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The inverse problem of finding a heat source under nonlocal boundary conditions

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Abstract: The paper considers the inverse problem of finding a time-dependent heat source in a parabolic equation with the nonlocal boundary condition. Under some assumption on the data the existence and uniqueness of the solution are shown by using the generalized Fourier method. Numerical procedure of this problem is given by using finite-difference method.

Keywords: Fourier method, heat source, inverse problems, nonlocal boundary condition, finite difference method.

1 Introduction

Let $T > 0$ be fixed number and let $D_T = \{(x,t): \ 0 < x < 1: \ 0 < t \leq T\}$.
Consider the inverse problem of finding a pair of functions (r,u) satisfying the system

$$u_t = u_{xx} + r(t)f(x,t), \quad (x,t) \in D_T, \tag{1}$$

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq 1, \tag{2}$$

$$u(0,t) = 0, \quad u_x(0,t) = u_x(1,t), \quad 0 \leq t \leq T, \tag{3}$$

$$u(1,t) = E(t), \quad 0 \leq t \leq T. \tag{4}$$

Definition 1. *The pair $\{r(t), u(x,t)\}$ from the class $C[0,T] \times C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T)$ for which conditions (1)-(4) are satisfied, is called a classical solution of the inverse problem (1)- (4)*

The inverse problem of finding the heat source power in a parabolic equation has been investigated for the heat source which is space-dependent in [1,2,3,6] and time-dependent in [2,4,5,10,11]. The inverse problems in these papers are similar from the mathematical point of view that local boundary and overdetermination conditions are used. The literature on inverse problems for parabolic equations under nonlocal boundary is not so vast, see [4,11]. The periodic nature of (3) type boundary conditions is demonstrated in [3,7]. The paper is organized as follows.

In Section 1, the existence and uniqueness of the solution of inverse problem (1)-(4) is proved by using the Fourier

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method. In Section 2, the numerical procedure for the solution of the inverse problem using the Crank-Nicolson finite-difference scheme combined with an iteration method is given.

2 Existence and uniqueness of the solution of the inverse problem

Proof. Consider the following system of functions on the interval [0,1]:

$$X_0(x) = x,\ X_{2k-1}(x) = x\cos(2\pi kx),\ X_{2k}(x) = \sin(2\pi kx),\ k = 1,2,... \tag{5}$$

$$Y_0(x) = 2,\ Y_{2k-1}(x) = 4\cos(2\pi kx),\ Y_{2k}(x) = 4(1-x)\sin(2\pi kx),\ k = 1,2,.... \tag{6}$$

The system of functions (5) and (6) arise in [3] for the solution of a nonlocal boundary value problem in heat conduction. It is easy to verify that the system of function (5) and (6) are biorthogonal on $[0,1]$. They are also Riesz bases in $L_2[0,1]$ [3].

We have the following assumptions on the data of the problem (1)-(4)

$(A_1) E(t) \in C^1[0,T]$;
$(A_2) \varphi(x) \in C^3[0,1]$, $\varphi(0) = 0$, $\varphi'(0) = \varphi'(1)$, $\varphi''(0) = 0$ and $\varphi(1) = E(0)$;
$(A_3) f(x,t) \in C(\overline{D}_T)$; $f(x,t) \in C^3[0,1]$, $\forall t \in [0,T]$; $f(0,t) = 0$, $f_x(0,t) = f_x(1,t)$, $f_{xx}(0,t) = 0$ and $f(1,t) \neq 0$, $\forall t \in [0,T]$.

The main result on existence and uniqueness of the solution of the inverse problem (1)-(4) is presented as follows.

Theorem 1. *Let the assumptions (A_1)-(A_3) satisfied. Then the inverse problem (1)-(4) has a unique solution.*

Proof. Any solution of the (1) can be represented as

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x), \tag{7}$$

where the functions $u_k(t)$, $k = 0,1,2,...$ satisfy the following system of equations:

$$u_0'(t) = r(t) f_0(t),$$

$$u_{2k-1}'(t) + (2\pi k)^2 u_{2k-1}(t) = r(t) f_{2k-1}(t),$$
$$u_{2k}'(t) + (2\pi k)^2 u_{2k}(t) + 4\pi k u_{2k-1}(t) = r(t) f_{2k}(t),\ k = 1,2,\dots .$$

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Substuting the solution of this system of equations and initial condition (2) in (7), we obtain the solution of the problem (1)-(3) in the following form

$$
\begin{aligned}
u(x,t) = & \left[\varphi_0 + \int_0^t r(\tau) f_0(\tau) d\tau\right] X_0(x) \\
& + \sum_{k=1}^{\infty} \left[\varphi_{2k-1} e^{-(2\pi k)^2} + \int_0^t r(\tau) f_{2k-1}(\tau) e^{-(2\pi k)^2} d\tau\right] X_{2k-1}(x) \\
& + \sum_{k=1}^{\infty} \left[(\varphi_{2k} - 4\pi k \varphi_{2k-1} t)\, e^{-(2\pi k)^2}\right] X_{2k}(x) \\
& + \sum_{k=1}^{\infty} \left[\int_0^t r(\tau)\,(f_{2k}(\tau) - 4\pi k f_{2k-1}(\tau)(t-\tau))\, e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau\right] X_{2k}(x).
\end{aligned}
\tag{8}
$$

where $\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx$ and $f_k(t) = \int_0^1 f(x,t) Y_k(x) dx$, $k = 0, 1, 2, \ldots$. Under the conditions (A_1), (A_2) and (A_3) the series (7) and its x-partial derivative converge uniformly in $\overline{D}_T$ since their majorizing sums are absolutely convergent. Therefore, their sums $u(x,t)$ and $u_x(x,t)$ are continuous in $\overline{D}_T$. In addition, the t-partial derivative and the xx-second order partial derivative series are uniformly convergent for $t \geq \varepsilon > 0$ (ε is an arbitrary positive number). Thus, $u(x,t) \in C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T)$ and satisfies the conditions (1)-(3). The formulas (3) and (4) yield a Volterra integral equation of the second kind

