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On large arbitrary girth of a semigroup

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Abstract: This paper is about girth of commuting graph of semigroup. Let S be a finite non-commutative semigroup, its commuting graph, denoted by $G(S)$, is a simple graph (which has no loops and multiple edges) whose sets of vertices are elements of S and whose sets of edges are those elements of S which commute with other elements i.e. for any $a,b \in S$ such that $ab = ba$ for $a \neq b$.
In this paper, we partly solve the the problem posted by J.Araújo, Kinyon M. and Konieczny that for all natural numbers $n \geq 3$ there is a semigroup S such that the girth of $G(S)$ is n.

Keywords: Commuting graph, girth, semigroup of full transformations.

1 Introduction

J.Araújo, Kinyon M. and Konieczny give a construction of band (semigroups of idempotents) in which one can find semigroup of any knit degree n , for some positive integer n , except $n = 3$ [1]. The construction of such type of semigroups also helps for finding semigroups for every $n \geq 2$ such that the diameter of commuting graph $G(S)$ is n. On the other hand, finding out the semigroups from a given graph is very important and difficult task in the theory of semigroups. In our paper, we construct semigroup S of large arbitrary even girth for all even natural numbers $n \geq 4$ such that girth of the commuting graph $G(S)$ is n.

Suppose that $G(S)$ is commuting graph of some non-commutative semigroup of S, then $G(S) = (V,E)$ where V is a finite vertex set and and E is a set of edges such that $E \subseteq \{\{u,v\} : u,v \in V for\ u \neq v\}$. If $v_1, v_2, ...v_k$ are the vertices in $G(S)$ then we write a path λ from v_1 to v_k as $\lambda = v_1 - v_2 - ...v_k$ of length $k-1$. A cycle is a path thats starts and ends on the same vertex. The girth of the graph is the length of shortest cycle in the graph.If the graph does not contain any cycle then its grith defined to be infinity.

Let S be a finite non-commutative semigroup whose centre is defined as $Z(S) = \{a \in S : ab = ba \quad \forall \quad b \in S\}$. The commuting graph of a finite non-commutative semigroup is a simple graph whose sets of vertices are from $S - Z(S)$ and whose sets of edges are the elements of S which commute with other elements.i.e for any $a,b \in$ S such that $ab = ba$ for $a \neq b$. This paper is actually the construction of band (of course non commutative) from given graph such that we can find the even girth of the commuting graph $G(S)$ of semigroup S.For the construction of such type of semigroups, our main focus will be on semigroup of full transformations $T(X)$ for a finite set X.

Let $T(X)$ be a semigroup of full transformations for a finite set X under the composition of function.Actually the semigroups $T(X)$ is the set of all functions from a finite set X to X. In this paper we consider transformations $a,b \in T(X)$ and define composition of functions as $(ab)(x) = a(b(x))$ from right instead of left i.e $(x)(ab) = ((x)a)b$ for $x \in X$.

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For $a \in T(X)$ we write image of a by $im(a)$ and kernel of a is defined as,

$$ker(a) = \{(x,y) \in X \times X : a(x) = a(y)\}$$

and rank of a as $rank(a) = |im(a)|$. Also $T(X)$ has n ideals I_1 ,$I_2 \ldots I_n$ where $1 \leq r \leq n$

$$I_r = \{a \in T(X) : rank(a) \leq r\}.$$

Clearly the ideal I_1 is of rank 1 i.e a constant transformation and hence its commuting graph will be isolated vertices.

2 Idempotents and zeros elements in semigroups

In this section, we describe about the notation of idempotent and constant transformations,bands,commuting transformations and left as well as right zero semigroups.

Definition 1. *Let S be a semigroup and $e \in S$ is an idempotent if $e^2 = e$.Also we define the sets of idempotents in S to be $E(S) = \{e \in S : e^2 = e\}$ Now,$E(S)$ may be empty or it may be $E(S) = S$.If $E(S) = S$ then S is a band.We construct band in our construction in the monoid $T(X)$.*

Definition 2. *Let $e \in T(X)$ be an idempotent and $\{A_1, A_2, \ldots .A_k\}$ be a partition of X and unique elements $x_1 \in A_1, x_2 \in A_2, ..x_k \in A_k$ such that for every i we have $A_i e = \{x_i\}$. Then the set $\{x_1, x_2, \ldots x_k\}$ is the image set of e.We use the following notation for e,*

$$e = (A_1, x_1\rangle (A_2, x_2\rangle \ldots (A_k, x_k\rangle$$

If e is a constant transformation with image set $\{x\}$ then we write $(X, x\rangle$ [1].

