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On Ricci pseudo-symmetric para-Kenmotsu manifolds

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Abstract: Considered a para-Kenmotsu manifold with the curvature condition $S(X,Y).R = 0$ and shown that it is an Einstein manifold. Further, we consider para-Kenmotsu manifolds with the conditions $R(X,Y).S = fQ(g,S)$ and $R(X,Y).R = fQ(S,R)$, known as the *Ricci* and *generalised Ricci* pseudo-symmetric manifolds respectively, and obtained the necessary conditions for these manifolds to be non-Einstein. The notations $S(X,Y)$ and $R(X,Y)$ denote the Ricci and Riemannian curvature tensors respectively.

Keywords: Para Kenmotsu manifold, Ricci pseudo-symmetric manifold, Einstein manifold, Ricci tensor.

1 Introduction

Sato [10] defined the notions of an almost para contact Riemannian manifold. After that, Adati and Matsumoto [1] defined and studied p-Sasakian and sp-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [9] defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [14] defined a class of almost para contact metric manifolds namely para Kenmotsu (briefly p-Kenmotsu) and special para Kenmotsu (briefly sp -Kenmotsu) manifolds.

As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. Locally symmetric, semisymmetric and pseudosymmetric para-Sasakian manifolds are widely studied by many geometers [2,5,6].

Motivated by these studies, Satyanarayana and Sai Prasad [12] studied Weyl semisymmetric para-Kenmotsu manifolds, and they prove that such a manifold is conformally flat and hence is an sp-Kenmotsu manifold. Further, they studied [13] Weyl-pseudosymmetric para-Kenmotsu manifolds which are the extended classes of Weyl-semisymmetric para-Kenmotsu manifolds. They showed that every n-dimensional, $n \geq 4$, para-Kenmotsu manifold is a Weyl-pesudosymmetric manifold of the form $R.\ C = -Q(g,C)$. Also, they studied para-Kenmotsu manifolds satisfying the condition $C(X,Y).S = 0$ where $C(X,Y)$ is the Weyl conformal curvature tensor and S is the Ricci tensor of the manifold [13].

In this study, our aim is to obtain the characterisations of Ricci-pseudosymmetric para-Kenmotsu manifolds and also the para-Kenmotsu manifold satisfying the curvature condition $S(X,Y).R = 0$ where $R(X,Y)$ is the curvature tensor and $S(X,Y)$ is the Ricci tensor of the manifold.

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2 Preliminaries

Let M_n be an n-dimensional differentiable manifold equipped with structure tensors (Φ, ξ, η) where Φ is a tensor of type (1, 1), ξ is a vector field, η is a 1-form such that

$$(a)\ \ \eta(\xi) = 1$$
$$(b)\ \ \Phi^2(X) = X - \eta(X)\xi;\ \overline{X} = \Phi X. \tag{1}$$

Then the manifold M_n is called an almost para contact manifold.

Let g be the Riemannian metric such that, for all vector fields X and Y on M_n

$$(a)\ \ g(X,\xi) = \eta(X)$$
$$(b)\ \ \Phi\xi = 0, \eta(\Phi X) = 0,\ \text{rank}\ \Phi = n-1 \tag{2}$$
$$(c)\ \ g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

Then the manifold M_n [10] is said to admit an almost para contact Riemannian structure (Φ, ξ, η, g).

A manifold of dimension n with Riemannian metric g admitting a tensor field Φ of type (1,1), a vector field ξ and a 1-form η satisfying (1), (2) along with

$$(a)\ \ (\nabla_X \eta)Y - (\nabla_Y \eta)X = 0$$
$$(b)\ \ (\nabla_X \nabla_Y \eta)Z = [-g(X,Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X,Y) + \eta(X)\eta(Y)]\eta(Z) \tag{3}$$
$$(c)\ \ \nabla_X \xi = \Phi^2 X = X - \eta(X)\xi$$
$$(d)\ \ (\nabla_X \Phi)Y = -g(X, \Phi Y)\xi - \eta(Y)\Phi X$$

is called a para-Kenmotsu manifold or briefly p-Kenmotsu manifold [14].

A p-Kenmotsu manifold admitting a 1-form η satisfying

$$(a)\ \ (\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y),\ \text{and} \tag{4}$$
$$(b)\ \ (\nabla_X \eta)Y = \varphi(\overline{X}, Y),\ \text{where } \varphi \text{ is an associate of } \Phi$$

is called a special para-Kenmotsu manifold or briefly sp-Kenmotsu manifold [14].

