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Additional results on group inverse of some 2×2 block matrices over Minkowski space $\mathscr{M}$

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Abstract: In this paper, we have established the existence and representation of group inverse for 2×2 block matrix over the field of complex numbers $\mathbb{C}$ in Minkowski space $\mathscr{M}$. Further, for various types of block matrices the characterization theorems for the existence of group inverse are determined.

Keywords: Idempotent matrix, block matrix, group inverse, Minkowski adjoint, Minkowski space.

1 Introduction

Let $\mathbb{C}$ be a field of complex numbers and let $F_n(\mathbb{C})$ be the set of all matrices over $\mathbb{C}$. For $A \in F_n(\mathbb{C})$, the matrix $X \in F_n(\mathbb{C})$ is said to be the group inverse of A if it holds that

$$AXA = A, \;\; XAX = X, \;\; AX = XA$$

It is well known that if the group inverse of a complex square matrix exists then it is unique and denoted by $X = A^{\sharp}$ [1].

The generalized inverse of block matrix has important applications in statistical probability, mathematical programming, numerical analysis, econometrics, game theory, control theory etc. For reference see [3,4,5]. The research on the existence and representation of the group inverse for block matrices in Euclidean space has been done in wide range. For the literature of the group inverse of block matrix in Euclidean space see [6,7,8,9,10,11,12].

In [13] the existence of anti-reflexive with respect to the generalized reflection anti-symmetric matrix $P^{\sim}$ and solution of the matrix equation $AXB = C$ in Minkowski space $\mathscr{M}$ is given. In [14] necessary and sufficient condition for the existence of Re-nnd solution has been established for the matrix equation $AXA^{\sim} = C$ where $P, Q \in F_n(C^n)$ and $C \in C^{n\times n}$. In [15] partitioned matrix $M^{\sim}$ in Minkowski space $\mathscr{M}$ was taken in the form $M^{\sim} = \begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1B^{\sim} & D^{*} \end{bmatrix}$ to yield a formula for the inverse of $M^{\sim}$ in terms of the Schur complement of D^{*}. In [17] the existence and representation of the group inverse for block matrix $M = \begin{bmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{bmatrix}$ $(P, Q \in K^{n\times n}$, where $P, Q \in F_n(C), (P^{\sim})^2 = P^{\sim})$ and for other matrices over skew fields in Minkowski space are given.

Minkowski Space $\mathscr{M}$ is an indefinite inner product space, in which the metric matrix associated with the indefinite inner product is denoted by G and is defined as

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}, \text{ satisfying } G^2 = I_n \text{ and } G^{*} = G.$$

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G is called the Minkowski metric matrix. I_n denotes the $n\times n$ identity matrix. In case $u\in\mathbb{C}^n$, indexed as $u=(u_0,u_1,\ldots,u_{n-1})$, G is called the Minkowski metric tensor and is defined as $Gu=(u_0,-u_1,\ldots,-u_{n-1})$ [13]. For any $P\in C^{n\times n}$, the Minkowski adjoint of $P\in F_n(\mathbb{C})$ denoted by $P^{\sim}$ is defined as $P^{\sim}=GP^*G$ where P^* is the usual Hermitian adjoint and G the Minkowski metric matrix of order n.

In this paper, we study the existence and representation of group inverse of some special type of block matrices formed from the set of matrices $\{F,F^{\sim},FF^{\sim}\}$, where F is an idempotent matrix and $F^{\sim}$ is the Minkowski adjoint of F and the cross-sectional block matrices formed from the above set of matrices are

$$\begin{bmatrix}FF^{\sim} & F\\ F & 0\end{bmatrix},\begin{bmatrix}F & F\\ FF^{\sim} & 0\end{bmatrix},\begin{bmatrix}FF^{\sim} & FF^{\sim}\\ F & 0\end{bmatrix},\begin{bmatrix}F & F\\ F^{\sim} & 0\end{bmatrix},\begin{bmatrix}F & FF^{\sim}\\ FF^{\sim} & 0\end{bmatrix},\begin{bmatrix}F & FF^{\sim}\\ F^{\sim} & 0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}F^{\sim} & F\\ F & 0\end{bmatrix}.$$

