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# On Monotone Generalized Quasi contraction mappings in modular metric spaces with a graph

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**Abstract:** One of the most popular result in Mathematics is the Banach Contraction principle in a complete metric space. Due to its wide range of applications, many mathematicians generalized the Banach contraction principle in different directions. One of the generalizations is due to Jachymski [Proc.Am. Math. Soc. 1(136),1359-1373], in which he considered a complete metric space with a graph structure. Alfraidan [Fixed Point Theory and Applications (2015) 2015:93. doi 10.1186/s13663-015-0341-2] generalized the work of Jachymski for quasi-contraction mappings in both metric and modular metric spaces with a graph structure. Modular metric spaces are more general than the usual metric spaces. In this paper, we extend Alfraidan's result to a generalized quasi contraction mappings.

**Keywords:** Modular metric space, graph structure, generalized quasi-contraction mappings.

## 1 Introduction

The abstract definition of a metric space was introduced by Frechet [13] in 1906, and was seen as a nonlinear version of a vector space endowed with a norm. Nakano [18], introduced modular function spaces in 1950 and Musielak and Orlicz [16] redefined the modular function space in such away that, a modular function space is seen as a vector space endowed with a modular function. It was proved in [15], that this space is a complete normed linear space with norm called Luxemburg norm. Therefore, it is natural to consider a nonlinear version of function modular spaces. Thus, the modular metric spaces considered in [8] are nonlinear versions of modular function spaces and are more general than metric spaces.

Fixed point theorems for monotone single valued mappings in a metric space endowed with partial orderings are first considered by Ran and Reurings [20] in 2004, and have been widely investigated. The theorem in [20] is a hybrid of the two independent fundamental theorems: Banach contraction principle [5] and Tarski's fixed point theorem [22].

A point $x \in X$ is called a fixed point of a self$-$mapping $T$ on $X$ if $x = T(x)$, for single valued mappings (and $x \in T(x)$ for set-valued mappings). The fixed point set of a mapping $T$ will be denoted by $Fix(T)$.

In 1922, Banach [5], proved the existence of a unique fixed point for contraction self-mappings in a complete metric space. Banach contraction principle [5] is a simple and powerful result with a wide range of applications, including iterative methods for solving linear and nonlinear differential, integral, and difference equations. Due to its applications in mathematics and other related disciplines, Banach contraction principle has been generalized and extended in many


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directions. One of the most influenced generalization of Banach's theorem is traced to Nadler [17]. In 1969, Nadler [17] extended Banach's contraction theorem to multivalued contraction mappings. A number of extensions and generalizations of Nadler's theorem were obtained by different authors. In 1974, *Ciriċ* [11], extended Nadler's theorem to quasi contraction mappings. Independent of Banach's result, Tarski and Knaster [22] in 1955, proved the existence of fixed points of single valued self-mappings in partially ordered sets. Ran and Reuring's [20], proved the existence of fixed points of single valued mappings in partially ordered metric space. Jachymski [14], investigated a new approach in metric fixed point theory by replacing an order structure with a graph structure on metric spaces. In this way, the results proved in ordered metric spaces are generalized (see for detail [14] and the reference therein). Beg et al.[4], extended Jachymski's theorem to multivalued mappings. Chifu and Petrusel [7], extended Jachymiski's theorem to a generalized contraction mappings and in [6], to Ciric $\delta-$contraction mappings. Chistyakov [10] proved the existence of fixed point theorems for $\omega-$contraction mappings in the setting of modular metric spaces. Abadoo and Khamsi [1] extended Chistyakov's fixed point theorem to a multivalued mappings, Cho et al. [23], to quasi contraction mappings and Rahimpoor et al.[19] to generalized $\omega-$quasi contractions in modular metric spaces. Alfraidan [2] made an extension of Jachymski's result to a more general class of spaces, modular metric spaces. In [3], Alfraidan extended the $G-\omega$ contraction mappings he considered in [2] to a more general class of mappings, Ciric$-$quasi contraction mappings in modular metric spaces.

It is our purpose in this paper to extend the work of Alfraidan [3] to a more general class of *generalized quasi contraction* mappings.

## 2 Preliminaries

Throughout this paper, $\mathbb{Z}^+$ and $\mathbb{N}$ will denote the set of positive integers and nonnegative integers, respectively. The terminology of graph theory instead of partial ordering gives a wider picture and yields interesting generalization of the Banach contraction principle. We give the basic notations in graph theory which will be used throughout (the detail will be found in [12]). A directed graph(digraph) is a pair: $G=(V,E)$ where $V$ is a nonempty set called vertices and $E=\{(u,v):u,v\in V\}$ is set of order pairs called edges.

Let $(X,d)$ be a metric space and $\Delta$ be the diagonal of $X\times X$. Let $G$ be a digraph such that the set $V(G)$ of its vertices coincide with $X$ and $\Delta\subseteq E(G)$. Assume that $G$ has no parallel Edges. We will suppose that $G$ can be identified with the pair $(V(G),E(G))$. If $x$ and $y$ are vertices of $G$, then a path in $G$ from $x$ to $y$ of length $k\in\mathbb{N}$ is a finite sequence $\{x_n\}_{n=0}^{n=k}$ of vertices such that $x_0=x,\ x_k=y$ and $(x_{i-1},x_i)\in E(G)$, for $i=1,2,3,...,k$.

By $G^{-1}$ we denote the conversion of $G$, i.e., the graph obtained from $G$ by reversing the direction of edges. Thus, we have $E(G^{-1})=\{(y,x)|(x,y)\in E(G)\}$. A digraph is called an oriented graph, if whenever $(u,v)\in E(G)$, then $(v,u)\notin E(G)$. Let us denote by $\tilde{G}$, the undirected graph obtained from $G$ by ignoring the direction of edges. Actually it is more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention, $E(\tilde{G})=E(G)\bigcup E(G^{-1})$. We say that $G$ is connected if there is a path between any two vertices and we say it is weakly connected if $\tilde{G}$ is connected. If $G$ is such that $E(G)$ is symmetric and $x$ is a vertex in $G$, then the subgraph $G_x$ consisting of all edges and vertices which are contained in some path beginning at $x$ is called the component of $G$ containing $x$. In this case $V(G_x)=[x]_G$, where $[x]_G$ is the equivalence class of the following relation $R$ defined on $V(G)$ by the rule: $yRz$ if there is a (directed) path in $G$ from $y$ to $z$. Clearly $G_x$ is connected.

Throughout this section, $X$ denotes a metric space $(X,d)$ and by $G$ we mean the graph $G$ with vertex $X$ and edge; $E(G)\subseteq X\times X$.

