New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.237

Complete lift of a tensor field of type (1,2) to semi-cotangent bundle

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Received: 16 October 2017, Accepted: 9 November 2017
Published online: 25 December 2017.

Abstract: The main purpose of this paper is to define the complete lift of a projectable tensor field of type (1,2) to semi-cotangent bundle t*M. Using projectable geometric objects on M, we examine lifting problem of projectable tensor field of type (1,2) to the semi-cotangent bundle. We also present the good square in the semi-cotangent bundle t*M.

Keywords: Complete lift, pull-back bundle, semi-cotangent bundle, vector field.

1 Introduction

Let M_n be a differentiable manifold of class C^∞ and finite dimension n, and let (M_n, π_1, B_m) be a differentiable bundle over B_m. We use the notation $(x^i) = (x^a, x^\alpha)$, where the indices $i, j, \ldots$ run from 1 to n, the indices $a, b, \ldots$ from 1 to $n-m$ and the indices $\alpha, \beta, \ldots$ from $n-m+1$ to n, x^α are coordinates in B_m, x^a are fibre coordinates of the bundle

$$\pi_1 : M_n \to B_m.$$

Let now $(T^*(B_m), \widetilde{\pi}, B_m)$ be a cotangent bundle [1] over base space B_m, and let M_n be differentiable bundle determined by a natural projection (submersion) $\pi_1 : M_n \to B_m$. The semi-cotangent bundle (pull-back [2], [3], [4], [5], [6]) of the cotangent bundle $(T^*(B_m), \widetilde{\pi}, B_m)$ is the bundle $(t^*(B_m), \pi_2, M_n)$ over differentiable bundle M_n with a total space

$$t^*(B_m) = \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}} \right) \in M_n \times T_x^*(B_m) : \pi_1 (x^a, x^\alpha) = \widetilde{\pi} \left(x^\alpha, x^{\overline{\alpha}} \right) = (x^\alpha) \right\} \subset M_n \times T_x^*(B_m)$$

and with the projection map $\pi_2 : t^*(B_m) \to M_n$ defined by $\pi_2(x^a, x^\alpha, x^{\overline{\alpha}}) = (x^a, x^\alpha)$, where $T_x^*(B_m)\,(x = \pi_1(\widetilde{x}), \widetilde{x} = (x^a, x^\alpha) \in M_n)$ is the cotangent space at a point x of B_m, where $x^{\overline{\alpha}} = p_\alpha$ $\left(\overline{\alpha}, \overline{\beta}, \ldots, = n+1, \ldots, 2n \right)$ are fibre coordinates of the cotangent bundle $T^*(B_m)$.

Where the pull-back (Pontryagin [7]) bundle $t^*(B_m)$ of the differentiable bundle M_n also has the natural bundle structure over B_m, its bundle projection $\pi : t^*(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$, and hence $\pi = \pi_1 \circ \pi_2$. Thus $(t^*(B_m), \pi_1 \circ \pi_2)$ is the composite bundle [[8], p.9] or step-like bundle [9]. Consequently, we notice the semi-cotangent bundle $(t^*(B_m), \pi_2)$ is a pull-back bundle of the cotangent bundle over B_m by π_1 [6].

If $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another local adapted coordinates in differentiable bundle M_n, then we have

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right). \end{cases} \tag{1}$$

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The Jacobian of (1) has components

$$\left(A_j^{i'}\right) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where $A_b^{a'} = \frac{\partial x^{a'}}{\partial x^b}$, $A_\beta^{a'} = \frac{\partial x^{a'}}{\partial x^\beta}$, $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$ [6].

To a transformation (1) of local coordinates of M_n, there corresponds on $t^*(B_m)$ the change of coordinate

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\alpha}'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} x^{\overline{\beta}}. \end{cases} \tag{2}$$

The Jacobian of coordinate system transformation (2) is:

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & p_\sigma A_\beta^{\beta'} A_{\beta'\alpha'}^\sigma & A_{\alpha'}^\beta \end{pmatrix}, \tag{3}$$

where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $I, J, ... = 1, ..., 2n$; $A_{\beta'\alpha'}^\sigma = \frac{\partial^2 x^\sigma}{\partial x^{\beta'} \partial x^{\alpha'}}$ [6].

Now, consider a diagram as

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow[\pi]{} & D \end{array}$$

A good square of vector bundles is a diagram as above verifying

(i) α and β are fibre bundles, but not necessarily vector bundles;
(ii) γ and π are vector bundles;
(iii) the square is commutative, i.e., $\pi \circ \alpha = \beta \circ \gamma$;
(iv) the local expression

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow[\pi]{} & D \end{array} \quad \begin{array}{ccc} U^n \times R^r \times G^s \times R^t & \rightarrow & U^n \times G^s \\ \downarrow & & \downarrow \\ U^n \times R^r & \rightarrow & U^n \end{array} \quad \begin{array}{ccc} (x^i, a^a, g^\lambda, b^\sigma) & \rightarrow & (x^i, g^\lambda) \\ \downarrow & & \downarrow \\ (x^i, a^a) & \rightarrow & (x^i) \end{array}$$

where G is a manifold and superindices denote the dimension of the manifolds [11].