2 Lemmas

Lemma 1. *[2] Suppose that $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{n\times m}$ and if the group inverses of AB and BA exists, then the following conditions hold.*

(i) $(AB)^{\sharp}=A((BA)^{\sharp})^2B,(BA)^{\sharp}=B((AB)^{\sharp})^2A$;

(ii) $(AB)^{\sharp}A=A((BA)^{\sharp}),(BA)^{\sharp}B=B(AB)^{\sharp}$;

(iii) $A(BA)^{\sharp}B=(AB)^{\sharp}AB,B(AB)^{\sharp}A=(BA)^{\sharp}BA$;

(iv) $AB(AB)^{\sharp}A=AB(AB)^{\sharp}A=A,BA(BA)^{\sharp}B=B$

Lemma 2. *Let $F\in F_n(\mathbb{C})$ then $(FF^{\sim})^{\sharp}$ and $(F-F^{\sim})^{\sharp}$ exists and*
(i) *there is a G-unitary matrix U such that*

$$F=U\begin{bmatrix}I_r & P\\ 0 & 0\end{bmatrix}U^{\sim}\tag{1}$$

$$FF^{\sim}=U\begin{bmatrix}G & 0\\ 0 & 0\end{bmatrix}U^{\sim},G=I_r-PP^*\tag{2}$$

$$(F-F^{\sim})^{\sharp}=U\begin{bmatrix}0 & P(P^*P)^{\sharp}\\ (P^*P)^{\sharp}P^* & 0\end{bmatrix}U^{\sim}\tag{3}$$

(ii) $F(FF^{\sim})^{\sharp}=(FF^{\sim})^{\sharp}=(FF^{\sim})^{\sharp}F^{\sim}$.

Proof. (i) By Theorem 2.3.1 of [16] we have a G-unitary matrix U such that

$$F=U\begin{bmatrix}Q & P\\ 0 & R\end{bmatrix}U^{\sim},$$

where Q is an $r\times r$ invertible upper triangular matrix and R is an upper triangular matrix with all diagonal elements zero. Then from $F^2=F$ we have $Q=Q^2$ and $R=R^2$, which gives $Q=I_r$ and $R=0$, hence (1) holds. Also,

$$F^{\sim}=U\begin{bmatrix}I_r & 0\\ -P^* & 0\end{bmatrix}U^{\sim}$$

Now

$$FF^{\sim}=U\begin{bmatrix} I_r-PP^* & 0\\ 0 & 0\end{bmatrix}U^{\sim}=U\begin{bmatrix} G & 0\\ 0 & 0\end{bmatrix}U^{\sim}, \quad G=I_r-PP^*$$

Since $FF^{\sim}$ and $F-F^{\sim}$ are both hermitian matrices and their group inverse exists. To prove (3) we have

$$F-F^{\sim}=U\begin{bmatrix} 0 & P\\ P^* & 0\end{bmatrix}U^{\sim}.$$

Let $X=\begin{bmatrix} 0 & P\\ P^* & 0\end{bmatrix}$ and $M=\begin{bmatrix} 0 & P(P^*P)^{\sharp}\\ (P^*P)^{\sharp}P^* & 0\end{bmatrix}$ then by Lemma (1) and by the direct computation we have

$$MX=XM=\begin{bmatrix} (PP^*)^{\sharp}PP^* & 0\\ 0 & (PP^*)^{\sharp}PP^*\end{bmatrix}.$$

Also we can check that $MXM=M$ and $XMX=X$ and hence (3) holds.

(ii) It is easy to check this conclusion so proof is omitted.