$$
r(t) = F(t) + \int_0^t K(t,\tau) r(\tau) d\tau, \tag{9}
$$

where

$$
F(t) = \frac{E'(t) + \sum_{k=1}^{\infty} (2\pi k)^2 \varphi_{2k-1} e^{-(2\pi k)^2 t}}{f_0(t) + \sum_{k=1}^{\infty} f_{2k}(t)}, \tag{10}
$$

$$
K(t,\tau) = \frac{\sum_{k=1}^{\infty} (2\pi k)^2 f_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau)}}{f_0(t) + \sum_{k=1}^{\infty} f_{2k}(t)}. \tag{11}
$$

Under the conditions $(A_1) - (A_3)$ the right-hand side $F(t)$ and the kernel $K(t,\tau)$ are continuous functions in $[0,T]$ and $[0,T] \times [0,T]$, respectively. We therefore obtain a single function $r(t)$, continuous on $[0,T]$, which, together with the solution of the problem (1)-(3) given by the Fourier series, form the unique solution of the inverse problem (1)-(4).

Theorem 1 has been proved.

3 Numerical procedure

We use the finite difference method with a predictor-corrector type approach, to the problem (1)-(4). We subdivide the intervals $[0,1]$ and $[0,T]$ into M and N subintervals of equal lengths $h = \frac{1}{M}$ and $\tau = \frac{T}{N}$, respectively.We choose the Crank-Nicolson scheme, which is absolutely stable and has a second order accuracy in both h and τ, [9]. The Crank-Nicolson scheme for (1)-(4) is as follows:

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$$2\frac{1}{\tau}\left(u_i^{j+1}-u_i^j\right)=\frac{1}{2h^2}\left[\left(u_{i-1}^j-2u_i^j+u_{i+1}^j\right)+\left(u_{i-1}^{j+1}-2u_i^{j+1}+u_{i+1}^{j+1}\right)\right] +\frac{1}{4}\left(r^j+r^{j+1}\right)\left(f_i^{j+1}+f_i^j\right), \tag{12}$$

$$u_i^0=\phi_i, \tag{13}$$

$$u_0^j=0, \tag{14}$$

$$u_{M+1}^j=u_1^j+u_M^j, \tag{15}$$

where $1\le i\le M$ and $0\le j\le N$ are the indices for the spatial and time steps respectively, $u_i^j=u(x_i,t_j)$, $\phi_i=\varphi(x_i)$, $f_i^j=f(x_i,t_j)$, $x_i=ih$, $t_j=j\tau$. At the $t=0$ level, adjustment should be made according to the initial condition and the compatibility requirements. Now, let us construct the predicting-correcting mechanism. First, we write $x=1$ in the equation (1) and using (??3) and (4), we obtain

$$r(t)=\frac{E'(t)-u_{xx}(1,t)}{f(1,t)}. \tag{16}$$

The finite difference approximation of (16) is

$$r^j=\frac{(E^{j+1}-E^j)/\tau+\left(u_{M-1}^j-2u_M^j+u_{M+1}^j\right)/h^2}{f_M^j}, \tag{17}$$

where $E^j=E(t_j)$, $j=0,1,...,N$. For $j=0$,

$$r^0=\frac{(E^1-E^0)/\tau+(\phi_{M-1}-2\phi_M-\phi_{M+1})/h^2}{f_M^0}. \tag{18}$$

We denote the values of r^j, u_i^j at the s-th iteration step $r^{j(s)}$, $u_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $r^{j+1(0)}=r^j$, $u_i^{j+1(0)}=u_i^j$, $j=0,1,2,....N$, $i=1,2,...,M$. At each $(s+1)$ -th iteration step we first determine $r^{j+1(s+1)}$ from the formula

$$r^{j+1(s+1)}=\frac{(E^{j+2}-E^{j+1})/\tau+(u_{M-1}^{j+1(s)}-2u_M^{j+1(s)}+u_{M+1}^{j+1(s)})/h}{f_M^{j+1}}. \tag{19}$$

Then from (12)-(15) we obtain

$$\begin{aligned}&\tfrac{1}{\tau}\left(u_i^{j+1(s+1)}-u_i^{j+1(s)}\right)=\tfrac{1}{2h^2}\left[\left(u_{i-1}^{j+1(s)}-2u_i^{j+1(s)}+u_{i+1}^{j+1(s)}\right)\right.\\&\left.+\tfrac{1}{2h^2}\left(u_{i-1}^{j+1(s+1)}-2u_i^{j+1(s+1)}+u_{i+1}^{j+1(s+1)}\right)\right]\\&+\tfrac{1}{4}\left(r^{j+1(s)}+r^{j+1(s+1)}\right)\left(f_i^{j+1}+f_i^j\right),\end{aligned} \tag{20}$$

$$u_0^{j+1(s)}=u_M^{j+1(s)}, \tag{21}$$

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$$u_{M+1}^{j+1(s)} = u_1^{j+1(s)} + u_M^{j+1(s)},\ s = 0,1,2,\ldots . \tag{22}$$

The system of equations (20)-(22) can be solved by the Gauss elimination method and $u_i^{j+1(s+1)}$ is determined.If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $r^{j+1(s+1)}$, $u_i^{j+1(s+1)}(i = 1,2,...,M)$ as r^{j+1}, $u_i^{j+1}(i = 1,2,...,M)$, on the $(j+1)$-th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

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