By means of above definition, we have

Theorem 1. *Let now $\pi : t^*(B_m) \rightarrow B_m$ be a semi-cotangent bundle and $\pi_1 : M_n \rightarrow B_m$ be a fibre bundle. Then, the following is a good square:*

$$\begin{array}{ccc} t^*(B_m) & \xrightarrow{\pi_2} & M_n \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ t^*(B_m) & \xrightarrow[\pi]{} & B_m \end{array} \quad \begin{array}{ccc} M_n \times T_x^*(B_m) & \xrightarrow{\pi_2} & M_n \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ M_n \times T_x^*(B_m) & \xrightarrow[\pi]{} & B_m \end{array} \quad \begin{array}{ccc} (x^a, x^\alpha, x^{\overline{\alpha}}) & \xrightarrow{\pi_2} & (x^a, x^\alpha) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ (x^a, x^\alpha, x^{\overline{\alpha}}) & \xrightarrow[\pi]{} & (x^\alpha) \end{array}$$

In this study, we continue to study the complete lifts of projectable tensor field of type (1,2) to semi-cotangent (pull-back) bundle $(t^*(B_m), \pi_2)$ initiated by F. Yildirim and A. Salimov [6].

We denote by $\Im_q^p(M_n)$ the set of all tensor fields of class C^∞ and of type (p,q) on M_n, i.e., contravariant degree p and covariant degree q. We now put $\Im(M_n)=\sum_{p,q=0}^{\infty}\Im_q^p(M_n)$, which is the set of all tensor fields on M_n. Smilarly, we denote by $\Im_q^p(B_m)$ and $\Im(B_m)$ respectively the corresponding sets of tensor fields in the base space B_m.

Let ω be a $1-$form with local components ω_α on B_m, so that ω is a $1-$form with local expression $\omega=\omega_\alpha dx^\alpha$. On putting [6]

$$^{vv}\omega=\begin{pmatrix}0\\0\\\omega_\alpha\end{pmatrix}, \tag{4}$$

we have a vector field $^{vv}\omega$ on $t^*(B_m)$. In fact, from (3) we easily see that $(^{vv}\omega)'=\overline{A}(^{vv}\omega)$. We call the vector field $^{vv}\omega$ the vertical lift of the $1-$form ω to $t^*(B_m)$.

Let $\widetilde{X}\in\Im_0^1(M_n)$ be a projectable vector field [10] with projection $X=X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X}=\widetilde{X}^a(x^a,x^\alpha)\partial_a+X^\alpha(x^\alpha)\partial_\alpha$. Now, consider $\widetilde{X}\in\Im_0^1(M_n)$, then $^{cc}\widetilde{X}$ (complete lift) has components on the semi-cotangent bundle $t^*(B_m)$ [6]

$$^{cc}\widetilde{X}=\left(^{cc}\widetilde{X}^\alpha\right)=\begin{pmatrix}\widetilde{X}^a\\X^\alpha\\-p_\varepsilon(\partial_\alpha X^\varepsilon)\end{pmatrix} \tag{5}$$

with respect to the coordinates $(x^a,x^\alpha,x^{\overline{\alpha}})$.

2 $\gamma-$operators

For any $F\in\Im_1^1(B_m)$, if we take account of (3), we can prove that $(\gamma F)'=\overline{A}(\gamma F)$, where γF is a vector field defined by [6]:

$$\gamma F=(\gamma F^I)=\begin{pmatrix}0\\0\\p_\beta F_\alpha^\beta\end{pmatrix} \tag{6}$$

with respect to the coordinates $(x^a,x^\alpha,x^{\overline{\alpha}})$ on $t^*(B_m)$.

For any $R\in\Im_3^1(B_m)$, if we take account of (3), we can prove that $\gamma R_{I'J'}^{K'}=A_K^{K'}A_{I'}^IA_{J'}^J\gamma R_{IJ}^K$, where γR has components $\overline{R}_{IJ}^K$ such that

$$\overline{R}_{\alpha\beta}^{\overline{\gamma}}=P_\varepsilon R_{\alpha\beta\gamma}^{\ \ \ \varepsilon}, \tag{7}$$

all the others being zero, with respect to the induced coordinates on $t^*(B_m)$. Where $R_{\alpha\beta\sigma}^{\ \ \ \gamma}$ are local components of R on B_m and also $I=(a,\alpha,\overline{\alpha})$, $J=(b,\beta,\overline{\beta})$, $K=(c,\gamma,\overline{\gamma})$.

Theorem 2. *If $\widetilde{X}$ and $\widetilde{Y}$ be a projectable vector fields on M_n with projection $X\in\Im_0^1(B_m)$ and $Y\in\Im_0^1(B_m)$. We have*

(i) $(\gamma R)(^{cc}\widetilde{X},{}^{cc}\widetilde{Y})=\gamma(R(X,Y))$,
(ii) $(\gamma R)(^{vv}\omega,{}^{vv}\theta)=0$,
(iii) $(\gamma R)(^{vv}\omega,{}^{cc}Y)=0$,
(iv) $(\gamma R)(^{vv}\omega,\gamma G)=0$,
(v) $(\gamma R)(^{cc}\widetilde{X},\gamma G)=0$,
(vi) $(\gamma R)(\gamma F,\gamma G)=0$

for any $\omega,\theta\in\Im_1^0(B_m)$, $F,G\in\Im_1^1(B_m)$ and $R\in\Im_3^1(B_m)$.