3 Main results

Theorem 1. *Suppose that $F\in F_n(\mathbb{C})$ and $M=\begin{bmatrix} FF^{\sim} & F\\ F & 0\end{bmatrix}$. Then $M^{\sharp}$ exists and*

$$M^{\sharp}=\begin{bmatrix} FF^{\sim}(I_n-F) & F\\ (FF^{\sim})^2(F-I_n)+F & -FF^{\sim}F\end{bmatrix} \tag{4}$$

Proof. Denote the right hand side of the equation (4) by X. Also we have $F^2=F$, then

$$\begin{aligned} MXM&=\begin{bmatrix} FF^{\sim} & F\\ F & 0\end{bmatrix}\begin{bmatrix} FF^{\sim}(I_n-F) & F\\ (FF^{\sim})^2(F-I_n)+F & -FF^{\sim}F\end{bmatrix}\begin{bmatrix} FF^{\sim} & F\\ F & 0\end{bmatrix}\\ &=\begin{bmatrix} F^3F^{\sim} & F^3\\ (FF^{\sim})^2-FF^{\sim}F^2F^{\sim}+F^2 & FF^{\sim}F-FF^{\sim}F^2\end{bmatrix}\\ &=\begin{bmatrix} FF^{\sim} & F\\ F & 0\end{bmatrix}\\ &=M\\ XMX&=\begin{bmatrix} FF^{\sim}(I_n-F) & F\\ (FF^{\sim})^2(F-I_n)+F & -FF^{\sim}F\end{bmatrix}\begin{bmatrix} FF^{\sim} & F\\ F & 0\end{bmatrix}\begin{bmatrix} FF^{\sim}(I_n-F) & F\\ (FF^{\sim})^2(F-I_n)+F & -FF^{\sim}F\end{bmatrix}\\ &=\begin{bmatrix} FF^{\sim}F-FF^{\sim}F^2+F^2F^{\sim}-F^2P^{\sim}F & F^2\\ (FF^{\sim})^2F^2-(FF^{\sim})^2F+F^2-FF^{\sim}F^2F^{\sim}F & -FF^{\sim}F^2\end{bmatrix}\\ &=\begin{bmatrix} FF^{\sim}(I_n-F) & F\\ (FF^{\sim})^2(F-I_n)+F & -FF^{\sim}F\end{bmatrix}\\ &=X\end{aligned}$$

$$
\begin{aligned}
MX &= \begin{bmatrix} FF^{\sim} & F \\ F & 0 \end{bmatrix} \begin{bmatrix} FF^{\sim}(I_n - F) & F \\ (FF^{\sim})^2(F - I_n) + F & -FF^{\sim}F \end{bmatrix} \\
&= \begin{bmatrix} F & 0 \\ FF^{\sim} - FF^{\sim}F & F \end{bmatrix} \\
XM &= \begin{bmatrix} FF^{\sim}(I_n - F) & F \\ (FF^{\sim})^2(F - I_n) + F & -FF^{\sim}F \end{bmatrix} \begin{bmatrix} FF^{\sim} & F \\ F & 0 \end{bmatrix} \\
&= \begin{bmatrix} F & 0 \\ FF^{\sim} - FF^{\sim}F & F \end{bmatrix} \\
MX &= XM
\end{aligned}
$$

Thus $M^{\sharp} = X$

Theorem 2. *Suppose that $F \in F_n(\mathbb{C})$ and $M = \begin{bmatrix} F & F \\ FF^{\sim} & 0 \end{bmatrix}$. Then $M^{\sharp}$ exists and*

$$
M^{\sharp} = \begin{bmatrix} (FF^{\sim})^{\sharp}(F - I_n) & (FF^{\sim})^{\sharp}F \\ FF^{\sim}(FF^{\sim})^{\sharp} - (FF^{\sim})^{\sharp}(F - I_n) & -(FF^{\sim})^{\sharp}F \end{bmatrix} \tag{5}
$$

Proof. Let X denote the right-hand side matrix of (5). By $F^2 = F$ i.e. F is idempotent, (iv) of Lemma 1 and by definition of group inverse we have

$$
MX = XM = \begin{bmatrix} FF^{\sim}(FF^{\sim})^{\sharp} & 0 \\ F - FF^{\sim}(FF^{\sim})^{\sharp} & F \end{bmatrix}
$$

Similarly we can see that $MXM = M$ and $XMX = X$. Thus $M^{\sharp} = X$.

