

New analytical solutions of the space fractional KdV equation in terms of Jacobi elliptic functions

Aysegul Dascioglu, Sevil Culha, Dilek Varol Bayram

Department of Mathematics, Faculty of Science and Arts, Pamukkale University, Denizli, Turkey

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Abstract: In this study, new families of analytical exact solutions of the space fractional Korteweg-de Vries (KdV) equation are presented. Here, the fractional derivative is considered in conformable sense. By utilizing the Jacobi elliptic function expansion method, the solutions are obtained in general form containing the hyperbolic, trigonometric, and rational functions. Also, the complex valued solutions are obtained and some solutions of this equation are demonstrated.

Keywords: Jacobi elliptic function, fractional differential equation, KdV equation.

1 Introduction

It is well known that the Korteweg-de Vries (KdV) equation arises as a model for one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media, as well as the evolution of weakly nonlinear ion acoustic waves in plasmas. So far, various methods are used for solving different types of fractional KdV equations. These methods can be listed as Adomian decomposition method [1,2], variational iteration method [2], homotopy perturbation method [3-7], differential transform method [8], modified direct algebraic method [9], (G'/G) -expansion method [10], improved (G'/G) -expansion method [10], homotopy perturbation transform method [11], function expansion method [12].

In this study, we consider the space fractional Korteweg-de Vries equation

$$u_t + uD_x^\alpha u + \beta D_x^\alpha D_x^\alpha D_x^\alpha u = 0, \quad 0 < \alpha \leq 1. \quad (1)$$

Here, D_x^α means conformable fractional derivative of function $u(x, t)$ with respect to x , and β is a constant. When $\alpha = 1$ the equation (1) becomes the known KdV equation of integer order.

For finding the analytical solutions of equation (1), we present an expansion method using the Jacobi elliptic functions. The main idea in this method is to find the solutions $u(\xi)$ in the form

$$u(\xi) = \sum_{j=0}^N a_j F^j(\xi) \quad (2)$$

where N , a_j ($j = 0, 1, 2, \dots, N$) are constants to be determined later and $F(\xi)$ is the solution of the Jacobi elliptic equation

$$(F')^2(\xi) = P + QF^2(\xi) + RF^4(\xi). \quad (3)$$

Here, $F' = \frac{dF}{d\xi}$, $\xi = \xi(t, x)$ and P , Q , R are constants. Eq.(3) has Jacobi elliptic function solutions for the different values of P , Q and R . The Jacobi elliptic functions corresponding to some values of P , Q and R can be found in [13].

2 Preliminaries

The basic Jacobian elliptic functions are

$$\operatorname{sn}\xi = \operatorname{sn}(\xi; m), \quad \operatorname{cn}\xi = \operatorname{cn}(\xi; m), \quad \operatorname{dn}\xi = \operatorname{dn}(\xi; m)$$

where $m(0 < m < 1)$ is the modulus of the elliptic function. When $m \rightarrow 0$ and $m \rightarrow 1$ Jacobi elliptic functions turn into trigonometric and hyperbolic functions [14]. There are different definitions of the fractional derivative such as Grunwald-Letnikov, Riemann-Liouville and Caputo [15-17]. However, there are some setbacks of these definitions [18]. To overcome these difficulties, Khalil et al. had given a new definition of fractional derivative named as conformable fractional derivative. This is the simplest definition to recognize the fractional derivative since it is the most familiar one to the definition of the usual derivative. Therefore, the KdV equation is considered in conformable sense. The definition and the properties of the conformable derivative are as follows:

Definition 1. (Conformable Fractional Derivative) [18] Let $f : [0, \infty) \rightarrow R$ be a function. The α -th order conformable fractional derivative of f is defined by

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0, \quad \alpha \in (0, 1).$$

If f is α -differentiable in some $(0, \alpha)$, $\alpha > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists, then we define $f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$.

Theorem 1. [18] Let $\alpha \in (0, 1]$ and suppose f, g are α -differentiable at point $t > 0$. Then, the following are satisfied:

- (i) $T_{\alpha}(cf + dg) = cT_{\alpha}(f) + dT_{\alpha}(g) \quad \forall c, d \in R$.
- (ii) $T_{\alpha}(t^p) = pt^{p-\alpha} \quad \forall p \in R$.
- (iii) $T_{\alpha}(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
- (iv) $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$.
- (v) $T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}$.
- (vi) If, in addition, f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}$.