Let (M_n, g) be an n-dimensional, $n \geq 3$, differentiable manifold of class C^∞ and let ∇ be its Levi-Civita connection. Then the Riemannian Christoffel curvature tensor R of type (1, 3) is given by [3]:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{BW}. \tag{5}$$

where $BW = [X,Y]^Z$, The Ricci operator S and the (0,2)-tensor S^2 is defined by

$$g(SX,Y) = S(X,Y), \tag{6}$$

and

$$S^2(X,Y) = S(SX,Y). \tag{7}$$

It is known that in a p-Kenmotsu manifold the following relations hold good [14]:

$$
\begin{aligned}
&(a)\ S(X,\xi) = -(n-1)\eta(X) \\
&(b)\ g[R(X,Y)Z,\xi] = \eta[R(X,Y,Z)] = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \\
&(c)\ R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi \\
&(d)\ R(X,Y,\xi) = \eta(X)Y - \eta(Y)X;\ \text{when } X \text{ is orthogonal to } \xi.
\end{aligned} \tag{8}
$$

If the Ricci curvature tensor S is of the form

$$S = aI_d + b\eta \otimes \xi, \tag{9}$$

where a and b are smooth functions on M_n, then the almost paracontact Riemannian manifold M_n is called as an η-Einstein manifold and if $b = 0$ then it is an Einstein manifold [2].

Furthermore we define the tensors $R(X,Y).S$ and $R(X,Y).R$ on (M_n, g) by

$$(R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W) \tag{10}$$

and

$$(R(X,Y).R)(Z,W)U = R(X,Y)R(Z,W)U - R(R(X,Y)Z,W)U - R(Z,R(X,Y)W)U - R(Z,W)R(X,Y)U; \tag{11}$$

where $X,Y,Z,W \in \chi(M_n)$, $\chi(M_n)$ being the Lie algebra of vector fields on M_n.

For a $(0,k)$-tensor field T, $k \geq 1$, on (M_n, g) we define the tensors $R.\ T$ and $Q(g,T)$ by

$$(R(X,Y).T)(X_1,X_2,...,X_k) = -T(R(X,Y)X_1,X_2,...,X_k) - \cdots - T(X_1,...,X_{k-1},R(X,Y)X_k), \tag{12}$$

$$Q(g,T)(X_1,X_2,...,X_k;X,Y) = -T((X\wedge Y)X_1,X_2,...,X_k) - \cdots - T(X_1,...,X_{k-1},(X\wedge Y)X_k) \tag{13}$$

respectively [8,15], where the endomorphism $(X \wedge Y)$ is defined by

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{14}$$

If the tensors $R(X,Y).S$ and $Q(g,S)$ are linearly dependent then the manifold M_n is called Ricci pseudo-symmetric [15]. This is equivalent to

$$R.S = f\ Q(g,S), \tag{15}$$

holding on the set U_S = {x$\in M_n$/S $\neq$ 0 at x}, where f is some function of U_S.

Analogously, if the tensors $R(X,Y).R$ and $Q(S,R)$ are linearly dependent then the manifold M_n is called Ricci generalised pseudo-symmetric [15]. This is equivalent to

$$R.R = f\ Q(S,R), \tag{16}$$

holding on the set U_R = {x∈ M_n/R $\neq 0$ at x}, where f is some function of U_R. An important subclass of this class of manifolds realizing the condition is:

$$R.R = Q(S,R). \tag{17}$$

For example, in the literature cited in [7], every three dimensional manifold satisfies the above equation identically. Also, the above equation will be satisfied by the semi-Riemannian manifolds which satisfies the equality $\omega(X)R(Y,Z)+\omega(Y)R(Z,X)+\omega(Z)R(X,Y) = 0$, where ω is a non-zero 1-form.

Moreover, the condition $R.R = Q(S,R)$ also appears in the theory of plane gravitational waves.

The paper is organized as follows: After preliminaries, in section 3 we characterise a para-Kenmotsu manifold satisfying the curvature condition $S(X,Y).R = 0$. In Section 4, we study Ricci pseudo-symmetric p-Kenmotsu manifold. It is shown that a p-Kenmotsu manifold is Ricci pseudo-symmetric if and only if it is an Einstein manifold provided $f \neq -1$. Further, the concept of Ricci pseudo-symmetric p-Kenmotsu manifold is generalised and shown that the Ricci generalised pseudo-symmetric p-Kenmotsu manifold is an Einstein manifold provided $nf \neq 1$.