Theorem 3. *Suppose that $F \in F_n(\mathbb{C})$ and $M = \begin{bmatrix} FF^{\sim} & FF^{\sim} \\ F & 0 \end{bmatrix}$. Then $M^{\sharp}$ exists and*

$$
M^{\sharp} = \begin{bmatrix} -F + FF^{\sim}(FF^{\sim})^{\sharp} & FF^{\sim}(FF^{\sim})^{\sharp} \\ F - FF^{\sim}(FF^{\sim})^{\sharp} + (FF^{\sim})^{\sharp}F & -FF^{\sim}(FF^{\sim})^{\sharp} \end{bmatrix} \tag{6}
$$

Proof. Denote $M^{\sharp}$ by X in (6). By the definition of group inverse and Lemma 1 and 2 we have

$$
MX = XM = \begin{bmatrix} F & 0 \\ FF^{\sim}(FF^{\sim})^{\sharp} - F & FF^{\sim}(FF^{\sim})^{\sharp} \end{bmatrix}
$$

Similarly by direct computation we can have $MXM = M$ and $XMX = X$. Thus $M^{\sharp} = X$.

Theorem 4. *Suppose that $F \in F_n(\mathbb{C})$ and $M = \begin{bmatrix} F & F \\ F^{\sim} & 0 \end{bmatrix}$. Then $M^{\sharp}$ exists and*

$$
M^{\sharp} = \begin{bmatrix} (FF^{\sim})^{\sharp}(F - I_n) & (FF^{\sim})^{\sharp}F \\ F^{\sim}(FF^{\sim})^{\sharp} + F^{\sim}((FF^{\sim})^{\sharp})^2(I_n - F) & -(F^{\sim}F)^{\sharp} \end{bmatrix} \tag{7}
$$

Proof. Denote $M^{\sharp}$ by X in equation (7). Using Lemma 1 and 2 and the fact that $F^2 = F$ we get

$$MX = XM = \begin{bmatrix} FF^{\sim}(FF^{\sim})^{\sharp} & 0 \\ F^{\sim}(FF^{\sim})^{\sharp}(F - I_n) & F^{\sim}(FF^{\sim})^{\sharp}F \end{bmatrix}$$

Similarly we can prove that $MXM = M$ and $XMX = X$. Thus $M^{\sharp} = X$.

Theorem 5. *Suppose that* $F \in F_n(\mathbb{C})$ *and* $M = \begin{bmatrix} F & FF^{\sim} \\ FF^{\sim} & 0 \end{bmatrix}$. *Then* $M^{\sharp}$ *exists and*

$$M^{\sharp} = \begin{bmatrix} ((FF^{\sim})^{\sharp})^2(F - I_n) & (FF^{\sim})^{\sharp} \\ ((FF^{\sim})^{\sharp})^3(I_n - F) + (FF^{\sim})^{\sharp} & -((FF^{\sim})^{\sharp})^2 \end{bmatrix} \tag{8}$$

Proof. Denote $M^{\sharp}$ by X in equation (8). By the definition of group inverse and Lemma 1 and 2 and by the fact that $F^2 = F$ we get

$$MX = XM = \begin{bmatrix} FF^{\sim}(FF^{\sim})^{\sharp} & 0 \\ (FF^{\sim})^{\sharp}(F - I_n) & FF^{\sim}(FF^{\sim})^{\sharp} \end{bmatrix}$$

Similarly we can prove that $MXM = M$ and $XMX = X$. Thus $M^{\sharp} = X$.

Theorem 6. *Suppose that* $F \in F_n(\mathbb{C})$ *and* $M = \begin{bmatrix} F & FF^{\sim} \\ F^{\sim} & 0 \end{bmatrix}$. *Then* $M^{\sharp}$ *exists and*

$$M^{\sharp} = \begin{bmatrix} FF^{\sim}(F - I_n) & FF^{\sim}(FF^{\sim})^{\sharp} \\ F^{\sim}(FF^{\sim})^{\sharp} - (F^{\sim}F)^{\sharp} + F^{\sim}((FF^{\sim})^{\sharp})^2 & -F^{\sim}(FF^{\sim})^{\sharp} \end{bmatrix} \tag{9}$$

Proof. Denote $M^{\sharp}$ by X in equation (9). By the definition of group inverse and Lemma 1 and 2 and by the fact that $F^2 = F$ we get

$$MX = XM = \begin{bmatrix} FF^{\sim}(FF^{\sim})^{\sharp} & 0 \\ F^{\sim}(FF^{\sim})^{\sharp}(F - I_n) & F^{\sim} \end{bmatrix}$$

Similarly we can prove that $MXM = M$ and $XMX = X$. Thus $M^{\sharp} = X$.