**Definition 1.** *A mapping $T : X \to X$ is called a $G-$monotone if $T$ preserves edges of $G$, i.e.,*

$$\forall x,y \in X((x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G)).$$

**Definition 2.** *A mapping $T : X \to X$ is called a Banach $G-$contraction or simply a $G-$contraction if $T$ preserves edges of $G$, i.e.,*

$$\forall x,y \in X((x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G))$$

*and $T$ decreases weight of edges of $G$ in the following way:*

$$there\ exists\ \alpha \in [0,1)\forall x,y \in X, (x,y) \in E(G) \Rightarrow d(T(x),T(y)) \leq \alpha d(x,y).$$

**Definition 3.** [11] *Let $C$ be a nonempty subset of a metric space $X$. A mapping $T : C \to C$ is called quasi contraction if there exists $k < 1$ such that for any $x,y \in C$, we have*

$$d(T(x),T(y)) \leq kmax\{d(x,y),d(x,T(x)),d(y,T(y)),d(x,T(y)),d(T(x),y)\}.$$

**Definition 4.** [3] *Let $C$ be a nonempty subset of $X$. A mapping $T : C \to C$ is called $G-$monotone quasi contraction if $T$ is $G-$monotone and there exists $k < 1$ such that for any $x,y \in C, (x,y) \in E(G)$, we have*

$$d(T(x),T(y)) \leq kmax\{d(x,y),d(x,T(x)),d(y,T(y)),d(x,T(y)),d(T(x),y)\}.$$

**Definition 5.** *A modular on a real linear space $X$ is a functional $\rho : X \to [0,\infty]$, satisfying the conditions:*

(i) $\rho(0) = 0$.
(ii) *If $x \in X$ and $\rho(\alpha x) = 0$ for $\alpha > 0$ then $x = 0$.*
(iii) *$\rho(-x) = \rho(x)$, for all $x \in X$.*
(iv) *$\rho(\alpha x+\beta y) \leq \rho(x)+\rho(y)$ for all $\alpha,\beta \geq 0$ with $\alpha+\beta = 1$ and $x,y \in X$. If the inequality*
(v) *$\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ holds in (iv), the modular $\rho$ is called convex.*

Due to the limitation of linear modular spaces with additional algebraic structures to solve certain problems from set valued analysis, such as supper position operators in [9], the modular theory on an arbitrary set was proposed.
Let $X$ be a nonempty set. Throughout, for a function $\omega : (0,\infty)\times X \times X \to [0,\infty]$, we will write $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $\lambda > 0$ and $x,y \in X$, so that

$$\omega = \{\omega_\lambda\}_{\lambda>0} \text{ with } \omega_\lambda : X \times X \to [0,\infty].$$

**Definition 6.** [8] *A function $\omega : (0,\infty)\times X \times X \to [0,\infty]$ is said to be a modular metric on $X$ if it satisfies the following axioms:*

(i) *Given $x,y \in X$, $x = y$ if and only if $\omega_\lambda(x,y) = 0$, for all $\lambda > 0$.*
(ii) *$\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$ and $x,y \in X$.*
(iii) *$\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z)+\omega_\mu(z,y)$, for all $\lambda,\mu > 0$, and $x,y,z \in X$. If instead of $(i)$ we have only the condition:*
(iv) *$\omega_\lambda(x,x) = 0$, for all $\lambda > 0$, $x \in X$, then $\omega$ is said to be pseudomodular (metric) on $X$.*

A modular metric $\omega$ on $X$ is said to be regular(or strict) if the following weaker version of $(i)$ is satisfied.

$$x = y\ if\ and\ only\ if\ \omega_\lambda(x,y) = 0,\ for\ some\ \lambda > 0.$$

$\omega$ is said to be convex if for all $\lambda, \mu > 0$ and $x, y, z \in X$, it satisfies the inequality:

$$\omega_{\lambda+\mu} \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

*Remark.* For a pseudomodular metric $\omega$ on a set $X$, and any $x,y \in X$, the function $\lambda \mapsto \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$, and so the limit from the right $\omega_{\lambda+0}(x,y)$ and the limit from the left $\omega_{\lambda-0}(x,y)$ exists in $[0,\infty]$ and satisfy the inequalities

$$\omega_{\lambda+0}(x,y) \leq \omega_\lambda(x,y) \leq \omega_{\lambda-0}(x,y).$$

Let $\omega$ be a (pseudo) modular metric on a set $X$. The binary relation $\overset{\omega}{\sim}$ on $X$ defined for $x,y \in X$ by $x \overset{\omega}{\sim} y$ if and only if $\lim_{\lambda\to\infty}\omega_\lambda(x,y) = 0$ is an equivalence relation on $X$. Let $X/\overset{\omega}{\sim}$ be the quotient set of $X$ under $\overset{\omega}{\sim}$. Given $x \in X$, the equivalence class of $x$ in $X/\overset{\omega}{\sim}$ is given by:

$$X_\omega(x) \equiv \widetilde{x} = \{y \in X : y \overset{\omega}{\sim} x\}.$$

We are interested in the equivalence classes $X_\omega(x)$ in $X/\overset{\omega}{\sim}$. According to Chistyakov [8], we fix an element $x_0 \in X$ arbitrarily and define the modular set:

$$X_\omega = \widetilde{x_0} = X_\omega(x_0).$$

**Definition 7.** [8] *Let $\omega$ be a pseudomodular on $X$. Fix $x_0 \in X$. The two sets:*

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \ as\ \lambda \to \infty\}$$

*and*

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0\ such\ that\ \omega_\lambda(x,x_0) < \infty\},$$

*are said to be Modular spaces.*

Here we see that $X_\omega \subseteq X^*_\omega$ and in [8], a counter example was given to show the inclusion is proper.

**Example 1.** Consider a metric space $(X,d)$ and define a functional

$$\omega : (0,\infty)\times X \times X \to [0,\infty) \qquad \text{by} \qquad \omega_\lambda(x,y) = \frac{d(x,y)}{\lambda},$$

then $\omega$ is a convex modular on $X$. For arbitrary fixed $x_0 \in X$, we see that $\lim_{\lambda\to\infty}\omega_\lambda(x,x_0) = 0$ for each $x \in X$, showing that $X_\omega = X$.

**Theorem 1.** [8] *If $\omega$ is a metric modular on $X$, then the modular set $X_\omega$ is a metric space with metric given by*

$$d_\omega(x,y) = inf\{\lambda > 0 : \omega_\lambda(x,y) \leq \lambda\},\ for\ any\ x,y \in X_\omega.$$

*Remark.* Let $\omega$ be a convex modular on a set $X$. Given $x,y \in X$,

(i) $\widehat{\omega}_\lambda(x,y) = \lambda\omega_\lambda(x,y)$, is a modular metric on $X$.
(ii) the functions $\lambda \mapsto \omega_\lambda(x,y)$ and $\lambda \mapsto \widehat{\omega}_\lambda(x,y)$ are nonincreasing on $(0,\infty)$.

If $\omega$ is convex modular on $X$ then the two modular spaces coincide, i.e., $X_\omega = X^*_\omega$, and this common set can be endowed with the metric $d^*_\omega$ given by

$$d^*_\omega(x,y) = inf\{\lambda > 0 : \omega_\lambda(x,y) \leq 1\}\ for\ any\ x,y \in X_\omega.$$

**Theorem 2.** [8] *Let $\omega$ be a convex modular on a set $X$. Then given $x,y \in X_\omega$ for the metrics $d^*_\omega$ and $d_\omega$, we have*

(a) *conditions $d_\omega < 1$ and $d^*_\omega < 1$ are equivalent and if at least one of them holds, then*

$$d^*_\omega(x,y) \le d_\omega(x,y) \le \sqrt{d^*_\omega(x,y)}.$$

(b) *conditions $d_\omega \ge 1$ and $d^*_\omega \ge 1$ are equivalent if at least one of them holds, then*

$$\sqrt{d^*_\omega(x,y)} \le d_\omega(x,y) \le d^*_\omega(x,y).$$

Let $\omega$ be a modular metric on a linear space $X$ and satisfies the following two conditions:

(i) $\omega_\lambda(\mu x, 0) = \omega_{\lambda/\mu}(x,0)$ for all $\lambda, \mu > 0$, and $x \in X$.