Definition 3. *Let $e = (A_1, x_1\rangle (A_2, x_2\rangle \ldots (A_k, x_k\rangle$ an idempotent in $T(X)$ and let $b \in T(X)$ then b commutes with e if and only if for every $i \in \{1, 2 \ldots k\}$ there is a $j \in \{1, 2 \ldots k\}$ such that $bx_i = x_j$ and $bA_i \subseteq A_j$ [1].*

Definition 4. *Let $e, f \in I_r$ be idempotents and suppose there is $x \in X$ such that $x \in im(e) \bigcap im(f)$ then $e - (X, x\rangle - f$ [1].*

Definition 5. *Let S be a semi group with some binary operation $*$ on it. A zero element is an element z in S such that for all s in S, we have, $z * s = s * z = z$. An element in S is called left zero, if all s in S, we have,$z * s = z$ and an element is called right zero, if all s in S, we have,$s * z = z$.*

Definition 6. *A semigroup in which every element is a left zero element is called a left zero semigroup.For example, the semigroup $S = \{a, b, c\}$ is a left zero semigroup. Then the Cayley table for S is as given below,*

$*$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Definition 7. *A semigroup in which every element is a right zero element is called a right zero semigroup.For example, The semi group $S = \{a, b, c\}$ is a right zero semigroup. Then the Cayley table for S is as given below,*

$*$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

Lemma 1. *Let $c_x, c_y, e \in T(X)$ such that e is an idempotent,then*
(1) $c_x e = ec_x$ *if and only if* $x \in im(e)$.
(2) $c_x e = c_y e$ *if and only if* $(x,y) \in ker(e)$.

Proof. **(1)** Consider $c_x e = ec_x$. As c_x and e commute with each other,therefore,there should be at least one element common in the images of c_x and e but c_x has only one element in the image set i.e x in $im(c_x)$. So $x \in im(c_x) \bigcap im(e)$ or $x \in \{x\} \bigcap im(e)$. This implies that $x \in im(e)$. Conversely, suppose that $x \in im(e)$.We can write it as $x \in \{x\} \bigcap im(e)$. This implies that $x \in im(c_x) \bigcap im(e)$. Thus we have $c_x e = ec_x$.
(2) Consider $c_x e = c_y e$. As $ker(e)$ is defined as $ker(e) = \{(x,y) \in X \times X : xe = ye\}$.Consider $c_x e = c_z$ and $c_y e = c_t$ for some t and z in$X$. Thus $c_z = c_t \Rightarrow z = t$ and hence $ze = te$. Therefore $(x,y) \in ker(e)$. Conversely,let $(x,y) \in ker(e)$ then by def. of $ker(e)$ we have $xe = ye$, implies $c_x e = c_y e$.

3 Construction of the girth of semigroup

Definition 8. *Let $k \geq 2$ be an integer.Let $X = \{y_1, y_2, y_3 \ldots y_{3k}\}$.We define idempotents $a_1, a_2, a_3 \ldots a_{3k}$ as follows.*
For $i \in \{1, 2 \ldots 3k\}$,let $n = \frac{|X|}{3}$,there will be three kernel classes of each a_i.
$class - 1 = \{y_1, y_2 \ldots y_n\}$
$class - 2 = \{y_{n+1}, y_{n+2} \ldots y_{2n}\}$
$class - 3 = \{y_{2n+1}, y_{2n+2} \ldots y_{3n}\}$
Let $X = A_1 \bigcup A_2 \bigcup A_3$ be the partition of X.Thus we have,$|X| = |A_1| + |A_2| + |A_3|$ such that
$class - 1 = \{y_1, y_2 \ldots y_n\} \subseteq A_1$
$class - 2 = \{y_{n+1}, y_{n+2} \ldots y_{2n}\} \subseteq A_2$
$class - 3 = \{y_{2n+1}, y_{2n+2} \ldots y_{3n}\} \subseteq A_3$
For even n,there will be $\frac{|X|}{3} + 1$,odd number of right-zero semigroups generators and even number (2times of odd) of constant generators.For example,for $X = 12$ we have $n = 4$,so there will be five right-zero semigroups generators and ten (2 times of 5) constant generators whose images sets are defined as;