3 *P*-Kenmotsu manifold satisfying the curvature condition $S(X,Y).R = 0$

In this section our aim is to find the characterisation of para-Kenmotsu manifolds satisfying the curvature condition $S(X,Y).R = 0$.

Theorem 1. *Let M_n $(n \geq 4)$ be an n-dimensional p-Kenmotsu manifold. If the condition $S(X,Y).R = 0$ holds on M_n then the manifold is an η-Einstein manifold.*

Proof. Assume that M_n is an n-dimensional, $n \geq 4$, p-Kenmotsu manifold satisfying the condition $S(X,Y).R = 0$. Then we have

$$(S(X,Y).R)(U,V)W = 0. \tag{18}$$

The equation (18) implies

$$(X \wedge_S Y)R(U,V)W + R((X \wedge_S Y)U,V)W + R(U,(X \wedge_S Y)V)W + R(U,V)(X \wedge_S Y)W = 0. \tag{19}$$

Then from (14), the above equation reduces to

$$\begin{aligned} S(Y,R(U,V)W)X - S(X,R(U,V)W)Y + S(Y,U)R(X,V)W - S(X,U)R(Y,V)W + S(Y,V)R(U,X)W \\ - S(X,V)R(U,Y)W + S(Y,W)R(U,V)X - S(X,W)R(U,V)Y = 0. \end{aligned} \tag{20}$$

By substituting $U = W = \xi$ in (20) and on using equations (c) *and* (d) of (8), we get

$$\begin{aligned} 2S(Y,V)X - 2S(X,V)Y + 2(n-1)\eta(V)\eta(Y)X - 2(n-1)\eta(X)\eta(V)Y - S(Y,V)\eta(X)\xi \\ + S(X,V)\eta(Y)\xi + (n-1)g(V,X)\eta(Y)\xi - (n-1)g(V,Y)\eta(X)\xi = 0. \end{aligned} \tag{21}$$

Now by taking the inner product of (21) with ξ, and on replacing X with ξ, then from equation (8)(a) we get that

$$S(Y,V) = (n-1)g(Y,V) - 2(n-1)\eta(Y)\eta(V). \tag{22}$$

This proves that the p-Kenmotsu manifold with the condition $S(X,Y).R = 0$ is an η-Einstein manifold.

4 Ricci pseudo-symmetric para-Kenmotsu manifolds

In this section, we consider the manifold satisfying the condition $R(X,Y).S = fQ(g,S)$, known as the Ricci pseudo-symmetric manifold. Let us assume that the manifold M_n $(n \geq 4)$ is an n-dimensional Ricci pseudo-symmetric para-Kenmotsu manifold and $X,Y,U,V \in \chi(M_n)$.

Then from (15), we have

$$(R(X,Y).S)(U,V) = fQ(g,S)(X,Y;U,V). \tag{23}$$

The above equation is equivalent to

$$(R(X,Y).S)(U,V) = f((X \wedge_g Y).S)(U,V). \tag{24}$$

Then by using (10), (13) and (24), we have

$$-S(R(X,Y)U,V) - S(U,R(X,Y)V) = f[-S((X \wedge_g Y)U,V) - S(U,(X \wedge_g Y)V)]. \tag{25}$$

Using (14), equation (25) reduces to

$$-S(R(X,Y)U,V) - S(U,R(X,Y)V) = f[-g(Y,U)S(X,V) + g(X,U)S(Y,V) - g(Y,V)S(U,X) + g(X,V)S(U,Y)]. \tag{26}$$

By substituting $X = U = \xi$ in (26) and on using the equations (a) and (c) of (8), we get

$$(1+f)[S(Y,V) + (n-1)g(Y,V)] = 0. \tag{27}$$

Then from (27), either $f = -1$ or the manifold is an Einstein manifold of the form $S(Y,V) = (1-n)g(Y,V)$. Hence, from the above result, we propose the following.

Proposition 1. *Every n-dimensional Ricci pseudo-symmetric para-Kenmotsu manifold M_n is of the form $R(X,Y).S = -Q(g,S)$, provided the manifold is non-Einstein.*

Conversely, if the manifold is an Einstein manifold of the form $S(Y,V) = (1-n)g(Y,V)$, then it is clear that $R(X,Y).S = fQ(g,S)$.

Thus we state the following.

Theorem 2. *An n-dimensional para-Kenmotsu manifold M_n is Ricci pseudo-symmetric if and only if the manifold is an Einstein manifold provided $f \neq -1$.*

In particular, if we consider $Q(g,S) = 0$, then we can state the following.