(ii) $\omega_\lambda(x+z, y+z) = \omega_\lambda(x,y)$ for all $x,y,z \in X$, and if we set:

$\rho(x) = \omega_1(x,0)$ and $\omega_\lambda(x,y) = \omega_1(\frac{x-y}{\lambda}, 0) = \rho(\frac{x-y}{\lambda})$, for all $\lambda > 0$ and $x,y \in X$, then

(i) $X_\rho = X_\omega$ is a linear subspace of $X$ and the functional:

$$|x|_\rho = d_\omega(x,0)\ x \in X \textit{ is an } F-\textit{norm on } X_\rho.$$

(ii) If $\omega$ is convex, $X^*_\rho = X^*_\omega = X_\rho$ is a linear subspace of $X$ and the functional:

$$|x|^*_\rho = d^*_\omega(x,0),\ x \in X^*_\omega \textit{ is a norm on } X_\rho.$$

*Remark.* Every metric space $(X,d)$ is a modular metric space. Indeed, if we define a functional $\omega : (0,\infty) \times X \times X \to [0,\infty)$ by $\omega_\lambda(x,y) = \frac{d(x,y)}{\lambda}$, then, we see that $\omega$ is a convex modular on $X$. So, $d(x,y) = \lambda\omega_\lambda(x,y) = \widehat{\omega}_\lambda(x,y)$ is a modular on $X$. On the other hand, a modular metric $\omega$ on a set $X$ is a metric if $\omega$ assumes only finite values and is independent of the parameter $\lambda$.

**Definition 8.** [1] *Let $(X,\omega)$ be a modular metric space.*

(1) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X_\omega$ is said to be $\omega-$convergent to $x \in X$ if and only if $\lim_{n\to\infty} \omega_1(x_n,x) = 0$. $x$ will be called the $\omega-$limit of $\{x_n\}$.*

(2) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X_\omega$ is said to be $\omega-$Cauchy if*

$$\lim_{n,m\to\infty} \omega_1(x_n,x_m) = 0.$$

(3) *A subset $M$ of $X_\omega$ is said to be $\omega-$closed, if the $\omega-$limit of $\omega-$convergent sequence of $M$ always belongs to $M$.*

(4) *A subset $M$ of $X_\omega$ is said to be $\omega-$complete, if any $\omega-$Cauchy sequence in $M$ is $\omega-$convergent sequence and its $\omega-$limit is in $M$.*

(5) *A subset $M$ of $X_\omega$ is said to be $\omega-$bounded if we have*

$$\delta_\omega(M) = sup\{\omega_1(x,y) : x,y \in M\} < \infty.$$

(6) *A subset $M$ of $X_\omega$ is said to be $\omega-$compact if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $M$, there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_n\}$ and $x \in M$ such that $\omega_1(x_{k_n},x) \to 0$ as $n \to \infty$.*

(7) *$\omega$ is said to satisfy the Fatou property if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X_\omega$, $\omega-$convergent to $x$, we have*

$$\omega_1(x,y) \le \liminf_{n\to\infty} \omega_1(x_n,y),\ \ for\ any\ y \in X_\omega.$$

In general, if $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$ for some $\lambda>0$, then we may not have $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$ for all $\lambda>0$.

**Definition 9.** [1] *Let $(X,\omega)$ be a modular metric space. $\omega$ is said to satisfy the $\Delta_2-$condition if and only if $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$ for some $\lambda>0$ implies that $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$ for all $\lambda>0$.*

**Theorem 3.** [8] *Let $\omega$ be a modular on a set $X$. Given a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X_\omega$ and $x\in X_\omega$, we have $d_\omega(x_n,x)\to 0$ as $n\to\infty$ if and only if $\omega_\lambda(x_n,x)\to 0$ as $n\to\infty$, for all $\lambda>0$.*

**Corollary 1.** *Let $\omega$ be a modular on a set $X$. Then, $\omega-$convergence and $d_\omega-$convergence are equivalent if and only if the modular $\omega$ satisfies the $\Delta_2-$condition.*

**Theorem 4.** [8] *Let $\omega$ be a convex modular on a set $X$. Given a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X^*_\omega$ and $x\in X^*_\omega$, we have $d^*_\omega(x_n,x)\to 0$ as $n\to\infty$ if and only if $\omega_\lambda(x_n,x)\to 0$ as $n\to\infty$, for all $\lambda>0$.*

Note that, if the modular $\omega$ is convex, then $d^*_\omega=d_\omega$ which implies
$\lim_{n\to\infty}d^*_\omega(x_n,x)=0$ *if and only if* $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0,\ \forall\lambda>0$ and for any sequence $\{x_n\}_{n\in N}\subseteq X_\omega$, and $x\in X_\omega$.

**Definition 10.** *Let $f:X\to(-\infty,\infty)$ be a function on a topological space $X$. Then $f$ is upper semi-continuous at the point $x\in X$ if and only if*

$$x_n\to x\Rightarrow\limsup_{n\to\infty}f(x_n)\le f(x).$$

Let $\Phi$ be the class of all functions $\varphi:[0,\infty)\to[0,\infty)$ such that:

(i) $\varphi$ is nondecreasing and upper semi-continuous.
(ii) $\varphi(t)<t,\ \forall t>0$. In the literature, $\Phi$ is called the class of comparison functions. If in addition $\varphi$ satisfies the condition:
(iii) ] $\sum_{n=1}^{\infty}\phi^n(t)<\infty$, then $\Phi$ is called the class of strong comparison functions .

**Lemma 1.** [21] *If $\varphi\in\Phi$, then $\varphi(0)=0$, and $\lim_{n\to\infty}\varphi^n(t)=0$ for each $t>0$, where $\varphi^n$ is the $n-$times composition of the function $\varphi$ with itself.*

As a consequence of property $(ii)$, we can see that the sequence $\{\varphi^n(t)\}$ is a nonincreasing sequence, for any $t>0$.

## 3 Generalized quasi contractions in metric spaces with a graph

Next, we give the definition of $G-$monotone generalized quasi contraction mappings in the setting of metric spaces.

**Definition 11.** *Let $C$ be a nonempty subset of a metric space $X$. $T:C\to C$ is called:*

(1) *$G-$monotone if $T$ is edge preserving. i.e., $(T(x),T(y))\in E(G)$, whenever $(x,y)\in E(G)$, for any $x,y\in C$.*
(2) *$G-$monotone generalized quasi contraction if $T$ is $G-$monotone and there exists a $\varphi\in\Phi$ such that for any $x,y\in C$ with $(x,y)\in E(G)$, we have*

$$d(T(x),T(y))\le max\{\varphi(d(x,y)),\varphi(d(x,T(x))),\varphi(d(y,T(y))),\varphi(d(x,T(y))),\varphi(d(T(x),y))\}.$$

For $x\in C$, we define the orbit $O(x)=\{x,T(x),T^2(x),\cdots\}$ and its diameter by $\delta(x)=sup\{d(T^n(x),T^m(x)):n,m\in\mathbb{N}\}$. We have the following technical lemma to prove one of our main results.