Proof. (i) If $R \in \Im_3^1(B_m)$, $\widetilde{X}$ and $\widetilde{Y}$ be a projectable vector fields on M_n with projection $X,Y \in \Im_0^1(B_m)$ and

$$\begin{pmatrix} [(\gamma R)({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})]^c \\ [(\gamma R)({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})]^\gamma \\ [(\gamma R)({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})]^{\overline{\gamma}} \end{pmatrix}$$

are components of $[(\gamma R)({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})]^K$ with respect to the coordinates $(x^c, x^\gamma, x^{\overline{\gamma}})$ on $t^*(B_m)$, then for $K = c$, we have

$$[(\gamma R)({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})]^c = \underbrace{(\overline{R}_{\alpha\beta}{}^c)}_{0}{}^{cc}\widetilde{X}^{\alpha\, cc}\widetilde{Y}^\beta = 0$$

because of (5) and (7). For $K = \gamma$, we have

$$[(\gamma R)({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})]^\gamma = \underbrace{(\overline{R}_{\alpha\beta}{}^\gamma)}_{0}{}^{cc}\widetilde{X}^{\alpha\, cc}\widetilde{Y}^\beta = 0$$

because of (5) and (7). For $K = \overline{\gamma}$, we have

$$[(\gamma R)({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})]^{\overline{\gamma}} = (\overline{R}_{\alpha\beta}{}^{\overline{\gamma}})\underbrace{{}^{cc}\widetilde{X}^\alpha}_{X^\alpha}\underbrace{{}^{cc}\widetilde{Y}^\beta}_{Y^\beta} = P_\varepsilon R_{\alpha\beta\gamma}{}^\varepsilon X^\alpha Y^\beta = P_\varepsilon (R(X,Y))_\gamma^\varepsilon$$

because of (5) and (7). It is well known that $\gamma(R(X,Y))$ have components

$$\gamma(R(X,Y)) = \begin{pmatrix} 0 \\ 0 \\ P_\varepsilon (R(X,Y))_\gamma^\varepsilon \end{pmatrix}$$

with respect to the coordinates $(x^c, x^\gamma, x^{\overline{\gamma}})$ on $t^*(B_m)$. Thus, we have $(\gamma R)({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y}) = \gamma(R(X,Y))$. Similarly, we can easily compute another equations of Theorem 2.

3 Complete lift of a tensor field of type (1,2) to semi-cotangent bundle

Let $\widetilde{S} \in \Im_2^1(M_n)$ be a projectable tensor field of type $(1,2)$ with projection $S = S_{ij}^k(x^a, x^\alpha)\,\partial_k \otimes dx^i \otimes dx^j$, i.e. $\widetilde{S}$ has componets such that

$${}^{cc}\widetilde{S}_{\alpha\beta}^c = S_{\alpha\beta}^c$$

with respect to the coordinates on M_n. Where $i = (a,\alpha)$, $j = (b,\beta)$, $k = (c,\gamma)$.

If we take account of (3), we can prove that ${}^{cc}\widetilde{S}_{I'J'}^{K'} = A_K^{K'} A_{I'}^I A_{J'}^J\, {}^{cc}\widetilde{S}_{IJ}^K$, where ${}^{cc}\widetilde{S}$ has components ${}^{cc}\widetilde{S}_{IJ}^K$ such that

$$\begin{cases} {}^{cc}\widetilde{S}_{\alpha\beta}^c = S_{\alpha\beta}^c \\ {}^{cc}\widetilde{S}_{\alpha\beta}^{\gamma} = S_{\alpha\beta}^{\gamma} \\ {}^{cc}\widetilde{S}_{\alpha\beta}^{\overline{\gamma}} = -p_\varepsilon(\partial_\alpha S_{\beta\gamma}^\varepsilon + \partial_\beta S_{\gamma\alpha}^\varepsilon + \partial_\gamma S_{\alpha\beta}^\varepsilon) \\ {}^{cc}\widetilde{S}_{\alpha\overline{\beta}}^{\overline{\gamma}} = S_{\alpha\gamma}^{\beta} \\ {}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^{\overline{\gamma}} = S_{\gamma\beta}^{\alpha} \end{cases}, \quad (8)$$

all the others being zero, with respect to the induced coordinates on $t^*(B_m)$. Where S_{IJ}^K are local components of S on M_n and also $I = (a,\alpha,\overline{\alpha})$, $J = (b,\beta,\overline{\beta})$, $K = (c,\gamma,\overline{\gamma})$.