Theorem 2. (Chain Rule) [18] Assume $f, g : (0, \infty) \rightarrow R$ be α -differentiable and for all t with $t \neq 0$ and $g(t) \neq 0$ we have

$$T_{\alpha}(h)(t) = T_{\alpha}(f)(g(t)) \cdot T_{\alpha}(g)(t) \cdot g(t)^{\alpha-1}.$$

If $t = 0$ we have

$$T_{\alpha}(h)(0) = \lim_{t \rightarrow 0} T_{\alpha}(f)(g(t)) \cdot T_{\alpha}(g)(t) \cdot g(t)^{\alpha-1}.$$

3 Solutions to the conformable space-fractional KdV equation

Let us consider the conformable space-fractional KdV equation (1). Changing of the variables in the equation as

$$u = u(\xi), \quad \xi = k \frac{x^{\alpha}}{\alpha} + lt \tag{4}$$

yields an ordinary differential equation (ODE) for $u(\xi)$

$$l \frac{du}{d\xi} + ku \frac{du}{d\xi} + \beta k^3 \frac{d^3 u}{d\xi^3} = 0 \tag{5}$$

where k and l are the arbitrary constants. Solving this nonlinear ordinary differential equation, the integer N in expression (2) can be determined by balancing the highest order linear term

$$O\left(u \frac{d^3 u}{d\xi^3}\right) = N + 3 \tag{6}$$

and the highest order nonlinear term

$$O\left(u \frac{du}{d\xi}\right) = 2N + 1 \tag{7}$$

in (5). As the result of this balancing, we obtain $N = 2$, thus the solution of Eq. (1) can be expressed as

$$u(\xi) = \sum_{j=0}^2 a_j F^j(\xi) = a_0 + a_1 F + a_2 F^2. \tag{8}$$

By differentiating Eq. (8) three times, we get

$$u'(\xi) = a_1 F' + 2a_2 F F', \tag{9}$$

$$u'''(\xi) = a_1 F''' + 6a_2 F' F''' + 2a_2 F F'' \tag{10}$$

and using Eq. (3), we have

$$F'' = 2RF^3 + QF, \tag{11}$$

$$F''' = 6RF^2 F' + QF'. \tag{12}$$

Substituting the equations (11) and (12) into Eq. (10) yields

$$u''' = a_1 QF' + 6a_1 RF^2 F' + 8a_2 QFF' + 24a_2 RF^3 F'. \tag{13}$$

Finally, substituting equations (9) and (13) into Eq. (5) and setting each coefficient of F to be zero, the following equations system arises

$$la_1 + ka_0 a_1 + \beta k^3 a_1 Q = 0$$

$$2la_2 + 2ka_0 a_2 + ka_1^2 + 8\beta k^3 a_2 Q = 0$$

$$3ka_1 a_2 + ka_1^2 + 6\beta k^3 a_1 R = 0$$

$$2ka_2^2 + 24\beta k^3 a_2 R = 0.$$

Solving this system, we get

$$a_0 = B + 4AQ, \quad a_1 = 0, \quad a_2 = 12AR \tag{14}$$

such that $A = -\beta k^2$, $B = -(l/k)$. Hence, the solution of the Eq. (5) is

$$u = B + 4AQ + 12ARF^2. \tag{15}$$

For the different values of the constants Q and R , we get the solutions of eq. (5) as in the following Table 1. By substituting ξ defined in (4), we obtain the exact solutions of the KdV equation (1).

Table 1: Solutions of Eq. (5) for the special values of P , Q and R .

