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More on $I-$cluster points of filters

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Abstract: This paper is an extension of our paper $I-$cluster points of filters [8]. In this paper, we have discussed the relationship between $I-$cluster points of filters and cluster points of nets and estabilished their equivalence. We have also estabilished the equivalence of $I-$cluster points of filters and nets.

Keywords: Filters, nets, convergence, cluster point, ideal.

1 Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by H. Fast [4] and I. J. Schoenberg [23]. Kostyrko et. al. in [9], [10] generalized the notion of statistical convergence and introduced the concept of $I-$convergence of real sequences which is based on the structure of the ideal I of subsets of the set of natural numbers. Mursaleen et. al. [15] defined and studied the notion of ideal convergence in random $2-$normed spaces and construct some interesting examples. Several works on $I-$convergence and statistical convergence have been done in [1], [3], [6], [7], [8], [9], [10], [11], [14], [15], [16], [17], [18], [22].

The idea of $I-$convergence has been extended from real number space to metric space [9] and to a normed linear space [21] in recent works. Later the idea of $I-$convergence was extended to an arbitrary topological space by B. K. Lahiri and P. Das [12]. It was observed that the basic properties remained preserved in a topological space. Lahiri and Das [13] introduced the idea of $I-$convergence of nets in a topological space and examined how far it affects the basic properties.

Taking the idea of [13], Jamwal et. al introduced the idea of $I-$convergence of filters and studied its various properties in [6]. Jamwal et. al reintroduced the idea of $I-$convergence of nets in a topological space in [7] and estabilished the equivalence of $I-$convergences of nets and filters on a topological space. Jamwal et. al introduced the idea of $I-$cluster points of filters in a topological space and studied their properties in [8].
We start with the following definitions:

Definition 1. *Let X be a non-empty set. Then a family $\mathscr{F} \subset 2^X$ is called a* **filter** *on X if*

(i) $\emptyset \notin \mathscr{F}$,
(ii) $A,B \in \mathscr{F}$ *implies* $A \cap B \in \mathscr{F}$ *and*
(iii) $A \in \mathscr{F}, B \supset A$ *implies* $B \in \mathscr{F}$.

Definition 2. *Let X be a non-empty set. Then a family $I \subset 2^X$ is called an* **ideal** *of X if*

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(i) $\emptyset \in I$,
(ii) $A, B \in I$ *implies* $A \cup B \in I$ *and*
(iii) $A \in I, B \subset A$ *implies* $B \in I$.

Definition 3. *Let X be a non-empty set. Then a filter $\mathscr{F}$ on X is said to be* **non-trivial** *if $\mathscr{F} \neq \{X\}$.*

Definition 4. *Let X be a non-empty set. Then an ideal I of X is said to be* **non-trivial** *if $I \neq \{\emptyset\}$ and $X \notin I$.*

Note(i) $\mathscr{F} = \mathscr{F}(I) = \{A \subset X : X \setminus A \in I\}$ is a filter on X, called the **filter associated with the ideal** I.
(ii) $I = I(\mathscr{F}) = \{A \subset X : X \setminus A \in \mathscr{F}\}$ is an ideal of X, called the **ideal associated with the filter** $\mathscr{F}$.
(iii) A non-trivial ideal I of X is called **admissible** if I contains all the singleton subsets of X. Several examples of non-trivial admissible ideals have been considered in [9].

We give a brief discussion on $I-$convergence of filters and nets in a topological space as given by [6], [7]. Throughout this paper, $X = (X, \tau)$ will stand for a topological space and $I = I(\mathscr{F})$ will be the ideal associated with the filter $\mathscr{F}$ on X.

Definition 5. *A filter $\mathscr{F}$ on X is said to be $I-$***convergent** *to $x_0 \in X$ if for each nbd U of x_0, $\{y \in X : y \notin U\} \in I$. In this case, x_0 is called an $I-$***limit of** *$\mathscr{F}$ and is written as $I - lim\mathscr{F} = x_0$.*

Notation In case more than one filters is involved, we use the notation $I(\mathscr{F})$ to denote the ideal associated with the corresponding filter $\mathscr{F}$.