**Lemma 2.** *Let $(X,d)$ be a metric space and $G$ be a reflexive and transitive digraph on $X$. Let $C$ be a nonempty subset of $X$ and $T:C\to C$ be a $G-$monotone generalized quasi contraction mapping. Let $x\in C$ be such that $(x,T(x))\in E(G)$ and $\delta(x)<\infty$, then for any $n\in\mathbb{N}$, we have*

$$\delta(T^n(x))\le\varphi^n(\delta(x))$$

*where $\varphi \in \Phi$ is the comparison function associated with the $G-$monotone generalized quasi contraction definition of $T$. Moreover, we have $d(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta(x))$ for each $n, m \in \mathbb{N}$.*

*Proof.* Since $T$ is $G-$monotone and $(x, T(x)) \in E(G)$, we have $(T^n(x), T^{n+1}(x)) \in E(G)$ for each $n \in \mathbb{N}$. By the transitivity of $G$, for each $n \in \mathbb{N}$, we also have

$$(T^n(x), T^{n+m}(x)) \in E(G) \text{ for any } m \in \mathbb{Z}^+. \tag{1}$$

As $G$ is reflexive, $(x, x) \in E(G)$ and using the $G-$monotonicity of $T$, we get

$$(T^n(x), T^n(x)) \in E(G) \text{ for any } n \in \mathbb{N} \tag{2}$$

and hence (1) holds true for $m = 0$. Thus, we obtain

$$(T^n(x), T^{n+m}(x)) \in E(G) \text{ for any } n, m \in \mathbb{N}. \tag{3}$$

First, we show that

$$\delta(T^n(x)) \leq \varphi^n(\delta(x)) \tag{4}$$

for each $n \in \mathbb{N}$. For $n = 0$, we have $\delta(x) = \varphi^0(\delta(x))$ and equality holds in this case. For $n = 1$, from (3) and using the monotonicity of $\varphi$, we have

$$\begin{aligned} d(T(x), T^{1+m}(x)) &\leq max\{\varphi(d(x, T^m(x))), \varphi(d(x, T(x))), \varphi(d(T^m(x), T^{1+m}(x))), \varphi(d(x, T^{1+m}(x))), \varphi(d(T(x), T^m(x)))\} \\ &\leq \varphi(\delta(x)), \end{aligned} \tag{5}$$

for each $m \in \mathbb{N}$. This shows that

$$\delta(T(x)) \leq \varphi(\delta(x)), \tag{6}$$

and from (6) and the monotonicity of $\varphi$, we get

$$\varphi(\delta(T(x))) \leq \varphi^2(\delta(x)). \tag{7}$$

In addition, inequalities (6), (7) and the monotonicity of $\varphi$, give

$$\delta(T^2(x)) = \delta(T(T(x))) \leq \varphi(\delta(T(x))) \leq \varphi^2(\delta(x)).$$

Whence, by induction we conclude that

$$\delta(T^n(x)) = \delta(T(T^{n-1}(x))) \leq \varphi(\delta(T^{n-1}(x))) \vdots \leq \varphi^n(\delta(x))$$

for each $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$,

$$\delta(T^n(x)) \leq \varphi^n(\delta(x)). \tag{8}$$

On the other hand, by using (3) and the definition of $\delta$, we obtain

$$d(T^n(x),T^{n+m}(x)) = d(T^n(x),T^n(T^m)(x)) \le \delta(T^n(x)) \tag{9}$$

for each $n,m \in \mathbb{N}$. From (8) and (9), we conclude that $d(T^n(x),T^{n+m}(x)) \le \varphi^n(\delta(x))$ for each $n,m \in \mathbb{N}$.

**Theorem 5.** *Let $(X,d)$ be a complete metric space and $G$ be a reflexive, transitive digraph defined on $X$ such that the triple $(X,d,G)$ has Property $(A)$, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $X$, if $x_n \to x$ and $(x_n,x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n,x) \in E(G)$ for each $n \in \mathbb{N}$. Let $C$ be a nonempty closed subset of $X$ and $T : C \to C$ be a $G-$monotone generalized quasi contraction mapping with $\varphi \in \Phi$, the associated comparison function. For $x \in C$ with $(x,T(x)) \in E(G)$ and $\delta(x) < \infty$, we have the following:*

(a) *There exists $p \in Fix(T)$ such that $\{T^n(x)\}$ converges to $p$. Moreover, we have*

$$(x,p) \in E(G) \text{ and } d(T^n(x),p) \le \varphi^n(\delta(x)), \ for\ each\ n \in \mathbb{N}.$$

(b) *If $u$ is any fixed point of $T$ such that $(x,u) \in E(G)$, then $u = p$.*

*Proof.* First we show $(a)$. By Lemma 2, we see that $\{T^n(x)\}$ is a Cauchy sequence in $C$. Since $X$ is a complete metric space and $C$ is closed subset of $X$, there exists $p \in C$ such that $\{T^n(x)\}$ converges to $p$. From (9), we have

$$d(T^n(x),T^{n+m}(x)) \le \varphi^n(\delta(x)) \tag{10}$$

for any $n,m \in \mathbb{N}$. Thus, from (10) (by letting $m \to \infty$), we get that $d(T^n(x),p) \le \varphi^n(\delta(x)),\ for\ n \in \mathbb{N}$. Moreover, since $T$ is $G-$monotone and $(x,T(x)) \in E(G)$, we have $(T^n(x),T^{n+1}(x)) \in E(G)$ for each $n \in \mathbb{N}$ and using Property $(A)$, we conclude that $(T^n(x),p) \in E(G)$ for each $n \in \mathbb{N}$. In particular, $(x,p) \in E(G)$. It remains to show that $p$ is a fixed point of $T$. From $(T^n(x),p) \in E(G)$ the $G-$monotonicity of $T$, we have $(T^{n+1}(x),T(p)) \in E(G)$ for each $n \in \mathbb{N}$. As $(T^n(x),T^{n+1}(x)) \in E(G)$ and $G$ is transitive, we obtain that

$$(T^n(x),T(p)) \in E(G) \text{ for each } n \in \mathbb{N}. \tag{11}$$

Thus, using (11) and the hypothesis that $T$ is a $G-$monotone generalized quasi contraction, we have a comparison function $\varphi$ satisfying

$$d(T^n(x),T(p)) \le \max\{\varphi(d(T^{n-1}(x),p)),\varphi(d(T^{n-1}(x),T^n(x))),\varphi(d(T^{n-1}(x),T(p))),\varphi(d(T^n(x),p)),\varphi(d(p,T(p)))\} \tag{12}$$

for each $n \in \mathbb{Z}^+$. Letting $n \to \infty$ in (12) and using the upper semi-continuity of $\varphi$, we get $d(p,T(p)) \le \varphi(d(p,T(p)))$ which implies that $d(p,T(p)) = 0$ and hence $p = T(p)$. Next we show $(b)$. Let $u \in C$ be any fixed point of $T$ such that $(x,u) \in E(G)$. Then for each $n \in \mathbb{N}$, as $T$ is $G-$monotone, we have $(T^n(x),u) \in E(G)$. Therefore,

$$d(T^n(x),u) \le \{\varphi(d(T^{n-1}(x),u)),\varphi(d(T^{n-1}(x),T^n(x))),\varphi(d(T^{n-1}(x),u)),\varphi(d(T^n(x),u))\} \text{ for each } n \in \mathbb{Z}^+. \tag{13}$$