	P	Q	R	Solutions
1	1	$-(1+m^2)$	m^2	$u_{1,1} = B - 4A(1+m^2) + 12Am^2\text{sn}^2\xi$
				$u_{1,2} = B - 4A(1+m^2) + 12Am^2\text{cd}^2\xi$
2	$1-m^2$	$2m^2-1$	$-m^2$	$u_2 = B + 4A(2m^2-1) - 12Am^2\text{cn}^2\xi$
3	m^2-1	$2-m^2$	-1	$u_3 = B + 4A(2-m^2) - 12A\text{dn}^2\xi$
4	m^2	$-(1+m^2)$	1	$u_{4,1} = B - 4A(1+m^2) + 12A\text{ns}^2\xi$
				$u_{4,2} = B - 4A(1+m^2) + 12A\text{dc}^2\xi$
5	$-m^2$	$2m^2-1$	$1-m^2$	$u_5 = B + 4A(2m^2-1) + 12A(1-m^2)\text{nc}^2\xi$
6	-1	$2-m^2$	m^2-1	$u_6 = B + 4A(2-m^2) + 12A(m^2-1)\text{nd}^2\xi$
7	1	$2-m^2$	$1-m^2$	$u_7 = B + 4A(2-m^2) + 12A(1-m^2)\text{sc}^2\xi$
8	1	$2m^2-1$	$-m^2+m^4$	$u_8 = B + 4A(2m^2-1) + 12A(-m^2+m^4)\text{sd}^2\xi$
9	$1-m^2$	$2-m^2$	1	$u_9 = B + 4A(2-m^2) + 12A\text{cs}^2\xi$
10	$-m^2+m^4$	$2m^2-1$	1	$u_{10} = B + 4A(2m^2-1) + 12A\text{ds}^2\xi$
11	$-\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$-\frac{1}{4}$	$u_{11} = B + 2A(1+m^2) - 3A(\text{mcn}\xi \mp \text{dn}\xi)^2$
12	$\frac{1}{4}$	$\frac{-2m^2+1}{2}$	$\frac{1}{4}$	$u_{12} = B - 2A(2m^2-1) + 3A(\text{ns}\xi \mp \text{cs}\xi)^2$
13	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$u_{13} = B + 2A(m^2+1) + 3A(1-m^2)(\text{nc}\xi \mp \text{sc}\xi)^2$
14	$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$u_{14} = B + 2A(m^2-2) + 3A(\text{ns}\xi \mp \text{ds}\xi)^2$
15	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$u_{15,1} = B + 2A(m^2-2) + 3Am^2(\text{sn}\xi \mp \text{icn}\xi)^2$
				$u_{15,2} = B + 2A(m^2-2) + 3Am^2 \frac{\text{dn}^2\xi}{1-m^2\text{sn}\xi \mp \text{cn}\xi}$
16	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$u_{16,1} = B + 2A(1-2m^2) + 3A(\text{mcn}\xi \mp \text{idn}\xi)^2$
				$u_{16,2} = B + 2A(m^2-2) + 3A\left(\frac{\text{sn}\xi}{1 \mp \text{cn}\xi}\right)^2$
17	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$u_{17} = B + 2A(m^2-2) + 3Am^2\left(\frac{\text{sn}\xi}{1 \mp \text{dn}\xi}\right)^2$
18	$\frac{m^2-1}{4}$	$\frac{1+m^2}{2}$	$\frac{m^2-1}{4}$	$u_{18} = B + 2A(1+m^2) + 3A(m^2-1)\left(\frac{\text{dn}\xi}{1 \mp \text{msn}\xi}\right)^2$
19	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$u_{19} = B + 2A(1+m^2) + 3A(1-m^2)\left(\frac{\text{cn}\xi}{1 \mp \text{sn}\xi}\right)^2$
20	$\frac{1}{4}$	$\frac{1+m^2}{2}$	$\frac{(1-m^2)^2}{4}$	$u_{20} = B + 2A(1+m^2) + 3A(1-m^2)^2\left(\frac{\text{sn}\xi}{\text{dn}\xi \mp \text{cn}\xi}\right)^2$
21	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$u_{21} = B + 2A(m^2-2) + 3Am^2 \frac{\text{cn}^2\xi}{1-m^2 \mp \text{dn}\xi}$

Considering the solutions in Table 1 and the behavior of the Jacobi elliptic functions for $m \rightarrow 0$ and $m \rightarrow 1$, the elementary function solutions can be given at Table 2.

Table 2: The behavior of the solutions in the Table 1 when $m \rightarrow 0$ and $m \rightarrow 1$.