Proof. For convenience sake we only consider ${}^{cc}\widetilde{S}_{\overline{\alpha}'\beta'}^{\overline{\gamma}'}$. In fact,

$$ {}^{cc}\widetilde{S}_{\overline{\alpha}'\beta'}^{\overline{\gamma}'} = A_{\overline{\gamma}}^{\overline{\gamma}'} A_{\overline{\alpha}'}^{\overline{\alpha}} A_{\beta'}^{\beta} \, {}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^{\overline{\gamma}} = A_{\gamma'}^{\gamma} A_{\alpha}^{\alpha'} A_{\beta'}^{\beta} S_{\gamma\beta}^{\alpha} = S_{\gamma'\beta'}^{\alpha'}. $$

Thus, we have ${}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^{\overline{\gamma}} = S_{\gamma\beta}^{\alpha}$. Similarly, from (3) and (8), we can easily find all other components of ${}^{cc}\widetilde{S}_{IJ}^{K}$ equal to zero, where $I = (a,\alpha,\overline{\alpha})$, $J = (b,\beta,\overline{\beta})$, $K = (c,\gamma,\overline{\gamma})$.

Theorem 3. *Let $\widetilde{S} \in \Im_2^1(M_n)$ be a projectable tensor field of type $(1,2)$. If $\widetilde{X},\widetilde{Y} \in \Im_0^1(M_n)$, $\omega,\theta \in \Im_1^0(B_m)$, $F,G \in \Im_1^1(B_m)$ then*

(i) ${}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta) = 0$,
(ii) ${}^{cc}\widetilde{S}({}^{vv}\omega, \gamma G) = 0$,
(iii) ${}^{cc}\widetilde{S}({}^{vv}\omega, {}^{cc}\widetilde{Y}) = -{}^{vv}(\omega \circ S_Y)$,
(iv) ${}^{cc}\widetilde{S}(\gamma F, \gamma G) = 0$,
(v) ${}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y}) = -\gamma(F \circ S_Y)$,
(vi) ${}^{cc}\widetilde{S}({}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}) = {}^{cc}(S(X,Y)) - \gamma((L_X S)_Y - (L_Y S)_X + S_{[X,Y]})$,

where $L_X S$ denotes the Lie derivative of S with respect to X.

Proof. **(i)** If $\omega,\theta \in \Im_1^0(B_m)$ and $\widetilde{S}$ is projectable tensor field of type $(1,2)$ on M_n with projection $S \in \Im_2^1(B_m)$ and

$$ \begin{pmatrix} \left({}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta)\right)^{c} \\ \left({}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta)\right)^{\gamma} \\ \left({}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta)\right)^{\overline{\gamma}} \end{pmatrix} $$

are components of $\left({}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta)\right)^{K}$ with respect to the coordinates $(x^c, x^\gamma, x^{\overline{\gamma}})$ on $t^*(B_m)$, then we have

$$ \left({}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta)\right)^{K} = {}^{cc}\widetilde{S}_{IJ}^{K} \, {}^{vv}\omega^{I} \, {}^{vv}\theta^{J} = {}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{K} \, {}^{vv}\omega^{\overline{\alpha}} \, {}^{vv}\theta^{\overline{\beta}} = {}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{K} \omega_\alpha \theta_\beta. $$

Firstly, if $K = c$, we have

$$ \left({}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta)\right)^{c} = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{c}}_{0} \omega_\alpha \theta_\beta = 0 $$

by virtue of (4) and (8). Secondly, if $K = \gamma$, we have

$$ \left({}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta)\right)^{\gamma} = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\gamma}}_{0} \omega_\alpha \theta_\beta = 0 $$

by virtue of (4) and (8). Thirdly, if $J = \overline{\beta}$, then we have

$$ \left({}^{cc}\widetilde{S}({}^{vv}\omega, {}^{vv}\theta)\right)^{\overline{\gamma}} = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}}}_{0} \omega_\alpha \theta_\beta = 0 $$

by virtue of (4) and (8). Thus (i) of Theorem 3 is proved.

(ii) If $G \in \Im_1^1(B_m)$ and $\widetilde{S}$ is projectable tensor field of type $(1,2)$ on M_n with projection $S \in \Im_2^1(B_m)$ and

$$\begin{pmatrix} \left({}^{cc}\widetilde{S}({}^{vv}\omega,\gamma G)\right)^{c} \\ \left({}^{cc}\widetilde{S}({}^{vv}\omega,\gamma G)\right)^{\gamma} \\ \left({}^{cc}\widetilde{S}({}^{vv}\omega,\gamma G)\right)^{\overline{\gamma}} \end{pmatrix}$$

are components of $\left({}^{cc}\widetilde{S}({}^{vv}\omega,\gamma G)\right)^{K}$ with respect to the coordinates $(x^c,x^\gamma,x^{\overline{\gamma}})$ on $t^*(B_m)$, then we have

$$\left({}^{cc}\widetilde{S}({}^{vv}\omega,\gamma G)\right)^{K} = {}^{cc}\,\widetilde{S}_{IJ}^{K}\,{}^{vv}\omega^{I}\gamma G^{J} = {}^{cc}\,\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{K}\,{}^{vv}\omega^{\overline{\alpha}}\gamma G^{\overline{\beta}} = {}^{cc}\,\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{K}\omega_\alpha p_\varepsilon G_\beta^\varepsilon.$$

