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Bounds on initial coefficients for a new subclass of bi-univalent functions

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Abstract: In recent years, the study of bi-univalent functions have gathered momentum mainly due to the pioneering work of Srivastava et al. [19], which has actually revived the study of the coefficient problems involving bi-univalent functions. With motivation from the work of Srivastava et al. [19], in the present paper we introduce a new subclass $\mathscr{T}_\Sigma[\phi]$ of the function class Σ of bi-univalent functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, for the functions in this subclass $\mathscr{T}_\Sigma[\phi]$ we obtain bounds on $|a_2|$, $|a_3|$ and $|a_4|$.

Keywords: Analytic function, Univalent function, Bi-univalent function, Coefficient bound.

1 Introduction

Let $\mathscr{H}$ denote the class of functions analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathscr{A}$ denote the class of functions in $\mathscr{H}$ given by:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

and $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of the functions univalent in $\mathbb{U}$. We have from the Koebe one quarter theorem [5] that the image of $\mathbb{U}$ under every function $f \in \mathscr{S}$ contains a disk of radius $1/4$. Hence, every function $f \in \mathscr{S}$ has an inverse f^{-1} such that $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and $f\left(f^{-1}(w)\right) = w$, $(|w| < r_0(f), r_0(f) \geq 1/4)$. In fact we have:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{2}$$

Let $\Sigma = \{f \in \mathscr{S} :$ both f and f^{-1} are univalent in $\mathbb{U}\}$ denote the class of bi-univalent functions in $\mathbb{U}$. See [19] (also see [2]) for brief information and examples on the class Σ. Recently, in their pioneering work on the subject of bi-univalent functions, Srivastava et al. [19] actually revived the study of the coefficient problems involving bi-univalent functions. Later, many researchers (viz [3], [4], [6], [8], [12], [13], [14], [15], [18], [20], [23], [24] etc.) obtained initial coefficient bounds for the functions in various subclasses of Σ. Also, some researchers (viz [16], [17], [21], [22] etc.) obtained initial coefficient bounds for subclasses of m-fold symmetric bi-univalent functions.

In 1972, Ozaki and Nunokawa [10] stated and proved the univalence criterion that if for $f(z) \in \mathscr{A}$,

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathbb{U}),$$

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then $f(z)$ is univalent in $\mathbb{U}$ and hence $f \in \mathscr{S}$. Also, let $\mathscr{T}(\mu)$ denote the class of functions $f(z) \in \mathscr{A}$ such that:

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < \mu \quad (z \in \mathbb{U}, 0 < \mu \le 1),$$

where μ is real and $\mathscr{T}(1) = \mathscr{T}$. Clearly, $\mathscr{T}(\mu) \subset \mathscr{T} \subset \mathscr{S}$.

For two analytic functions f and g, we say that the function f is subordinate to g, written as $f(z) \prec g(z)$; $(z \in \mathbb{U})$ if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{U}$, then we have the equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let ϕ be an analytic function with positive real part in $\mathbb{U}$ such that $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Hence we have,

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \; (B_1 > 0). \tag{3}$$

In 1967, Lewin [7] investigated the class Σ and proved that $|a_2| < 1.51$, after which Netanyahu [9] showed that $max_{f\in\Sigma}|a_2| = 4/3$ and further, Brannan and Clunie [1] conjectured that $|a_2| \le \sqrt{2}$. In recent years, after a seminal paper by Srivastava et al. [19], the study of coefficient problems of bi-univalent functions have gathered momentum. But still the problem of coefficient bound for $|a_n|$, $(n = 3, 4, \cdots)$ is an open problem.

The object of the present paper is to introduce the subclass $\mathscr{T}_\Sigma[\phi]$ of the bi-univalent function class Σ defined on the open unit disk $\mathbb{U}$ and to prove that $|a_2| \le min\left\{1, \sqrt{B_1}\right\}$, $|a_3| \le B_1$ and $|a_4| \le 3B_1/2$ for the functions in this subclass.

We need to recall the following lemma (see [11]) to prove our main result.