Lemma 1. *Let $\mathscr{F}$ and $\mathscr{G}$ be two filters on X. Then $\mathscr{F} \subset \mathscr{G}$ if and only if $I(\mathscr{F}) \subset I(\mathscr{G})$.*

Proposition 1. *Let $\mathscr{F}$ be a filter on X such that $I - \lim\mathscr{F} = x_0$. Then every filter $\mathscr{F}'$ on X finer than $\mathscr{F}$ also $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Proposition 2. *Let $\mathscr{F}$ be a filter on X such that $I - \lim\mathscr{F} = x_0$. Then every filter $\mathscr{F}'$ on X coarser than $\mathscr{F}$ also $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Proposition 3. *Let $\mathscr{F}$ be a filter on X and $\mathscr{G}$ be any other filter on X finer than $\mathscr{F}$. Then $I(\mathscr{F}) - \lim\mathscr{G} = x_0$ implies $I(\mathscr{G}) - \lim\mathscr{G} = x_0$.*

Proposition 4. *Let τ_1 and τ_2 be two topologies on X such that τ_1 is coarser than τ_2. Let $\mathscr{F}$ be a filter on X such that $I - \lim\mathscr{F} = x_0$ w.r.t τ_2. Then $I - \lim\mathscr{F} = x_0$ w.r.t τ_1.*

Lemma 2. *Let $\mathscr{M} = \{\mathscr{G} : \mathscr{G}\ is\ a\ filter\ on\ X\}$. Then $\mathscr{F} = \cap_{\mathscr{G}\in\mathscr{M}}\mathscr{G}$ if and only if $I(\mathscr{F}) = \cap_{\mathscr{G}\in\mathscr{M}}I(\mathscr{G})$.*

Proposition 5. *Let $\mathscr{M}$ be a collection of all those filters $\mathscr{G}$ on a space X which $I(\mathscr{G})-$converges to the same point $x_0 \in X$. Then the intersection $\mathscr{F}$ of all the filters in $\mathscr{M}$ $I(\mathscr{F})-$converges to x_0.*

Theorem 1. *A filter $\mathscr{F}$ on X $I-$converges to $x_0 \in X$ if and only if every derived net λ of $\mathscr{F}$ converges to x_0.*

Theorem 2. *A filter $\mathscr{F}$ on X $I-$converges to $x_0 \in X$ if and only if $\mathscr{F}$ converges to x_0.*

Definition 6. *Let I be a non-trivial ideal of subsets of X. Let $\lambda : \mathscr{D} \to X$ be a net in X, where $\mathscr{D}$ is a directed set. Then λ is said to be $I-$***convergent** *to x_0 in X if for each nbd U of x_0, $\{\lambda(c) \in X : \lambda(c) \notin U\} \in I$.*

Theorem 3. *A filter $\mathscr{F}$ on X $I-$converges to $x_0 \in X$ if and only if every derived net λ of $\mathscr{F}$ $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Lemma 3. *A filter $\mathscr{F}$ on X converges to x_0 in X if and only if every derived net λ of $\mathscr{F}$ $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Theorem 4. *Let $\lambda : \mathscr{D} \to X$ be a net in X and $\mathscr{F}$ be a derived filter of λ. Then λ $I-$converges to x_0 in X if and only if the derived filter $\mathscr{F}$ $I-$converges to x_0, where $I = I(\mathscr{F})$.*

We now recall some of the results discussed in our paper $I-$cluster points of filters [8].

Definition 7. *A point $x_0 \in X$ is called an $I-$***cluster point** *of a filter $\mathscr{F}$ on X if for each nbd U of x_0, $\{y \in X : y \in U\} \notin I$. In other words, $x_0 \in X$ is called an $I-$cluster point of $\mathscr{F}$ if $U \notin I$, for each nbd U of x_0.*

Equivalently, x_0 is an $I-$cluster point of $\mathscr{F}$ if for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \subset V\} \nsubseteq I$.

Notation Let $I(C_{\mathscr{F}})$ and $I(L_{\mathscr{F}})$ respectively denotes the set of all $I-$cluster points and the set of all $I-$limits of a filter $\mathscr{F}$ on X.