Theorem 7. *Suppose that* $F \in F_n(\mathbb{C})$ *and* $M = \begin{bmatrix} F^{\sim} & F \\ F & 0 \end{bmatrix}$. *Then* $M^{\sharp}$ *exists and*

$$M^{\sharp} = \begin{bmatrix} ((F - F^{\sim})^{\sharp})^2(I_n - F) & (F - F^{\sim})^{\sharp}F^{\sim}F + F \\ FF^{\sim}(FF^{\sim})^{\sharp}[I_n - ((F - F^{\sim})^{\sharp})^2 + (I_n - F)] & -F^{\sim}(F - F^{\sim})^{\sharp}F^{\sim}F - F^{\sim}F \end{bmatrix} \tag{10}$$

Proof. Let us denote the right-hand side matrix in (10) by $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$, where $X_i \in \mathbb{C}^{n\times n}, i = 1,2,3,4$. Since by Lemma (2) there is a unitary matrix U such that $(1),(2)and(3)$ holds and also $F^2 = F$. Now by Lemma 1 and 2 we have

$$X_1 = U \begin{bmatrix} 0 & (P^{\sim}P)^{\sharp}P \\ 0 & -(P^{\sim}P)^{\sharp} \end{bmatrix} U^{\sim},$$

$$X_2 = U \begin{bmatrix} I_n - P(P^{\sim}P)^{\sharp}P^{\sim} & 0 \\ (P^{\sim}P)^{\sharp}P^{\sim} & (P^{\sim}P)^{\sharp}P^{\sim}F \end{bmatrix} U^{\sim},$$

$$X_3 = U \begin{bmatrix} I_n & -(P^{\sim}P)^{\sharp}P \\ 0 & 0 \end{bmatrix} U^{\sim},$$

$$X_4 = U \begin{bmatrix} I_n - P(P^{\sim}P)^{\sharp}P^{\sim} & 0 \\ 0 & 0 \end{bmatrix} U^{\sim},$$

So, we can rewrite M and X as following

$$M=\begin{bmatrix} U & 0\\ 0 & U\end{bmatrix}\begin{bmatrix} I_r & 0 & I_r & P\\ -P^* & 0 & 0 & 0\\ I_r & P & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} U^{\sim} & 0\\ 0 & U^{\sim}\end{bmatrix}$$

$$X=\begin{bmatrix} U & 0\\ 0 & U\end{bmatrix}\begin{bmatrix} 0 & P(P^*P)^\sharp & I_r-P(P^*P)^\sharp P^* & 0\\ 0 & -(P^*P)^\sharp & (P^*P)^\sharp P^* & (P^*P)^\sharp P^*P\\ I_r & -P(P^*P)^\sharp & -I_r+P(P^*P)^\sharp P^* & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} U^{\sim} & 0\\ 0 & U^{\sim}\end{bmatrix}$$

Now by using Lemma 1 and 2, we get

$$MX=XM=\begin{bmatrix} U & 0\\ 0 & U\end{bmatrix}\begin{bmatrix} I_r & 0 & 0 & 0\\ 0 & (P^*P)^\sharp P^*P & 0 & 0\\ 0 & 0 & I_r & P\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} U^{\sim} & 0\\ 0 & U^{\sim}\end{bmatrix}$$

It is now easy to check that $MXM=M$ and $XMX=X$ and the proof is complete.

4 Conclusion

We extend the main results of [9] on block matrices from Euclidean space into Minkowski space. We also extend the Hermitian adjoint into Minkowski adjoint to get the group inverse of block matrices in Minkowski space.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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