$\mathrm{Im}(a_1)$	y_1	y_{n+1}	y_{2n+1}
$\mathrm{Im}(a_2)$	y_1	y_{n+2}	y_{2n+2}
$\mathrm{Im}(a_3)$	y_2	y_{n+2}	y_{2n+3}
$\mathrm{Im}(a_4)$	y_2	y_{n+3}	y_{2n+4}
$\mathrm{Im}(a_5)$	y_3	y_{n+3}	y_{2n+1}

Table 1: Images of the generators when n is even

For odd n,there will be $\frac{|X|}{3} + 1$,even number of right-zero semigroups generators and even number (2times of even) of constant generators.For example,for $X = 9$ we have $n = 3$,there will be four right-zero semigroups generators and eight (2times of even) whose images sets are defined as;

$\mathrm{Im}(a_1)$	y_1	y_{n+1}	y_{2n+1}
$\mathrm{Im}(a_2)$	y_1	y_{n+2}	y_{2n+2}
$\mathrm{Im}(a_3)$	y_2	y_{n+2}	y_{2n+3}
$\mathrm{Im}(a_4)$	y_2	y_{n+3}	y_{2n+1}

Table 2: Images of the generators when n is odd

Let $i \in \{1,2\dots n\}$ we define the constant transformations as $e_i = (X,y_i\rangle = cy_i$ i.e all the elements of X maps on the single element y_i. The semigroup S generated by $S =< a_1,e_1,a_2,e_2\dots a_{n+1},e_{n+1} >$ is our required semigroup of girth $2(\frac{|X|}{3}+1)$ i.e 2(no. of generators a_i).It is easy to see that $Z(S) = \phi$.

Lemma 2. *Let $1 \le i < j$,*
(1) $a_ia_i = a_i$,$a_ja_j = a_j$, $e_ie_j = e_j$, $e_je_i = e_i$
(2) $a_ia_j = a_j$, $a_ja_i = a_i$
(3) $a_ie_i = e_i$, $a_ie_j = e_j$
(4) $a_je_i = e_i$, $a_je_j = e_j$
(5) $e_ia_i = e_i$, $e_ja_i = e_j$
(6) $e_ie_j = e_j$, $e_je_i = e_i$

Proof. **(1)** Proof is obvious by definition 2 that a_i , a_j , c_i and c_j are idempotents.
(2) Since the element of the image set of a_i properly lies in each A_i of a_j,therefore,a_j maps all the elements of $im(a_i)$ to $im(a_j)$,thus $a_ia_j = a_j$ forming it right zero semigroup.Similary,$a_ja_i = a_i$. Other proof are similar.

Lemma 3. *For each $i, j \in \{1,2\dots n\}$,for the adjacent transformations a_i and a_{i+1} , there will be constant transformation such that $a_i - e_j - a_{i+1}$ and there will be no path between a_i and a_{i+1}.*

Proof. Since from Lemma 2,we have $a_ie_j = e_j$ and $e_ja_i = e_j$ also $a_je_j = e_j$ and $e_ja_j = e_j$.This implies $a_i - e_j - a_{i+1}$.Similarly,$a_ia_j = a_j$ and $a_ja_i = a_i$.This implies there will be no path between a_i and a_{i+1}.

Lemma 4. *For each $i, j \in \{1,2\dots n\}$,the cycle $\Pi = a_i - e_j - a_{i+1} - e_j - a_{i+2} - e_j \dots e_j - a_i$ is unique and minimal cycle in $G(S)$ of length $2n$.*

Proof. Suppose that there is another cycle λ in $G(S)$ whose length is $2n+1$ more that $2n$ such that $\lambda = a_i - e_j - a_{i+1} - e_j - a_{i+2} - e_j \dots e_j - e_{j+1} - a_i$. or $\lambda = a_i - e_j - a_{i+1} - e_j - a_{i+2} - e_j \dots e_j - a_{i-1} - a_i$. We claim that length of λ is greater than the length of Π. Consider the case-1 when $\lambda = a_i - e_j - a_{i+1} - e_j - a_{i+2} - e_j \dots e_j - e_j - a_i$. From lemma 2,$e_je_{j+1} = e_{j+1}$ and $e_{j+1}e_j = e_j$ thus $e_je_{j+1} \neq e_{j+1}e_j$ i.e two constant transformations never commute with each other.Therefore,there will be no path between e_j and e_{j+1}.Thus there will be no cycle whose length is more than Π.So $\lambda = \Pi$. Now consider that case-2,$\lambda = a_i - e_j - a_{i+1} - e_j - a_{i+2} - e_j \dots e_j - a_{i-1} - a_i$. From lemma 3, there will be no path between a_i and $_{i-1}$.Therefore,such type of cycle does not exist.This implies that,$\lambda = \Pi$. So our claim in both cases is wrong.Thus,the cycle $\Pi = a_i - e_j - a_{i+1} - e_j - a_{i+2} - e_j \dots e_j - a_i$ is unique and minimal cycle in $G(S)$ of length $2n$.
Now the only case left for the girth of four which is present in the following proposition.