Lemma 1. *If $p(z) \in \mathscr{P}$, then $|p_n| \le 2$ for each $n \in \mathbb{N}$ where $\mathscr{P}$ be the class of functions analytic in $\mathbb{U}$ with $\Re(p(z)) > 0$ and $p(z)$ have the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ for $z \in \mathbb{U}$.*

2 Main result

Definition 1. *A function $f(z) \in \Sigma$ given by (1) is said to be in the class $\mathscr{T}_\Sigma[\phi]$ if the following conditions are satisfied:*

$$\left(\frac{z^2 f'(z)}{(f(z))^2}\right) \prec \phi(z) \quad (z \in \mathbb{U})$$

and

$$\left(\frac{w^2 g'(w)}{(g(w))^2}\right) \prec \phi(w) \quad (w \in \mathbb{U})$$

where the functions g and ϕ are defined by (2) and (3) respectively.

Theorem 1. *Let the function $f(z) \in \Sigma$ given by (1) be in the class $\mathscr{T}_\Sigma[\phi]$. Then,*

$$|a_2| \le min\left\{1, \sqrt{B_1}\right\}, \quad |a_3| \le B_1, \quad |a_4| \le 3B_1/2.$$

Proof. Using Definition 1 we can write:

$$\frac{z^2 f'(z)}{(f(z))^2} = \phi(u(z)) \quad (z \in \mathbb{U}) \tag{4}$$

and

$$\frac{w^2 g'(w)}{(g(w))^2} = \phi(v(w)) \quad (w \in \mathbb{U}), \tag{5}$$

where the functions $u, v : \mathbb{U} \to \mathbb{U}$ are analytic with $u(0) = v(0) = 0$.
Define the functions s and t as:

$$s(z) = \frac{1+u(z)}{1-u(z)} = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \ (z \in \mathbb{U})$$

and

$$t(w) = \frac{1+v(w)}{1-v(w)} = 1 + t_1 w + t_2 w^2 + t_3 w^3 + \cdots, \ (w \in \mathbb{U}).$$

Clearly s and t both satisfies the conditions of Lemma 1 and hence,

$$|s_n| \le 2, \ |t_n| \le 2, \ (n \in \mathbb{N}). \tag{6}$$

Solving for $u(z)$ and $v(w)$, we get:

$$u(z) = \frac{1}{2}\left[s_1 z + \left(s_2 - \frac{s_1^2}{2}\right) z^2 + \left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right) z^3 + \cdots\right], \ (z \in \mathbb{U})$$

and

$$v(w) = \frac{1}{2}\left[t_1 w + \left(t_2 - \frac{t_1^2}{2}\right) w^2 + \left(t_3 - t_1 t_2 + \frac{t_1^3}{4}\right) w^3 + \cdots\right], \ (w \in \mathbb{U}).$$

Using these expansions in equation (3), we obtain:

$$\begin{aligned}\phi(u(z)) = 1 + \frac{1}{2}B_1 s_1 z + \left[\frac{1}{2}B_1\left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4}B_2 s_1^2\right] z^2 + \\ \left[\frac{1}{2}B_1\left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right) + \frac{1}{2}B_2\left(s_1 s_2 - \frac{s_1^3}{2}\right) + \frac{1}{8}B_3 s_1^3\right] z^3 + \cdots\end{aligned}$$

and

$$\begin{aligned}\phi(v(w)) = 1 + \frac{1}{2}B_1 t_1 w + \left[\frac{1}{2}B_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}B_2 t_1^2\right] w^2 + \\ \left[\frac{1}{2}B_1\left(t_3 - t_1 t_2 + \frac{t_1^3}{4}\right) + \frac{1}{2}B_2\left(t_1 t_2 - \frac{t_1^3}{2}\right) + \frac{1}{8}B_3 t_1^3\right] w^3 + \cdots.\end{aligned}$$

Also by using equation (1) and (2), we obtain:

$$\frac{z^2 f'(z)}{(f(z))^2} = 1 + \left(a_3 - a_2^2\right) z^2 + 2\left(a_2^3 + a_4 - 2a_2 a_3\right) z^3 + \cdots$$

and

$$\frac{w^2 g'(w)}{(g(w))^2} = 1 - \left(a_3 - a_2^2\right) w^2 - 2\left(2a_2^3 + a_4 - 3a_2 a_3\right) w^3 + \cdots.$$