Corollary 1. *An n-dimensional para-Kenmotsu manifold M_n satisfies the condition $Q(g,S) = 0$ if and only if M_n is an Einstein manifold.*

5 Ricci generalised pseudo-symmetric para-Kenmotsu manifolds

In this section, we consider the manifold satisfying the condition $R(X,Y).R = fQ(S,R)$, known as the Ricci generalised pseudo-symmetric manifold. Let us assume that the manifold M_n $(n \geq 4)$ is an n-dimensional generalised Ricci pseudo-symmetric para-Kenmotsu manifold and $X,Y,U,V \in \chi(M_n)$.

Then from (16), we have

$$(R(X,Y).R)(U,V)W = fQ(S,R)(X,Y;U,V)W. \tag{28}$$

The above equation is equivalent to

$$(R(X,Y).R)(U,V)W = f((X\wedge_S Y).R)(U,V)W. \tag{29}$$

Then by using (11), (13) and (29), we get

$$\begin{aligned} R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W = f[(X\wedge_S Y)R(U,V)W \\ - R((X\wedge_S Y)U,V)W - R(U,(X\wedge_S Y)V)W - R(U,V)(X\wedge_S Y)W]. \end{aligned} \tag{30}$$

Using (14), equation (30) reduces to

$$\begin{aligned} R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W = f[S(Y,R(U,V)W)X \\ - S(X,R(U,V)W)Y - S(Y,U)R(X,V)W + S(X,U)R(Y,V)W - S(Y,V)R(U,X)W \\ + S(X,V)R(U,Y)W - S(Y,W)R(U,V)X + S(X,W)R(U,V)Y. \end{aligned} \tag{31}$$

By substituting $X = U = \xi$ in (31) and on using the equations (a), (c) and (d) of (8), we get

$$\begin{aligned} -g(V,W)Y + g(V,W)\eta(Y)\xi - R(Y,V)W + \eta(Y)\eta(W)V - g(V,W)\eta(Y)\xi - \eta(Y)\eta(W)V + g(Y,W)V \\ = f[\eta(W)S(Y,V)\xi - (n-1)g(V,W)Y - (n-1)R(Y,V)W - S(Y,W)V \\ + (n-1)g(Y,W)\eta(V)\xi + S(Y,W)\eta(V)\xi + (n-1)g(V,Y)\eta(W)\xi]. \end{aligned} \tag{32}$$

Now, by taking the inner product of (32), we get

$$\begin{aligned} -g(V,W)g(Y,Z) - g(R(Y,V)W,Z) + g(Y,W)g(V,Z) = f[S(Y,V)\eta(W)\eta(Z) - (n-1)g(V,W)g(Y,Z) \\ - (n-1)g(R(Y,V)W,Z) - S(Y,W)g(V,Z) + (n-1)g(Y,W)\eta(V)\eta(Z) \\ + S(Y,W)\eta(V)\eta(Z) + (n-1)g(V,Y)\eta(W)\eta(Z)]. \end{aligned} \tag{33}$$

Let $\{e_i\}$ $(1\le i\le n)$ be an orthonormal basis of the tangent space at any point. Now, by taking the summation over $i = 1,2,..,n$ of the relation (33) for $V = W = e_i$, we get

$$S(Y,Z) + (n-1)g(Y,Z) = n\, f[S(Y,Z) + (n-1)g(Y,Z)]. \tag{34}$$

The above equation implies either $f = 1/n$ or the manifold is an Einstein manifold of the form $S(Y,Z) = -(n-1)$. Hence, from the above result, we propose the following.

Proposition 2. *Every n-dimensional Ricci generalised pseudo-symmetric para-Kenmotsu manifold M_n is of the form $R(X,Y).R = 1/n\ Q(S,R)$, provided the manifold is non-Einstein.*

Thus we state the following.

Theorem 3. *An n-dimensional Ricci generalised para-Kenmotsu manifold M_n is an Einstein manifold provided $nf \neq 1$.*

In particular, if we consider $Q(S,R) = 0$, then from the above theorem we state the following.

Corollary 2. *If an n-dimensional para-Kenmotsu manifold M_n satisfies the condition $Q(S,R) = 0$ then the manifold M_n is an Einstein manifold.*

Remark. The above findings are quite in similar to the results obtained for Ricci pseudo-symmetric para-Sasakian manifolds [7].

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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