Proposition 1. *For all even natural numbers $n \ge 4$,there is a semigroup S of such that girth of $G(S)$ is n.*

Proof. Let $n = 4$.Consider $X = \{y_1,y_2,y_3,y_4,y_5,y_6\}$. Define the idempotents a_1 and a_2 as;
$im(a_1) = \{y_1,y_3,y_5\}$ and $im(a_2) = \{y_1,y_3,y_6\}$.So there will be three kernel classes of each a_i such that,
$class - 1 = \{y_1,y_2\}$
$class - 2 = \{y_3,y_4\}$
$class - 3 = \{y_5,y_6\}$
By lemma 2,$a_1a_1 = a_1$,$a_2a_2 = a_2$, $a_1a_2 = a_2$,$a_2a_1 = a_1$ and $a_1cy_1 = cy_1$, $a_3cy_1 = cy_3$, $a_1cy_3 = cy_3$, $a_2cy_3 = cy_3$, $cy_1a_1 = cy_1$, $cy_1a_2 = cy_1$,$cy_3a_1 = cy_3$, $cy_3a_2 = cy_3$, $cy_1cy_3 = cy_3$,$cy_3cy_1 = cy_1$, $cy_1cy_i = cy_1$, $cy_3cy_3 = cy_3$. Thus $S =< a_1,e_1,a_2,e_3 >$ is the semigroup in which unique and minimal length of cycle $a_1 - e_1 - a_2 - e_3 - a_1$ is 4.So girth of S is 4.

Example 1. Let $X = \{y_1,y_2,y_3,y_4,y_5,y_6\}$.By definition 8, Since $n = \frac{|X|}{3}$ is even so there will be $\frac{|X|}{3}+1$ three right zero semigroups generators namely a_1,a_2,a_3 and three constant generators namely e_1,e_4,e_5 whose sets of images are defined as;

$\text{Im}(a_1)$	y_1	y_3	y_5
$\text{Im}(a_2)$	y_1	y_4	y_6
$\text{Im}(a_3)$	y_2	y_4	y_5

Table 3: Images of the generators.

Kernel classes of each a_1, a_2, a_3 is defined as,
$class - 1 = \{y_1, y_2\}$
$class - 2 = \{y_3, y_4\}$
$class - 3 = \{y_5, y_6\}$
The products defined in lemma 2,we have $S =< a_1, e_1, a_2, e_4, a_3, e_5 >$ is the semigroup in which the unique and minimal cycle $a_1 - e_1 - a_2 - e_4 - a_3 - e_5 - a_1$ is of length 6 i.e $2(\frac{|X|}{3} + 1)$.Thus girth of $S = 6$. The Table 4 presents the Cayley's table for S.

	a_1	a_2	a_3	e_1	e_4	e_5	e_2	e_3	e_6
a_1	a_1	a_2	a_3	e_1	e_4	e_5	e_2	e_3	e_6
a_2	a_1	a_2	a_3	e_1	e_4	e_5	e_2	e_3	e_6
a_3	a_1	a_2	a_3	e_1	e_4	e_5	e_2	e_3	e_6
e_1	e_1	e_1	e_2	e_1	e_4	e_5	e_2	e_3	e_6
e_4	e_3	e_4	e_4	e_1	e_4	e_5	e_2	e_3	e_6
e_5	e_5	e_6	e_5	e_1	e_4	e_5	e_2	e_3	e_6
e_2	e_1	e_1	e_2	e_1	e_4	e_5	e_2	e_3	e_6
e_3	e_3	e_4	e_4	e_1	e_4	e_5	e_2	e_3	e_6
e_6	e_5	e_6	e_5	e_1	e_4	e_5	e_2	e_3	e_6

Table 4: Cayley table for S

Conjecture!. There are no semigroups of a circle and straight line.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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