	Solutions	$m \rightarrow 0$	$m \rightarrow 1$
1	$u_{1,1}$	$B - 4A$	$B - 8A + 12A \tanh^2 \xi$
	$u_{1,2}$	$B - 4A$	$B + 4A$
2	u_2	$B - 4A$	$B + 4A - 12A \operatorname{sech}^2 \xi$
3	u_3	$B - 4A$	$B + 4A - 12A \operatorname{sech}^2 \xi$
4	$u_{4,1}$	$B - 4A + 12A \operatorname{csc}^2 \xi$	$B - 8A + 12A \operatorname{coth}^2 \xi$
	$u_{4,2}$	$B - 4A + 12A \operatorname{sec}^2 \xi$	$B + 4A$
5	u_5	$B - 4A + 12A \operatorname{sec}^2 \xi$	$B + 4A$
6	u_6	$B - 4A$	$B + 4A$
7	u_7	$B + 8A + 12A \tan^2 \xi$	$B + 4A$
8	u_8	$B - 4A$	$B + 4A$
9	u_9	$B + 8A + 12A \cot^2 \xi$	$B + 4A + 12A \operatorname{csch}^2 \xi$
10	u_{10}	$B - 4A + 12A \operatorname{csc}^2 \xi$	$B + 4A + 12A \operatorname{csch}^2 \xi$
11	u_{11}	$B - A$	$B + 4A$
12	u_{12}	$B + 2A + 3A(\operatorname{csc} \xi \mp \cot \xi)^2$	$B - 2A + 3A(\operatorname{csch} \xi \mp \operatorname{coth} \xi)^2$
13	u_{13}	$B + 2A + 3A(\operatorname{sec} \xi \mp \tan \xi)^2$	$B + 4A,$ $B + 4A - 12A \operatorname{sech}^2 \xi$
14	u_{14}	$B - 4A,$	$B - 2A + 3A(\operatorname{coth} \xi \mp \operatorname{csch} \xi)^2$
		$B - 4A + 12A \operatorname{csc}^2 \xi$	
15	$u_{15,1}$	$B - 4A$	$B - 2A + 3A(\tanh \xi \mp i \operatorname{sech} \xi)^2$
	$u_{15,2}$	$B - 4A$	$B - 2A + 3A \frac{\operatorname{sech}^2 \xi}{1 - \tanh \xi \mp \operatorname{sech} \xi}$
16	$u_{16,1}$	$B - A$	$B - 2A + 3A((1 \mp i) \operatorname{sech} \xi)^2$
	$u_{16,2}$	$B - 4A + 3A \left(\frac{\sin \xi}{1 \mp \cos \xi} \right)^2$	$B - 2A + 3A \left(\frac{\sinh \xi}{\cosh \xi \mp 1} \right)^2$
17	u_{17}	$B - 4A$	$B - 2A + 3A \left(\frac{\sinh \xi}{\cosh \xi \mp 1} \right)^2$
18	u_{18}	$B - A$	$B + 4A$
19	u_{19}	$B + 2A + 3A \left(\frac{\cos \xi}{1 \mp \sin \xi} \right)^2$	$B + 4A$
20	u_{20}	$B + 2A + 3A \left(\frac{\sin \xi}{1 \mp \cos \xi} \right)^2$	$B + 4A$
21	u_{21}	$B - 4A$	$B - 2A \mp 3A \operatorname{sech} \xi$

4 Demonstrations of some solutions

In this section, we give 8 figures demonstrating two solutions from Table 1 and two solutions from Table 2 in both 2D and 3D plots. In all figures, the solutions are considered for $k = l = \beta = 1$. All graphics in figures are drawn by the aid of

Mathematica. Firstly, let us reconsider solution u_2 in Table 1,

$$u_2 = 3 - 8m^2 + 12m^2cn^2\xi$$

We demonstrate the above solution for $0 \leq \xi \leq 5$ when $0 \leq m \leq 1$ and $m = 0.5$ in Figure 1 and in Figure 2, respectively.

