N}$. Then x_n is a fixed point of T and the proof is complete. Now, we assume that $x_n \notin Tx_n$, so we obtain a sequence $\{x_n\} \subset X$ such that $x_{n+1} \in Tx_n$ and (13) holds for all $n \in \mathbb{N}$. By using similar way as in the proof of Theorem 2, we get that $\{x_n\}$ is a Cauchy sequence. Therefore since is complete then there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Finally, we will show that z is a fixed point of T. From (11) with $x = x_n$ and $y = z$, we have

$$\begin{aligned} d(z,Tz) &\leq d(z,x_{n+1}) + d(x_{n+1},Tz) \leq d(z,x_{n+1}) + H(Tx_n,Tz) \\ &\leq d(z,x_{n+1}) + \left(\frac{d(x_n,Tz) + d(z,Tx_n) + d\left(z,T^2x_n\right) + r}{d(x_n,Tx_n) + d\left(x_n,T^2x_n\right) + d(z,Tz) + m} + k \right) d(x_n,z) \\ &\leq d(z,x_{n+1}) + \left(\frac{d(z,Tz) + d(z,x_{n+1}) + d(z,x_{n+2}) + r}{d(x_n,x_{n+1}) + d(x_n,x_{n+2}) + d(z,Tz) + m} + k \right) d(x_n,z). \end{aligned} \tag{14}$$

On taking limit as $n \to +\infty$ on both sides of (14), we get $d(z,Tz) \leq 0$. This implies that $d(z,Tz) = 0$. As Tz is closed, we get that $z \in Tz$, that is, z is a fixed point of T.

Using the similar method as in the proof of Theorem 6, we have the following theorem.

Theorem 7. *Let (X,d) be a complete metric space and $T : X \to K(X)$ be a nonexpansive multi-valued mapping such that*

$$H(Tx,Ty) \leq \left(\frac{d(x,Ty) + d(y,Tx) + d(y,T^2x) + r}{d(x,Tx) + d(x,T^2x) + d(y,Ty) + m} + k \right) d(x,y) + Ld(y,Tx)$$

for all $x,y \in X$, where $k \in [0,1)$, $m,r,L \in \mathbb{R}^+$ such that $r < m$. If there exists $x_0 \in X$ and (12) holds, then T has a fixed point.

Remark. If in Theorems 6 and 7 we take the parameter $r = 0$ and $m = 1$ then we get that the generalizations of Theorems 5.1 and 5.2 in [15].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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