Firstly, if $K=c$, we have

$$\left({}^{cc}\widetilde{S}({}^{vv}\omega,\gamma G)\right)^{c} = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{c}}_{0}\omega_\alpha p_\varepsilon G_\beta^\varepsilon = 0$$

by virtue of (4), (6) and (8). Secondly, if $K=\gamma$, we have

$$\left({}^{cc}\widetilde{S}({}^{vv}\omega,\gamma G)\right)^{\gamma} = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\gamma}}_{0}\omega_\alpha p_\varepsilon G_\beta^\varepsilon = 0$$

by virtue of (4), (6) and (8). Thirdly, if $J=\overline{\beta}$, then we have

$$\left({}^{cc}\widetilde{S}({}^{vv}\omega,\gamma G)\right)^{\overline{\gamma}} = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}}}_{0}\omega_\alpha p_\varepsilon G_\beta^\varepsilon = 0$$

by virtue of (4), (6) and (8). Thus (ii) of Theorem 3 is proved.

(iii) If $\widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{S}$ is projectable tensor field of type $(1,2)$ on M_n with projection $S \in \Im_2^1(B_m)$ and

$$\begin{pmatrix} \left({}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})\right)^{c} \\ \left({}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})\right)^{\gamma} \\ \left({}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})\right)^{\overline{\gamma}} \end{pmatrix}$$

are components of $\left({}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})\right)^{K}$ with respect to the coordinates $(x^c,x^\gamma,x^{\overline{\gamma}})$ on $t^*(B_m)$, then we have

$$\left({}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})\right)^{K} = {}^{cc}\,\widetilde{S}_{IJ}^{K}\,({}^{vv}\omega)^{I}\left({}^{cc}\widetilde{Y}\right)^{J} = {}^{cc}\,\widetilde{S}_{\overline{\alpha}b}^{K}\,({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{b} + {}^{cc}\,\widetilde{S}_{\overline{\alpha}\beta}^{K}\,({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{\beta} + {}^{cc}\,\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{K}\,({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}}.$$

Firstly, if $K=c$, we have

$$\left({}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})\right)^{c} = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}b}^{c}}_{0}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{b} + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^{c}}_{0}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{\beta} + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{c}}_{0}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}} = 0$$

by virtue of (4), (5) and (8). Secondly, if $K=\gamma$, we have

$$\left({}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})\right)^{\gamma}=\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}b}^{\gamma}}_{0}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{b}+\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^{\gamma}}_{0}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{\beta}+\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\gamma}}_{0}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}}=0$$

by virtue of (4), (5) and (8). Thirdly, if $K=\overline{\gamma}$, then we have

$$\begin{aligned}\left({}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})\right)^{\overline{\gamma}}&=\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}b}^{\overline{\gamma}}}_{0}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{b}+\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^{\overline{\gamma}}}_{S_{\gamma\beta}^{\alpha}=-S_{\beta\gamma}^{\alpha}}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{\beta}+\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}}}_{0}({}^{vv}\omega)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}}\\&=-S_{\beta\gamma}^{\alpha}\omega_{\alpha}Y^{\beta}=-S_{\beta\gamma}^{\alpha}\omega_{\alpha}Y^{\beta}=-(\omega\circ S_Y)_{\gamma}\end{aligned}$$

by virtue of (4), (5) and (8). On the other hand, we know that ${}^{vv}(\omega\circ S_Y)$ have components

$$ {}^{vv}(\omega\circ S_Y)=\begin{pmatrix}0\\0\\(\omega\circ S_Y)_{\gamma}\end{pmatrix}$$

with respect to the coordinates $(x^c,x^{\gamma},x^{\overline{\gamma}})$ on $t^*(B_m)$. Thus, we have ${}^{cc}\widetilde{S}({}^{vv}\omega,{}^{cc}\widetilde{Y})=-{}^{vv}(\omega\circ S_Y)$.

(iv) If $F,G\in\mathfrak{I}_1^1(B_m)$ and $\widetilde{S}$ is projectable tensor field of type $(1,2)$ on M_n with projection $S\in\mathfrak{I}_2^1(B_m)$ and

$$\begin{pmatrix}\left({}^{cc}\widetilde{S}(\gamma F,\gamma G)\right)^{c}\\\left({}^{cc}\widetilde{S}(\gamma F,\gamma G)\right)^{\gamma}\\\left({}^{cc}\widetilde{S}(\gamma F,\gamma G)\right)^{\overline{\gamma}}\end{pmatrix}$$

are components of $\left({}^{cc}\widetilde{S}(\gamma F,\gamma G)\right)^{K}$ with respect to the coordinates $(x^c,x^{\gamma},x^{\overline{\gamma}})$ on $t^*(B_m)$, then we have

$$\left({}^{cc}\widetilde{S}(\gamma F,\gamma G)\right)^{K}={}^{cc}\widetilde{S}_{IJ}^{K}\gamma F^{I}\gamma G^{J}={}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{K}(\gamma F)^{\overline{\alpha}}(\gamma G)^{\overline{\beta}}={}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{K}(p_{\varepsilon}F_{\alpha}^{\varepsilon})\left(p_{\varepsilon}G_{\beta}^{\varepsilon}\right).$$