If

$$max\{\varphi(d(T^{n-1}(x),u)),\varphi(d(T^{n-1}(x),T^n(x))),\varphi(d(T^n(x),u))\} = \varphi(d(T^n(x),u)) \tag{14}$$

for some $n \in \mathbb{Z}^+$, then from (13), we have

$$d(T^n(x),u)) \le \varphi(d(T^n(x),u)).$$

Thus, by property $(ii)$ of $\varphi$, we get

$$d(T^n(x),u)=0 \text{ which implies } T^n(x)=u.$$

Hence,

$$T^{n+m}(x)=T^m(T^n(x))=T^m(u)=u \text{ for each } m\in\mathbb{N},$$

which shows that the sequence $T^n(x)\to u$ as $n\to\infty$. By the uniqueness of the limit we conclude that $u=p$. Otherwise,

$$max\{\varphi(d(T^{n-1}(x),u)),\varphi(d(T^{n-1}(x),T^n(x))),\varphi(d(T^n(x),u))\}\neq\varphi(d(T^n(x),u)), \text{ for all } n\in\mathbb{Z}^+.$$

Again, from (13) we must have

$$d(T^n(x),u)\le max\{\varphi(d(T^{n-1}(x),u)),\varphi(d(T^{n-1}(x),T^n(x)))\}\le\varphi(d(T^{n-1}(x),u))+\varphi(d(T^{n-1}(x),T^n(x))) \tag{15}$$

for all $n\in\mathbb{Z}^+$.
If we take limit superior of (15), and use the upper-semi continuity of $\varphi$, we obtain

$$d(p,u)\le\limsup_{n\to\infty}\varphi(d(T^{n-1}(x),u))+\limsup_{n\to\infty}\varphi(d(T^n(x),T^{n-1}(x)))\le\varphi(d(p,u)).$$

Which gives that

$$d(p,u)\le\varphi(d(p,u)). \tag{16}$$

Using property $(ii)$ of $\varphi$ and (16), we conclude that

$$d(u,p)=0 \text{ and hence } u=p.$$

*Remark.* If we take $\varphi(t)=kt$, where $k\in[0,1)$, then Theorem 5 is reduced to the result of Alfuraidan [[3], Theorem 3.1].

From Remark 30, we observe that the class of $G-$monotone generalized quasi contraction mappings contains the class of $G-$ monotone quasi contraction mappings. The following example illustrates that the inclusion is proper.

**Example 2.** Let $X=\mathbb{R}$ with metric $d$, the usual absolute value metric, i.e., $d(x,y)=|x-y|$. We see that $(X,d)$ is a complete metric space. Let $C=[0,\infty)\subset\mathbb{R}$ which is closed. Define a map

$$T:C\to C \text{ by } T(x)=\frac{x}{1+x}.$$

Consider a graph $G$ on $X$ with $E(G)=X\times X$, then $G$ is connected, reflexive and transitive digraph. We are going to show that $T$ is a $G-$monotone generalized quasi contraction mapping, but not a $G-$monotone quasi contraction mapping. Now, consider a function

$$\varphi:[0,\infty)\to[0,\infty) \text{ given by } \varphi(t)=\frac{t}{1+t}.$$

We observe that $\varphi$ is a comparison function. Let $x,y \in C$. Without loss of generality we assume that $x < y$ (as $d$ is symmetric, the same result follows for $x > y$). Then we have

$$d(T(x),T(y)) = \frac{y-x}{(1+x)(1+y)} = \frac{y-x}{1+x+y+xy} \leq \frac{y-x}{1+y+x} \leq \frac{y-x}{1+y-x} = \varphi(y-x) = \varphi(d(x,y)) \leq \varphi(M(x,y)),$$

where,

$$M(x,y) = max\{d(x,y), d(x,T(x)), d(y,T(y)), d(x,T(y)), d(T(x),y)\}.$$

Therefore,

$$d(T(x),T(y)) \leq \varphi(M(x,y)), \tag{17}$$

which shows that $T$ is a $G-$monotone generalized quasi contraction mapping. Next we show that $T$ is not a $G-$monotone quasi contraction mapping. Indeed,

$$d(x,T(x)) = x - \frac{x}{1+x} = \frac{x^2}{1+x}, \tag{18}$$

$$d(y,T(y)) = y - \frac{y}{1+y} = \frac{y^2}{1+y}, \tag{19}$$

$$d(x,T(y)) = |x - \frac{y}{1+y}| = \frac{|x-y+xy|}{1+y} \leq \frac{y-x+xy}{1+y} \leq \frac{y-x+xy}{1+x}, \tag{20}$$

and

$$d(T(x),y) = |y - \frac{x}{1+x}| = \frac{y-x+xy}{1+x}. \tag{21}$$

Now, we claim that

$$M(x,y) = \frac{y-x+yx}{1+x}. \tag{22}$$

Using the fact that the function

$$f(x) = \frac{x}{1+x}$$

is strictly increasing function and the assumption that $x < y$, we have

$$\frac{x^2}{1+x} = x\frac{x}{1+x} \leq x\frac{y}{1+y} \leq \frac{y^2}{1+y}. \tag{23}$$

In addition, if $y-x > 0$, then we have

$$y-x+yx-yx+y^2+y^2x > yx-yx+y^2+y^2x,$$

which implies that

$$(y-x)(1+y)+yx(1+y) \geq y^2(1+x),$$

and hence

$$\frac{y-x+yx}{1+x} \geq \frac{y^2}{1+y}. \tag{24}$$

On the other hand

$$d(x,T(y)) \leq \frac{y-x+yx}{1+y} \leq \frac{y-x+yx}{1+x}. \tag{25}$$

Again, as $-x^2 \leq 0$, we have

$$y-x+yx-x^2 \leq y-x+xy,$$

which implies that

$$(y-x)(1+x) \leq y-x+yx$$

and hence

$$y-x \leq \frac{y-x+yx}{1+x}. \tag{26}$$

From (20)-(26), we conclude that

$$M(x,y) = \frac{y-x+yx}{1+x} = d(x,Ty). \tag{27}$$

Now, if there is a constant number $k \in [0,1)$ such that for each $x,y \in C$

$$d(T(x),T(y)) = \frac{y-x}{(1+x)(1+y)} \leq kM(x,y) = k\frac{(y-x+yx)}{1+x} \tag{28}$$

then we must have

$$\frac{1}{1+y} \leq k(1+\frac{yx}{y-x}). \tag{29}$$

Letting $y \to 0$ (and hence $x \to 0$) in (29), we get $1 \leq k$, which is a contradiction. Hence, $T$ is not a $G-$ monotone quasi contraction mapping. Actually, $T$ has a unique fixed point and for each $x \in C$,

$$T^n(x) = \frac{x}{1+nx} \leq \frac{x}{nx} = \frac{1}{n} \to 0$$

as $n \to \infty$. Here 0 is a unique fixed pint of $T$, where the $G-$monotone quasi contraction mapping could not be applied to guarantee the existence of such fixed points.