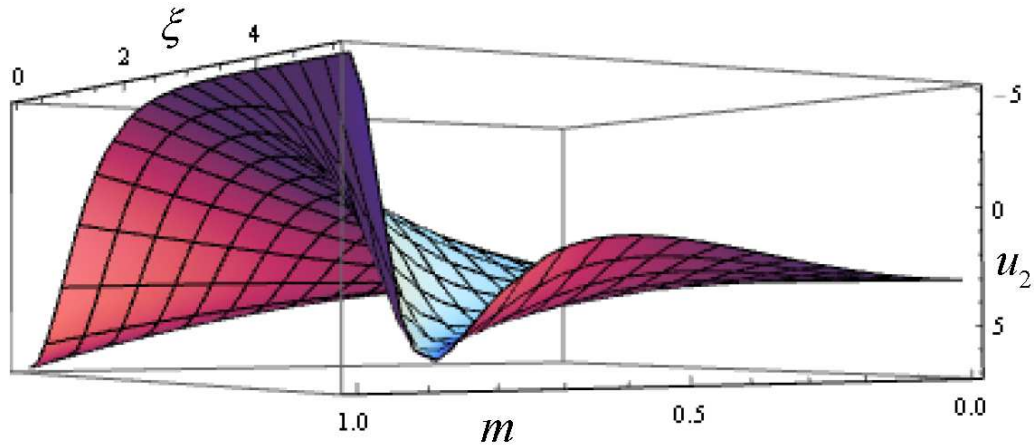


Fig. 1: 3D plot of the obtained solution $u_2(\xi)$ when $0 \leq m \leq 1$

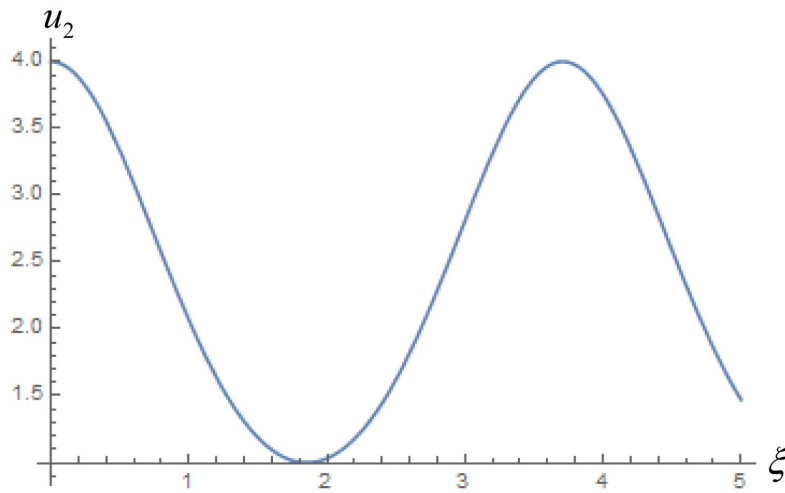


Fig. 2: 2D plot of the solution $u_2(\xi)$ when $m = 0.5$

When $m \rightarrow 0$, $u_2 = 3$ and when $m \rightarrow 1$, $u_2 = -5 + 12\text{sech}^2(2\sqrt{x} + t)$ for $\alpha = 0.5$. We demonstrate the second case for $0 \leq x \leq 4$ when $0 \leq t \leq 1$ and at $t = 1$ in Figure 3 and in Figure 4, respectively.

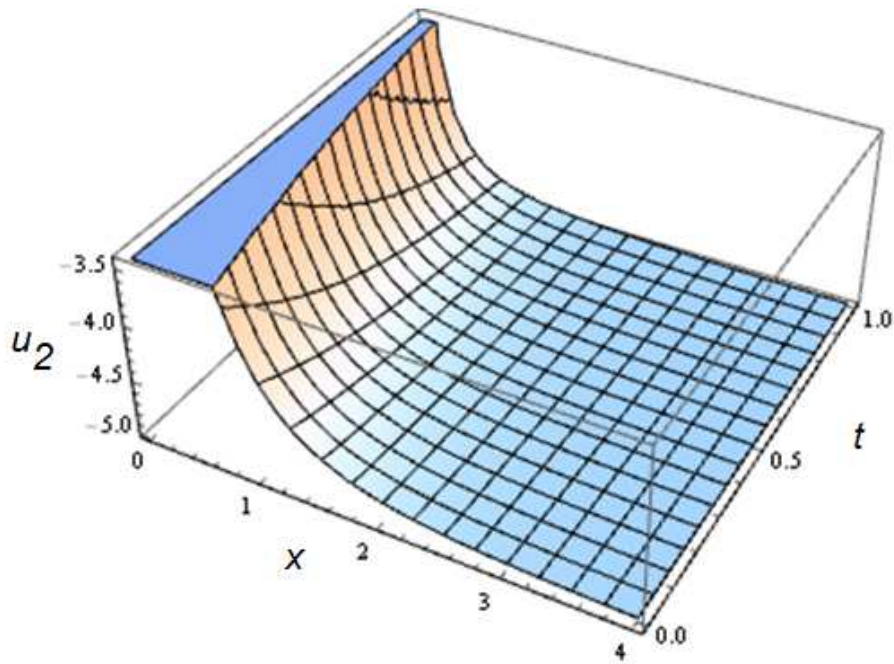


Fig. 3: 3D plot of the obtained solution $u_2(x,t)$ when $m \rightarrow 1$.