Firstly, if $K=c$, we have

$$\left({}^{cc}\widetilde{S}(\gamma F,\gamma G)\right)^{c}=\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{c}}_{0}(p_{\varepsilon}F_{\alpha}^{\varepsilon})\left(p_{\varepsilon}G_{\beta}^{\varepsilon}\right)=0$$

by virtue of (6) and (8). Secondly, if $K=\gamma$, we have

$$\left({}^{cc}\widetilde{S}(\gamma F,\gamma G)\right)^{\gamma}=\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\gamma}}_{0}(p_{\varepsilon}F_{\alpha}^{\varepsilon})\left(p_{\varepsilon}G_{\beta}^{\varepsilon}\right)=0$$

by virtue of (6) and (8). Thirdly, if $J=\overline{\beta}$, then we have

$$\left({}^{cc}\widetilde{S}(\gamma F,\gamma G)\right)^{\overline{\gamma}}=\underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}}}_{0}(p_{\varepsilon}F_{\alpha}^{\varepsilon})\left(p_{\varepsilon}G_{\beta}^{\varepsilon}\right)=0$$

by virtue of (6) and (8). Thus (iv) of Theorem 3 is proved.

(v) If $\widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{S}$ is projectable tensor field of type $(1,2)$ on M_n with projection $S \in \Im_2^1(B_m)$ and

$$\begin{pmatrix} \left({}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y})\right)^c \\ \left({}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y})\right)^\gamma \\ \left({}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y})\right)^{\overline{\gamma}} \end{pmatrix}$$

are components of $\left({}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y})\right)^K$ with respect to the coordinates $(x^c, x^\gamma, x^{\overline{\gamma}})$ on $t^*(B_m)$, then we have

$$\left({}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y})\right)^K = {}^{cc}\widetilde{S}_{IJ}^K (\gamma F)^I \left({}^{cc}\widetilde{Y}\right)^J = {}^{cc}\widetilde{S}_{\overline{\alpha}b}^K (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^b + {}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^K (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^\beta + {}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^K (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}}.$$

Firstly, if $K = c$, we have

$$\left({}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y})\right)^c = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}b}^c}_{0} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^b + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^c}_{0} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^\beta + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^c}_{0} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}} = 0$$

by virtue of (5), (6) and (8). Secondly, if $K = \gamma$, we have

$$\left({}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y})\right)^\gamma = \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}b}^\gamma}_{0} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^b + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^\gamma}_{0} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^\beta + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^\gamma}_{0} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}} = 0$$

by virtue of (5), (6) and (8). Thirdly, if $K = \overline{\gamma}$, then we have

$$\begin{aligned} \left({}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y})\right)^{\overline{\gamma}} &= \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}b}^{\overline{\gamma}}}_{0} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^b + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^{\overline{\gamma}}}_{S_{\gamma\beta}^{\alpha} = -S_{\beta\gamma}^{\alpha}} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^\beta + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}}}_{0} (\gamma F)^{\overline{\alpha}} \left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}} \\ &= -S_{\beta\gamma}^{\alpha} p_\varepsilon F_\alpha^\varepsilon Y^\beta = -p_\varepsilon \left(S_{\beta\gamma}^{\alpha} F_\alpha^\varepsilon Y^\beta\right) = -p_\varepsilon (F \circ S_Y)_\gamma^\varepsilon \end{aligned}$$

by virtue of (5), (6) and (8). On the other hand, we know that $\gamma(F \circ S_Y)$ have components

$$\gamma(F \circ S_Y) = \begin{pmatrix} 0 \\ 0 \\ p_\varepsilon (F \circ S_Y)_\gamma^\varepsilon \end{pmatrix}$$

with respect to the coordinates $(x^c, x^\gamma, x^{\overline{\gamma}})$ on $t^*(B_m)$. Thus, we have ${}^{cc}\widetilde{S}(\gamma F, {}^{cc}\widetilde{Y}) = -\gamma(F \circ S_Y)$.

(vi) If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{S}$ is projectable tensor field of type $(1,2)$ on M_n with projection $S \in \Im_2^1(B_m)$ and

$$\begin{pmatrix} \left({}^{cc}\widetilde{S}({}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y})\right)^c \\ \left({}^{cc}\widetilde{S}({}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y})\right)^\gamma \\ \left({}^{cc}\widetilde{S}({}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y})\right)^{\overline{\gamma}} \end{pmatrix}$$

are components of $\left({}^{cc}\widetilde{S}({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})\right)^K$ with respect to the coordinates $(x^c, x^\gamma, x^{\overline{\gamma}})$ on $t^*(B_m)$, then we have

$$\left({}^{cc}\widetilde{S}({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})\right)^K = {}^{cc}\widetilde{S}_{IJ}^K\left({}^{cc}\widetilde{X}\right)^I\left({}^{cc}\widetilde{Y}\right)^J = {}^{cc}\widetilde{S}_{\alpha\beta}^K\left({}^{cc}\widetilde{X}\right)^\alpha\left({}^{cc}\widetilde{Y}\right)^\beta + {}^{cc}\widetilde{S}_{\alpha\overline{\beta}}^K\left({}^{cc}\widetilde{X}\right)^\alpha\left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}} + {}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^K\left({}^{cc}\widetilde{X}\right)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^\beta.$$