## 4 Generalized quasi contractions in modular metric spaces with a graph

Next, we are going to discuss the validity of the previous results in the setting of modular metric spaces.

**Definition 12.** *Let $(X,\omega)$ be a modular metric space and $G$ be a reflexive, transitive digraph defined on $X$. Let $C$ be a nonempty subset of $X$. A mapping $T : C \to C$ is said to be: $G-$monotone generalized $\omega-$quasi contraction if $T$ is $G-$monotone and there exists $\varphi \in \Phi$ such that for any $x,y \in C$, and $(x,y) \in E(G)$, we have*

$$\omega_1(T(x),T(y)) \leq max\{\varphi(\omega_1(x,y)),\varphi(\omega_1(x,T(x))),\varphi(\omega_1(y,T(y))),\varphi(\omega_1(x,T(y))),\varphi(\omega_1(T(x),y))\}$$

Let $C$ be a nonempty subset of $X$ and let $T : C \to C$ be any self mapping. For any $x \in C$ we define, the orbit $O(x) = \{x,T(x),T^2(x),\dots\}$ and its diameter by

$$\delta_\omega(x) = sup\{\omega_1(T^n(x),T^m(x)) : n,m \in \mathbb{N}\}.$$

Throughout, we assume that $\omega$ is regular and satisfy the Fatou property.

**Lemma 3.** *Let $(X,\omega)$ be a modular metric space and $G$ be a reflexive and transitive digraph on $X$ . Let $C$ be a nonempty subset of $X$ and $T : C \to C$ be a $G-$monotone generalized $\omega-$quasi contraction mapping. For $x \in C$ with $(x,T(x)) \in E(G)$ and $\delta_\omega(x) < \infty$, we have $\delta_\omega(T^n(x)) \leq \varphi^n(\delta_\omega(x))$ for each $n \in \mathbb{N}$, where $\varphi$ is the comparison function associated with the $G-$monotone generalized $\omega-$quasi contraction definition of $T$. Moreover, we have*

$$\omega_1(T^n(x),T^{n+m}(x)) \leq \varphi^n(\delta_\omega(x)) \text{ for any } n,m \in \mathbb{N}.$$

*Proof.* Since $T$ is $G-$monotone and $(x,T(x)) \in E(G)$, we have

$$(T^n(x),T^{n+1}(x)) \in E(G) \text{ for any } n \in \mathbb{N}.$$

By the transitivity of the graph $G$, for each $n \in \mathbb{N}$, we have

$$(T^n(x),T^{n+m}(x)) \in E(G) \text{ for any } m \in \mathbb{Z}^+. \tag{30}$$

Again as $G$ is reflexive, $(x,x) \in E(G)$ and using the monotonocity of $G$, we obtain

$$(T^n(x),T^n(x)) \in E(G) \text{ for each } n \in \mathbb{N}. \tag{31}$$

Thus, from (30) and (31), we infer that

$$(T^n(x),T^{n+m}(x)) \in E(G) \text{ for each } n,m \in \mathbb{N}. \tag{32}$$

Since $T$ is a $G-$monotone generalized $\omega-$quasi contraction, and using (32), there is a comparison function $\varphi \in \Phi$ such that

$$\begin{aligned}\omega_1(T^n(x),T^{n+m}(x)) \leq max\{&\varphi(\omega_1(T^{n-1}(x),T^{n+m-1}(x))),\varphi(\omega_1(T^{n-1}(x),T^n(x))),\\ &\varphi(\omega_1(T^{n+m-1}(x),T^{n+m}(x))),\varphi(\omega_1(T^{n-1}(x),T^{n+m}(x))),\\ &\varphi(\omega_1(T^n(x),T^{n+m-1}(x)))\}\end{aligned} \tag{33}$$

for any $n \in \mathbb{Z}^+$ and any $m \in \mathbb{N}$. We need to show that

$$\delta_\omega(T^n(x)) \le \varphi^n(\delta_\omega(x)) \text{ for each } n \in \mathbb{N}. \tag{34}$$

To see this, if $n = 0$, the result is obvious; in this case equality holds. Now, if we take $n = 1$, in (33), we get

$$\begin{aligned}\omega_1(T(x), T^{1+m}(x)) \le\, & max\{\varphi(\omega_1(x, T^m(x))), \varphi(\omega_1(x, T(x))), \varphi(\omega_1(T^m(x), T^{1+m}(x))), \\ & \varphi(\omega_1(x, T^{1+m}(x))), \varphi(\omega_1(T(x), T^m(x)))\} \\ & \le \varphi(\delta_\omega(x)) \end{aligned} \tag{35}$$

for any $m \in \mathbb{N}$. Thus

$$\omega_1(T(x), T^{1+m}(x)) \le \varphi(\delta_\omega(x)) \tag{36}$$

for each $m \in \mathbb{N}$.
By the definition of $\delta$ and (36), we obtain

$$\delta_\omega(T(x)) \le \varphi(\delta_\omega(x)). \tag{37}$$

Using the monotonicity of $\varphi$ and (37), we obtain

$$\varphi(\delta_\omega(T(x))) \le \varphi^2(\delta_\omega(x)). \tag{38}$$

Using (37) and (38), we get

$$\delta_\omega(T^2(x)) = \delta_\omega(T(T(x)) \le \varphi(\delta_\omega(T(x))) \le \varphi^2(\delta_\omega(x)). \tag{39}$$

Repeated use of the monotonicity of $\varphi$ and (37), gives

$$\begin{aligned}\delta_\omega(T^n(x)) = \delta_\omega(T(T^{n-1}(x))) &\le \varphi(\delta_\omega(T^{n-1}(x))) \\ &\vdots \le \varphi^n(\delta_\omega(x)).\end{aligned} \tag{40}$$

On the other hand, from (33) and using the definition of $\delta_\omega$, we obtain

$$\omega_1(T^n(x), T^{n+m}(x)) = \omega_1(T^n(x), T^m(T^n(x))) \le \delta_\omega(T^n(x)) \tag{41}$$

for each $n, m \in \mathbb{N}$. Hence, from (40) and (41), we get that

$$\omega_1(T^n(x), T^{n+m}(x)) \le \varphi^n(\delta_\omega(x))$$

for each $n, m \in \mathbb{N}$.

**Lemma 4.** *Let $(X, \omega)$ be a modular metric space and let $C$ be a nonempty $\omega-$complete subset of $X$. Let $T : C \to C$ be a $G-$monotone generalized $\omega-$quasi contraction mapping and $x \in C$ be such that $\delta_\omega(x) < \infty$. Then $\{T^n(x)\}$ $\omega-$converges to a point $u \in C$. Moreover, one has*

$$\omega_1(T^n(x), u) \le \varphi^n(\delta_\omega(x))$$

*for all* $n \in \mathbb{N}$.