Now, equating the coefficients in equation (4) and (5), we get $s_1 = t_1 = 0$ and hence the further equalities becomes:

$$\left(a_3 - a_2^2\right) = \frac{1}{2}B_1 s_2, \tag{7}$$

$$2\left(a_2^3 + a_4 - 2a_2a_3\right) = \frac{1}{2}B_1s_3, \tag{8}$$

$$-\left(a_3 - a_2^2\right) = \frac{1}{2}B_1t_2, \tag{9}$$

$$-2\left(2a_2^3 + a_4 - 3a_2a_3\right) = \frac{1}{2}B_1t_3. \tag{10}$$

Equation (7) and (9) in light of (6) gives:

$$\left|a_3 - a_2^2\right| = \left|\frac{1}{2}B_1s_2\right| = \frac{1}{2}B_1\left|s_2\right| \leq B_1 \tag{11}$$

and

$$\left|a_2^2 - a_3\right| = \left|\frac{1}{2}B_1t_2\right| = \frac{1}{2}B_1\left|t_2\right| \leq B_1 \tag{12}$$

respectively.
By adding equation (8) in (10), we obtain:

$$2a_2\left(a_3 - a_2^2\right) = \frac{1}{2}B_1\left(s_3 + t_3\right), \tag{13}$$

which, by using equation (6) gives:

$$\left|a_2\left(a_3 - a_2^2\right)\right| = \left|a_2\right|\left|a_3 - a_2^2\right| \leq B_1. \tag{14}$$

See that, equation (11) and (14) together yields:

$$\left|a_2\right| \leq 1. \tag{15}$$

Also, by using the triangle inequality:

$$\left|\left|z_1\right| - \left|z_2\right|\right| \leq \left|z_1 - z_2\right| \tag{16}$$

in equation (12), we get:

$$\left|a_2^2\right| - \left|a_3\right| \leq \left|a_2^2 - a_3\right| \leq B_1,$$

which implies that:

$$\left|a_2^2\right| \leq B_1. \tag{17}$$

Equation (15) and (17) together yields:

$$\left|a_2\right| \leq min\left\{1, \sqrt{B_1}\right\}.$$

Now, using the inequality (16) in equation (11), we can write:

$$\left|a_3\right| - \left|a_2^2\right| \leq \left|a_3 - a_2^2\right| \leq B_1,$$

from which, it is obvious that:

$$\left|a_3\right| \leq B_1.$$

Next, by subtracting equation (10) from (8), we get:

$$2\left(3a_2^3 + 2a_4 - 5a_2a_3\right) = \frac{1}{2}B_1\left(s_3 - t_3\right). \tag{18}$$

Eliminating a_2^3 by using equation (13) and (18), we get:

$$4\left(a_4-a_2a_3\right)=\frac{1}{2}B_1\left(s_3-t_3\right)+3\frac{1}{2}B_1\left(s_3+t_3\right)=\frac{1}{2}B_1\left(4s_3+2t_3\right)$$

which, by using equation (6) gives:

$$\left|a_4-a_2a_3\right|\le\frac{3}{2}B_1. \tag{19}$$

Finally, by using the inequality (16) in equation (19), we get:

$$|a_4|\le 3B_1/2.$$

This completes the proof of Theorem 1.

Remark.[1] If the function ϕ is given by

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2z^2+\cdots;\quad \left(z\in\mathbb{U},0<\alpha\le 1\right),$$

then the bounds are:

$$|a_2|\le\begin{cases}\sqrt{2\alpha} & ;\quad \left(0<\alpha<\frac{1}{2}\right)\\ 1 & ;\quad \left(\frac{1}{2}\le\alpha\le 1\right),\end{cases}$$

$|a_3|\le 2\alpha$ and $|a_4|\le 3\alpha$.

Remark.[2] If the function ϕ is given by

$$\phi(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots;\quad \left(z\in\mathbb{U},0\le\beta<1\right),$$

then the bounds are:

$$|a_2|\le\begin{cases}1 & ;\quad \left(0\le\beta\le\frac{1}{2}\right)\\ \sqrt{2(1-\beta)} & ;\quad \left(\frac{1}{2}<\beta<1\right),\end{cases}$$

$|a_3|\le 2(1-\beta)$ and $|a_4|\le 3(1-\beta)$.

3 Conclusion

For the bi-univalent function class Σ, Lewin [7] proved that $|a_2|<1.51$, Netanyahu [9] proved that $max_{f\in\Sigma}|a_2|=4/3$ and Brannan and Clunie [1] conjectured that $|a_2|\le\sqrt{2}$; whereas for the function class $\mathscr{T}_\Sigma[\phi]$, we obtain an improved result $|a_2|\le min\left\{1,\sqrt{B_1}\right\}$ which is even better estimate than $|a_2|\le 1$. Also, we have an interesting problem here that, can we generalize this theorem to $|a_n|\le(n-1)B_1/2$, for $n\ge 3$?

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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