Firstly, if $K = c$, we have

$$\begin{aligned}\left({}^{cc}\widetilde{S}({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})\right)^c &= \underbrace{{}^{cc}\widetilde{S}_{\alpha\beta}^c}_{S_{\alpha\beta}^c}\underbrace{\left({}^{cc}\widetilde{X}\right)^\alpha}_{X^\alpha}\underbrace{\left({}^{cc}\widetilde{Y}\right)^\beta}_{Y^\beta} + \underbrace{{}^{cc}\widetilde{S}_{\alpha\overline{\beta}}^c}_{0}\left({}^{cc}\widetilde{X}\right)^\alpha\left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}} + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^c}_{0}\left({}^{cc}\widetilde{X}\right)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^\beta \\ &= S_{\alpha\beta}^c X^\alpha Y^\beta = (S(X,Y))^c\end{aligned}$$

by virtue of (5) and (8). Secondly, if $K = \gamma$, we have

$$\begin{aligned}\left({}^{cc}\widetilde{S}({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})\right)^\gamma &= \underbrace{{}^{cc}\widetilde{S}_{\alpha\beta}^\gamma}_{S_{\alpha\beta}^\gamma}\underbrace{\left({}^{cc}\widetilde{X}\right)^\alpha}_{X^\alpha}\underbrace{\left({}^{cc}\widetilde{Y}\right)^\beta}_{Y^\beta} + \underbrace{{}^{cc}\widetilde{S}_{\alpha\overline{\beta}}^\gamma}_{0}\left({}^{cc}\widetilde{X}\right)^\alpha\left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}} + \underbrace{{}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^\gamma}_{0}\left({}^{cc}\widetilde{X}\right)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^\beta \\ &= S_{\alpha\beta}^\gamma X^\alpha Y^\beta = (S(X,Y))^\gamma\end{aligned}$$

by virtue of (5) and (8). Thirdly, if $K = \overline{\gamma}$, then we have

$$\begin{aligned}\left({}^{cc}\widetilde{S}({}^{cc}\widetilde{X},{}^{cc}\widetilde{Y})\right)^{\overline{\gamma}} &= {}^{cc}\widetilde{S}_{\alpha\beta}^{\overline{\gamma}}\left({}^{cc}\widetilde{X}\right)^\alpha\left({}^{cc}\widetilde{Y}\right)^\beta + {}^{cc}\widetilde{S}_{\alpha\overline{\beta}}^{\overline{\gamma}}\left({}^{cc}\widetilde{X}\right)^\alpha\left({}^{cc}\widetilde{Y}\right)^{\overline{\beta}} + {}^{cc}\widetilde{S}_{\overline{\alpha}\beta}^{\overline{\gamma}}\left({}^{cc}\widetilde{X}\right)^{\overline{\alpha}}\left({}^{cc}\widetilde{Y}\right)^\beta \\ &= -p_\varepsilon(\partial_\alpha S_{\beta\gamma}^\varepsilon + \partial_\beta S_{\gamma\alpha}^\varepsilon + \partial_\gamma S_{\alpha\beta}^\varepsilon)X^\alpha Y^\beta - p_\varepsilon S_{\alpha\gamma}^\beta X^\alpha\partial_\beta Y^\varepsilon - p_\varepsilon S_{\gamma\beta}^\alpha\partial_\alpha X^\varepsilon Y^\beta \\ &= -p_\varepsilon\partial_\alpha S_{\beta\gamma}^\varepsilon X^\alpha Y^\beta - p_\varepsilon\partial_\beta S_{\gamma\alpha}^\varepsilon X^\alpha Y^\beta - p_\varepsilon\partial_\gamma S_{\alpha\beta}^\varepsilon X^\alpha Y^\beta - p_\varepsilon S_{\alpha\gamma}^\beta X^\alpha\partial_\beta Y^\varepsilon - p_\varepsilon S_{\gamma\beta}^\alpha\partial_\alpha X^\varepsilon Y^\beta \\ &= -\underbrace{p_\alpha\partial_\beta S_{\varepsilon\gamma}^\alpha X^\beta Y^\varepsilon}_{A1} - \underbrace{p_\alpha\partial_\varepsilon S_{\gamma\beta}^\alpha X^\beta Y^\varepsilon}_{A2} - \underbrace{p_\alpha\partial_\gamma S_{\beta\varepsilon}^\alpha X^\beta Y^\varepsilon}_{A3} - \underbrace{p_\varepsilon S_{\alpha\gamma}^\beta X^\alpha\partial_\beta Y^\varepsilon}_{A4} + \underbrace{p_\varepsilon S_{\beta\gamma}^\alpha\partial_\alpha X^\varepsilon Y^\beta}_{A5}\end{aligned}$$