*Proof.* By Lemma 3, $\{T^n(x)\}$ is $\omega-$Cauchy sequence in $C$. Since $C$ is $\omega-$complete, then there exists $u \in C$ such that $\{T^n(x)\}$ $\omega-$converges to $u$. Again, from (40), we have

$$\omega_1(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta_\omega(x)) \tag{42}$$

for $n, m \in \mathbb{N}$. As $\omega$ satisfies the Fatou property, letting $m \to \infty$ in (42) to obtain

$$\omega_1(T^n(x), u) \leq \varphi^n(\delta_\omega(x), \text{ for each } n \in \mathbb{N}. \tag{43}$$

**Theorem 6.** *Let* $(X, \omega)$ *be a modular metric space and let G be a reflexive transitive digraph defined on* $X$ *such that the triple* $(X, \omega, G)$ *have Property (B):*
*for any sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $X$, *if* $x_n \overset{\omega}{\to} x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$, *then* $(x_n, x) \in E(G)$ *for each* $n \in \mathbb{N}$. *Let* $C$ *be a nonempty* $\omega-$*complete subset of* $X$ *and* $T : C \to C$ *be a* $G-$*monotone generalized* $\omega-$*quasi contraction mapping with the associated comparison function* $\varphi \in \Phi$, *the strong comparison function. For* $x \in C$ *with* $(x, Tx) \in E(G)$ *and* $\delta_\omega(x) < \infty$, *we have the following:*

(a) *The sequence* $\{T^n(x)\}$ $\omega-$*converges to some point* $u$ *in* $C$.
*Moreover, we have* $(x, u) \in E(G)$ *and* $\omega_1(T^n(x), u) \leq \varphi^n(\delta_\omega(x))$
*for each* $n \in \mathbb{N}$. *If in addition,* $\omega_1(u, T(u)) < \infty$ *and* $\omega_1(x, T(u)) < \infty$,
*then* $u$ *is a fixed point of* $T$.
(b) *If* $u^*$ *is any fixed point of* $T$ *in* $C$ *such that* $(x, u^*) \in E(G)$ *and*
$\omega_1(T^n(x), u^*) < \infty$ *for any* $n \in \mathbb{N}$, *then* $u = u^*$.

*Proof.* First we prove (a). By Lemma 4, the sequence $\{T^n(x)\}$ is $\omega-$Cauchy in $C$. Since $C$ is $\omega-$complete, there is a $u \in C$ such that $T^n(x) \overset{\omega}{\longrightarrow} u$ as $n \to \infty$. From (40), we have

$$\omega_1(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta_\omega(x)) \tag{44}$$

for each $m \in \mathbb{N}$, and by assumption $\omega$ satisfies the Fatou property, we get (by letting $m \to \infty$) in (44)

$$\omega_1(T^n(x), u) \leq \liminf_{m\to\infty} \omega_1(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta_\omega(x)) \text{ for any } n \in \mathbb{N}.$$

From property $(B)$, we have $(T^n(x), u) \in E(G)$, for $n \in \mathbb{N}$. In particular $(x, u) \in E(G)$. Since $T$ is $G-$monotone, we have $(T(x), T(u)) \in E(G)$ and hence

$$\omega_1(T(x), T(u)) \leq max\{\varphi(\omega_1(x, u)), \varphi(\omega_1(x, T(x))), \varphi(\omega_1(u, T(u)), \varphi(\omega_1(x, T(u)), \varphi(\omega_1(T(x), u))\}. \tag{45}$$

Since $\omega$ satisfies the Fatou property, and using Lemma 6, we have

$$\omega_1(x, u) \leq \liminf_{m\to\infty} \omega_1(x, T^m(x)) \leq \delta_\omega(x), \omega_1(x, T(x)) \leq \delta_\omega(x),$$

and

$$\omega_1(T(x), u) \leq \liminf_{m\to\infty} \omega_1(T(x), T^m(x)) \leq \delta_\omega(T(x)) \leq \varphi(\delta_\omega(x)).$$

Substituting these values in (45), we obtain

$$\omega_1(T(x), T(u)) \leq max\{\varphi(\delta_\omega(x)), \varphi(\omega_1(u, T(u))), \varphi(\omega_1(x, T(u))), \varphi^2(\delta_\omega(x))\}. \tag{46}$$

Since $T$ is a $G-$monotone and $(T^n(x),u) \in E(G)$, for each $n \in \mathbb{N}$, we have that

$$(T^{n+1}(x),T(u)) \in E(G). \tag{47}$$

As $(x,T(x)) \in E(G)$ and $T$ is a $G-$monotone, we also have

$$(T^n(x),T^{n+1}(x)) \in E(G)$$

for each $n \in \mathbb{N}$, and using the transitivity property of $G$, we get

$$(T^n(x),T(u)) \in E(G).$$

Now, we assume that

$$\begin{aligned} \omega_1(T^n(x),T(u)) \leq max\{&\varphi^n(\delta_\omega(x)),\varphi(\omega_1(u,(T(u)))),\varphi^2(\omega_1(u,T(u))),\\ &\cdots,\varphi^n(\omega_1(u,T(u))),\varphi^n(\omega_1(x,T(u)),\varphi^{n+1}(\delta_\omega(x))\}. \end{aligned} \tag{48}$$

Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \omega_1(T^{n+1}(x),T(u)) &= \omega_1(T^n(T(x),T(u))\\ &\leq max\{\varphi^n(\delta_\omega(T(x))),\varphi(\omega_1(u,T(u))),\varphi^2(\omega_1(u,(T(u)))),\\ &\cdots,\varphi^n(\omega_1(u,T(u))),\varphi^n(\omega_1(T(x),T(u)),\\ &\varphi^{n+1}(\delta_\omega(T(x)))\}. \end{aligned} \tag{49}$$

From (46) and the monotonicity of $\varphi$, we get

$$\varphi^n(\omega_1(T(x),T(u))) \leq max\{\varphi^{n+1}(\delta_\omega(x)),\varphi^{n+1}(\omega_1(u,T(u)),\varphi^{n+1}(\omega_1(x,T(u))),\varphi^{n+2}(\delta_\omega(x))\}. \tag{50}$$

Combining (50) and (49), we obtain

$$\begin{aligned} \omega_1(T^{n+1}(x),T(u)) \leq max\{&\varphi^{n+1}(\delta_\omega(x)),\varphi^{n+2}(\delta_\omega(x)),\varphi(\omega_1(u,T(u))),\varphi^2(\omega_1(u,(T(u)))),\\ &\cdots,\varphi^n(\omega_1(u,T(u))),\varphi^{n+1}(\omega_1(x,T(u))\}. \end{aligned}$$

Hence, by induction we conclude that

$$\begin{aligned} \omega_1(T^n(x),T(u)) \leq max\{&\varphi^n(\delta_\omega(x)),\varphi^{n+1}(\delta_\omega(x)),\varphi(\omega_1(u,T(u))),\varphi^2(\omega_1(u,T(u))),\\ &\cdots,\varphi^n(\omega_1(u,T(u))),\varphi^n(\omega_1(x,T(u))\} \end{aligned}$$

for each $n \in \mathbb{N}$. Since $\varphi^n(t)$ is a nonincreasing sequence for any $t > 0$ and $\omega_1(u,T(u)) < \infty$, we have that

$$\varphi^n(\omega_1(u,T(u))) \leq \varphi(\omega_1(u,T(u))) \tag{51}$$

for each $n \in \mathbb{Z}^+$.
Thus, from (49) and (51), we have

$$\begin{aligned}\omega_1(T^n(x),T(u)) &\leq max\{\varphi^n(\delta_\omega(x)),\varphi^{n+1}(\delta_\omega(x)),\varphi(\omega_1(u,T(u))),\varphi^n(\omega_1(x,T(u))\} \\ &\leq \varphi^n(\delta_\omega(x))+\varphi^{n+1}(\delta_\omega(x))+\varphi^n(\omega_1(x,T(u))+\varphi(\omega_1(u,T(u))).\end{aligned} \tag{52}$$