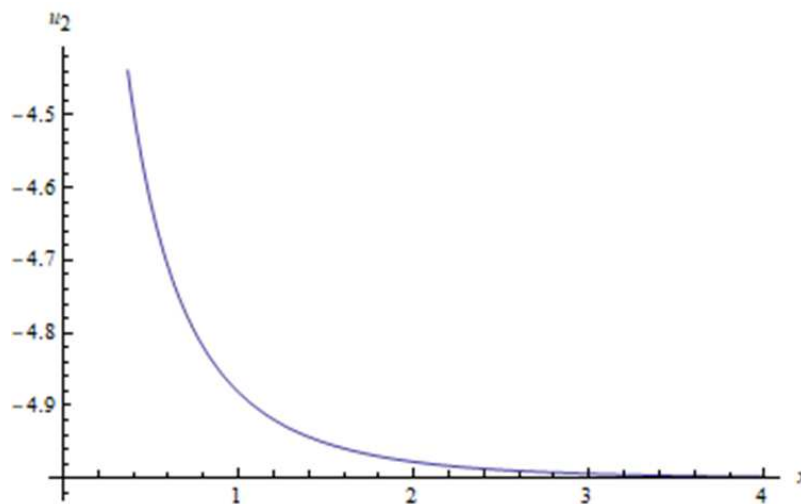


Fig. 4: 2D plot of the solution $u_2(x,1)$ when $m \rightarrow 1$.

Secondly, we analyze the solution $u_{4,2}$ in Table 1

$$u_{4,2} = 3 + 4m^2 - 12dc^2\xi.$$

We demonstrate the considered solution for $1 \leq \xi \leq 4$ when $0 \leq m \leq 1$ and $m = 0.5$ in Figure 5 and in Figure 6, respectively.

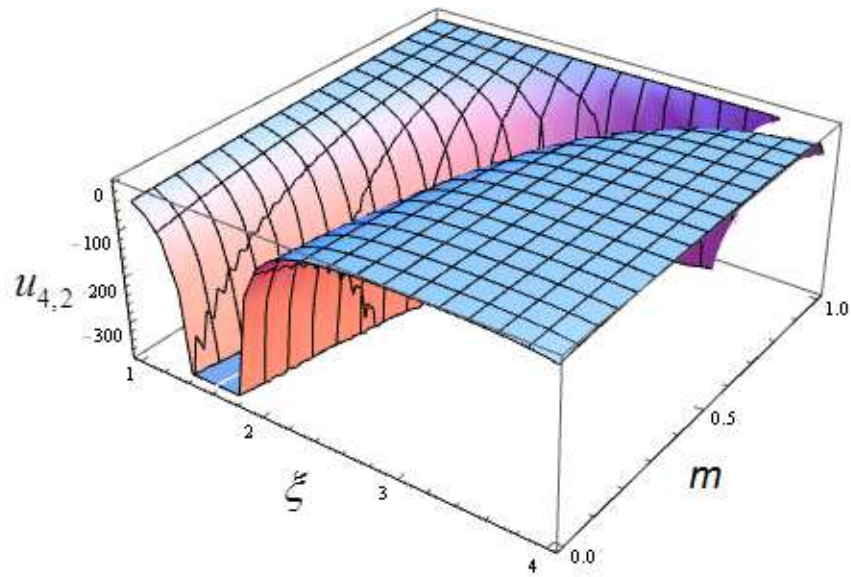


Fig. 5: 3D plot of the obtained solution $u_{4,2}(\xi)$ when $0 \leq m \leq 1$.

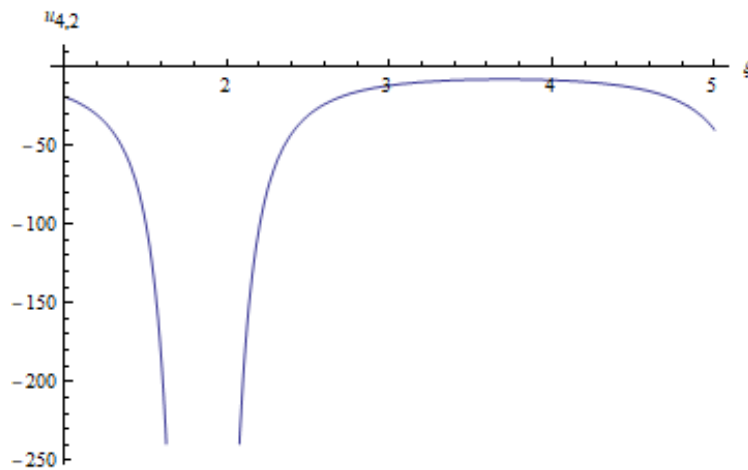


Fig. 6: 2D plot of the obtained solution $u_{4,2}(\xi)$ when $m = 0.5$.