by virtue of (5) and (8). We know that ${}^{cc}(S(X,Y))^{\overline{\gamma}}$, $p_\alpha((L_X S)_Y)_\gamma^\alpha$, $-p_\alpha((L_Y S)_X)_\gamma^\alpha$ and $p_\alpha\left(S_{[X,Y]}\right)_\gamma^\alpha$ have respectively, components on $t^*(B_m)$

$$\begin{aligned}{}^{cc}(S(X,Y))^{\overline{\gamma}} &= -p_\alpha\partial_\gamma(S_{\beta\varepsilon}^\alpha X^\beta Y^\varepsilon) = -p_\alpha\left(\partial_\gamma S_{\beta\varepsilon}^\alpha\right)X^\beta Y^\varepsilon - p_\alpha\left(\partial_\gamma X^\beta\right)S_{\beta\varepsilon}^\alpha Y^\varepsilon - p_\alpha\left(\partial_\gamma Y^\varepsilon\right)S_{\beta\varepsilon}^\alpha X^\beta \\ {}^{cc}(S(X,Y))^{\overline{\gamma}} &= -p_\alpha\left(\partial_\gamma S_{\beta\varepsilon}^\alpha\right)X^\beta Y^\varepsilon + p_\alpha\left(\partial_\gamma X^\beta\right)S_{\varepsilon\beta}^\alpha Y^\varepsilon - p_\alpha\left(\partial_\gamma Y^\varepsilon\right)S_{\beta\varepsilon}^\alpha X^\beta \\ {}^{cc}(S(X,Y))^{\overline{\gamma}} &= \underbrace{-p_\alpha\left(\partial_\gamma S_{\beta\varepsilon}^\alpha\right)X^\beta Y^\varepsilon}_{A3} + \underbrace{p_\alpha\left(\partial_\gamma X^\beta\right)S_{\varepsilon\beta}^\alpha Y^\varepsilon}_{A6} - \underbrace{p_\alpha\left(\partial_\gamma Y^\varepsilon\right)S_{\beta\varepsilon}^\alpha X^\beta}_{A7} \\ p_\alpha((L_X S)_Y)_\gamma^\alpha &= \underbrace{p_\alpha X^\beta\partial_\beta S_{\varepsilon\gamma}^\alpha Y^\varepsilon}_{A1} + \underbrace{p_\alpha\partial_\varepsilon X^\beta S_{\beta\gamma}^\alpha Y^\varepsilon}_{A8} + \underbrace{p_\alpha\partial_\gamma X^\beta S_{\varepsilon\beta}^\alpha Y^\varepsilon}_{A6} - \underbrace{p_\alpha\partial_\beta X^\alpha S_{\varepsilon\gamma}^\beta Y^\varepsilon}_{A5}, \\ -p_\alpha((L_Y S)_X)_\gamma^\alpha &= -\underbrace{p_\alpha Y^\beta\partial_\beta S_{\varepsilon\gamma}^\alpha X^\varepsilon}_{A2} - \underbrace{p_\alpha\partial_\varepsilon Y^\beta S_{\beta\gamma}^\alpha X^\varepsilon}_{A9} - \underbrace{p_\alpha\partial_\gamma Y^\beta S_{\varepsilon\beta}^\alpha X^\varepsilon}_{A7} + \underbrace{p_\alpha\partial_\beta Y^\alpha S_{\varepsilon\gamma}^\beta X^\varepsilon}_{A4}, \\ p_\alpha\left(S_{[X,Y]}\right)_\gamma^\alpha &= p_\alpha S_{\beta\gamma}^\alpha(X^\varepsilon\partial_\varepsilon Y^\beta - Y^\varepsilon\partial_\varepsilon X^\beta) = \underbrace{p_\alpha S_{\beta\gamma}^\alpha X^\varepsilon\partial_\varepsilon Y^\beta}_{A9} - \underbrace{p_\alpha S_{\beta\gamma}^\alpha Y^\varepsilon\partial_\varepsilon X^\beta}_{A8}\end{aligned}$$

with respect to the coordinates $(x^c, x^\gamma, x^{\overline{\gamma}})$. Where the same equations are denoted by $A1, A2, \ldots, A9$. On the other hand, we know that ${}^{cc}(S(X,Y))$ and $\gamma((L_X S)_Y - (L_Y S)_X + S_{[X,Y]})$ have respectively, components

$$
{}^{cc}(S(X,Y)) = \begin{pmatrix} (S(X,Y))^c \\ (S(X,Y))^\gamma \\ -p_\varepsilon \partial_\gamma (S(X,Y))^\varepsilon \end{pmatrix},
$$

$$
\gamma((L_X S)_Y - (L_Y S)_X + S_{[X,Y]}) = \begin{pmatrix} 0 \\ 0 \\ p_\alpha ((L_X S)_Y - (L_Y S)_X + S_{[X,Y]})^\alpha_\gamma \end{pmatrix}
$$

with respect to the coordinates $(x^c, x^\gamma, x^{\overline{\gamma}})$ on $t^*(B_m)$. Thus, we have

$$
{}^{cc}\widetilde{S}({}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}) = {}^{cc}(S(X,Y)) - \gamma((L_X S)_Y - (L_Y S)_X + S_{[X,Y]})
$$

by the necessary simplifications made in equalities.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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