Thus, taking limit superior of (52), gives

$$\begin{aligned}\limsup_{n\to\infty}\omega_1(T^n(x),T(u)) &\leq \limsup_{n\to\infty}(\varphi^n(\delta_\omega(x))+\varphi^{n+1}(\delta_\omega(x))+\varphi^n(\omega_1(x,T(u))+\varphi(\omega_1(u,T(u))) \\ &\leq \limsup_{n\to\infty}(\varphi^n(\delta_\omega(x))+\limsup_{n\to\infty}(\varphi^{n+1}(\delta_\omega(x))+\limsup_{n\to\infty}\varphi^n(\omega_1(x,T(u)))+\varphi(\omega_1(u,T(u)) \\ &= \varphi(\omega_1(u,T(u)).\end{aligned}$$

Whence, using the Fatou property, we obtain

$$\omega_1(u,T(u)) \leq \liminf_{n\to\infty}\omega_1(T^n(x),T(u)) \leq \limsup_{n\to\infty}\omega_1(T^n(x),T(u)) \leq \varphi(\omega_1(u,T(u)).$$

Since $\omega_1(u,T(u)) < \infty$, and using property $(ii)$ of $\varphi$, we must have $\omega_1(u,T(u)) = 0$. Since $\omega$ is regular, we conclude that $u = T(u)$. Next we prove (b). Let $u^*$ be any fixed point of $T$. Suppose that $(x,u^*) \in E(G)$, and $\omega_1(T^n(x),u^*) < \infty$ for any $n \in \mathbb{N}$. By induction, and using the $G-$monotonicty of $T$, we have

$$(T^n(x),u^*) \in E(G) \text{ for each } n \in \mathbb{N}.$$

By hypothesis, there is a $\varphi \in \Phi$, strong comparison function, such that

$$\omega_1(T^n(x),u^*) \leq max\{\varphi(\omega_1(T^{n-1}(x),u^*)),\varphi(\omega_1(T^{n-1}(x),T^n(x))),\varphi(\omega_1(T^n(x),u^*))\}$$

for each $n \in \mathbb{Z}^+$. If

$$max\{\varphi(\omega_1(T^{n-1}(x),u^*),\varphi(\omega_1(T^{n-1}(x),T^n(x))),\varphi(\omega_1(T^n(x),u^*))\} = \varphi(\omega_1(T^n(x),u^*))$$

for some $n \in \mathbb{Z}^+$, then

$$\omega_1(T^n(x),u^*) \leq \varphi(\omega_1(T^n(x),u^*)).$$

Using property of $\varphi$, once again, we get

$$\omega_1(T^n(x),u^*) = 0.$$

Thus, as $\omega$ is regular, we have $T^n(x) = u^*$ which implies that $T^n(x) \xrightarrow{\omega} u^*$ as $n \to \infty$. Thus,

$$\omega_2(u,u^*) \leq \omega_1(u,T^n(x)) + \omega_1(T^n(x),u^*) \xrightarrow{\omega} 0$$

as $n \to \infty$ and the regularity of of $\omega$ provides $u = u^*$.
Now, suppose that

$$max\{\varphi(\omega_1(T^{n-1}(x),u^*),\varphi(\omega_1(T^{n-1}(x),T^n(x)),\varphi(\omega_1(T^n(x),u))\} \neq \varphi(\omega_1(T^n(x),u)),$$

for each $n \in \mathbb{Z}^+$. Then,

$$\begin{aligned}\omega_1(T^n(x),u^*) &\leq max\{\varphi(\omega_1(T^{n-1}(x),u^*)),\varphi(\omega_1(T^{n-1}(x),T^n(x)))\}\\ &\leq max\{\varphi(\omega_1(T^{n-1}(x),u^*)),\varphi^n(\delta_\omega(x))\}\\ &\leq max\{\omega_1(T^{n-1}(x),u^*)),\varphi^n(\delta_\omega(x))\}\\ &\leq \omega_1(T^{n-1}(x),u^*))+\varphi^n(\delta_\omega(x))\end{aligned} \tag{53}$$

for each $n \in \mathbb{Z}^+$. Similarly, we have

$$\omega_1(T^{n-1}(x),u^*) \leq \omega_1(T^{n-2}(x),u^*))+\varphi^{n-1}(\delta_\omega(x)).$$

Following the same procedure, we obtain

$$\omega_1(T^n(x),u^*) \leq \sum_{j=2}^{n}\varphi^j(\delta_\omega(x))+\omega_1(T(x),u^*).$$

and

$$\begin{aligned}\omega_1(T(x),u^*) &\leq max\{\varphi(\omega_1(x,u^*)),\varphi(\omega_1(x,T(x)))\}\\ &\leq max\{\omega_1(x,u^*)),\varphi(\omega_1(\delta_\omega(x)))\}\\ &\leq \omega_1(x,u^*))+\varphi(\omega_1(\delta_\omega)).\end{aligned}$$

So, we obtain

$$\omega_1(T^n(x),u^*) \leq \sum_{j=1}^{n}\varphi^j(\delta_\omega(x))+\omega_1(x,u^*). \tag{54}$$

Thus,

$$\limsup_{n\to\infty}\omega_1(T^n(x),u^*) \leq \sum_{j=1}^{\infty}\varphi^j(\delta_\omega(x))+\omega_1(x,u^*) < \infty$$

as $\varphi$ was assumed to be a strong comparison function.
Now, if we set

$$\gamma(u^*) := \limsup_{n\to\infty}\omega_1(T^n(x),u^*),$$

then we have $\gamma(u^*) < \infty$. Then, from (53), we get

$$\omega_1(T^n(x),u^*) \leq max\{\varphi^n(\delta_\omega(x)),\varphi(\omega_1(T^{n-1}(x),u^*))\} \tag{55}$$

and taking limit superior as $n \to \infty$ in (55), we obtain, $\gamma(u^*) \leq \varphi(\gamma(u^*))$ which implies that $\gamma(u^*) = 0$. i.e., $\limsup_{n\to\infty} \omega_1(T^n(x), u^*) = 0$. Therefore, applying Fatou property once again, we get

$$\omega_1(u,u^*) \leq \liminf_{n\to\infty} \omega_1(T^n(x),u^*) \leq \limsup_{n\to\infty} \omega_1(T^n(x),u^*) = 0,$$

and hence since $\omega$ is regular, we obtain $u = u^*$.

*Remark.* If we assume that $(u,u^*) \in E(G)$ for any fixed point $u^*$ in $C$ and $\omega_1(u,u^*) < \infty$, then $(T(u),T(u^*)) \in E(G)$ and $\omega_1(u,u^*) = \omega_1(T(u),T(u^*)) \leq \varphi(\omega_1(u,u^*))$. This clearly shows that $u = u^*$.

*Remark.* If we take $\varphi(t) = kt$, where $k \in [0,1)$, then Theorem 6 is reduced to the result of Alfuraidan [[3],Theorem 4.1]

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## Competing interests

The authors declare that they have no competing interests.

## Authors contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript

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