When $m \rightarrow 1$, $u_{4,2} = -5$ and when $m \rightarrow 0$, $u_{4,2} = 3 - 12\sec^2(2\sqrt{x} + t)$ for $\alpha = 0.5$. We demonstrate the second case for $1 \leq x \leq 4$ when $0 \leq t \leq 1$ and at $t = 0.5$ in Figure 7 and in Figure 8, respectively.

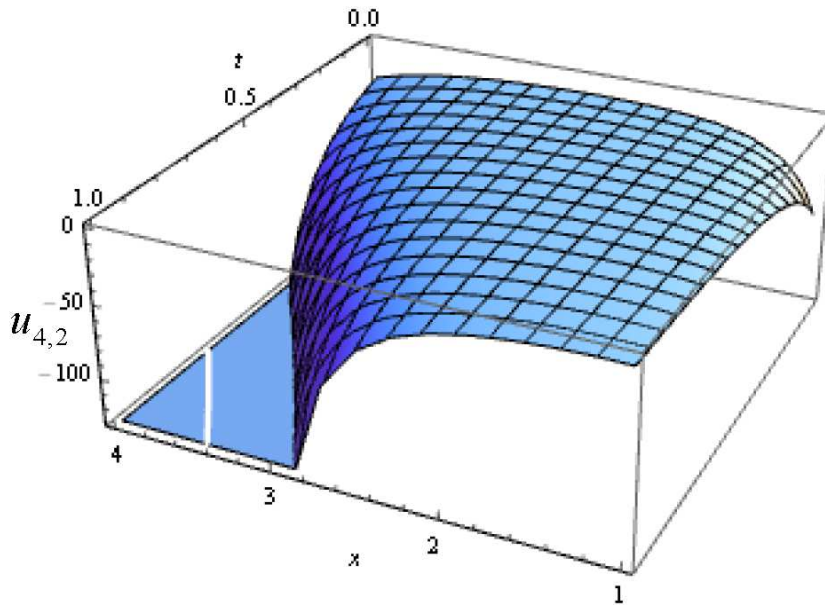


Fig. 7: 3D plot of the obtained solution $u_{4,2}(x,t)$ when $m \rightarrow 0$.

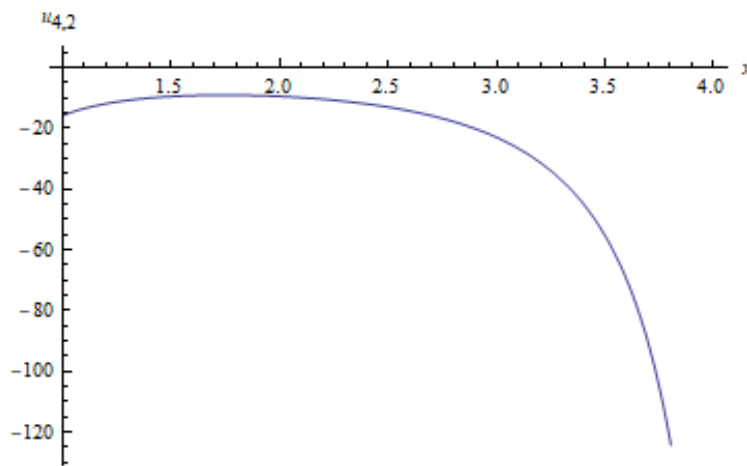


Fig. 8: 2D plot of the obtained solution $u_{4,2}(x, 0.5)$ when $m \rightarrow 0$.

5 Conclusion

In this study, the Jacobi elliptic function expansion method is used to obtain the exact solutions of the conformable space-fractional KdV equation. This method obtains the results directly, quickly and needs simple algorithms in programming. By this powerful method, the solutions are found in hyperbolic, trigonometric, and rational function form involving some parameters; thus, by this single method we cover the solutions of several different methods at the same time. Additionally, by using conformable fractional derivative definition, fractional wave equations can be solved easily rather than the other definitions.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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