Theorem 5. *With usual notations, $I(L_{\mathscr{F}}) \subset I(C_{\mathscr{F}})$. But not conversely.*

Theorem 6. *Let $\mathscr{F}$ be a filter on X. If x_0 is an $I-$cluster point of $\mathscr{F}$, then for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I$.*

Proposition 6. *Let $\mathscr{F}$ be a filter on a non-discrete space X. Then $\mathscr{F}$ has x_0 as an $I(\mathscr{F})-$cluster point if and only if there is a filter $\mathscr{G}$ on X finer than $\mathscr{F}$ such that $I(\mathscr{G}) - \lim \mathscr{G} = x_0$.*

Proposition 7. *Let $\mathscr{F}$ be a filter on X and $\mathscr{G}$ be a filter on X finer than $\mathscr{F}$. Then $\mathscr{F}$ has x_0 as an $I(\mathscr{G})-$cluster point if and only if $I(\mathscr{G}) - \lim \mathscr{G} = x_0$.*

Proposition 8. *Let $\mathscr{F}$ be a filter on X such that $\mathscr{F}$ has $x_0 \in X$ as an $I-$cluster point. Then every filter $\mathscr{F}'$ finer than $\mathscr{F}$ also has x_0 as an $I-$cluster point, where $I = I(\mathscr{F})$.*

Proposition 9. *Let $\mathscr{F}$ be a filter on X such that $\mathscr{F}$ has $x_0 \in X$ as an $I-$cluster point. Then every filter $\mathscr{F}'$ coarser than $\mathscr{F}$ also has x_0 as an $I-$cluster point, where $I = I(\mathscr{F})$.*

Remark. Let $\mathscr{F}$ be a filter on X and $\mathscr{F}'$ be a filter on X finer than $\mathscr{F}$. Then $I(\mathscr{F})-$cluster point of $\mathscr{F} = x_0$ need not imply that $I(\mathscr{F}')-$cluster point of $\mathscr{F}' = x_0$.

Proposition 10. *Let τ_1 and τ_2 be two topologies on X such that τ_1 is coarser than τ_2. Let $\mathscr{F}$ be a filter on X such that x_0 is an $I-$cluster point of $\mathscr{F}$ w.r.t τ_2. Then x_0 is also an $I-$cluster point of $\mathscr{F}$ w.r.t τ_1. But not conversely.*

Proposition 11. *Let $\mathscr{M}$ be a collection of all those filters $\mathscr{G}$ on a space X which have $x_0 \in X$ as an $I(\mathscr{G})-$cluster point. Then the intersection $\mathscr{F}$ of all the filters in $\mathscr{M}$ also has x_0 as an $I(\mathscr{F})-$cluster point.*

Theorem 7. *Let X be a Lindelöf space such that every filter on X has an $I-$cluster point, where I is an admissible ideal of X. Then X is compact.*

Theorem 8. *A topological space X is compact if and only if every filter on X has an $I-$cluster point.*

2 Equivalence of $I-$cluster points of filters and cluster points of nets

We have the following definition of cluster point of a net in a space X as given by [24].

Definition 8. *Let $\lambda : \mathscr{D} \to X$ be a net in X. Then a point $x_0 \in X$ is called a* **cluster point** *of λ if λ is frequently in each nbd of x_0. By λ frequently in each nbd of x_0, we mean that for each nbd U of x_0 and each $d \in \mathscr{D}$, there is $c \geq d$ in $\mathscr{D}$ for which $\lambda(c) \in U$. Equivalently, x_0 is a cluster point of λ if each tail of λ is contained in U, for each nbd U of x_0, where $\Lambda_d = \{\lambda(c) : c \geq d \text{ in } \mathscr{D}\}$ is a tail of the net λ in X. In other words, we can say that x_0 is a cluster point of λ if $\Lambda_d \cap U \neq \emptyset$, for each nbd U of x_0 and each tail Λ_d of the net λ in X.*

Theorem 9. *A net $\lambda : \mathscr{D} \to X$ has $x_0 \in X$ as a cluster point if and only if x_0 is an $I-$cluster point of the derived filter on X, where $I = I(\mathscr{F})$.*

Proof. Suppose x_0 is a cluster point of a net $\lambda : \mathscr{D} \to X$. Then for each nbd U of x_0 and each $d \in \mathscr{D}$, there is $c \geq d$ in $\mathscr{D}$ such that $\lambda(c) \in U$. That is, each tail of λ is contained in U. Let $\mathscr{F}$ be the derived filter of λ. That is, $\mathscr{F} = \{F \subset X :$ some tail of λ is contained in $F\}$. Since each tail of λ is contained in U, we find that $U \in \mathscr{F}$, for each nbd U of x_0.

Thus $\mathscr{U}_{x_0} \subset \mathscr{F}$, where $\mathscr{U}_{x_0}$ is the nbd filter at x_0. This implies that $\mathscr{F}$ is convergent to x_0. Hence by Theorem 2, $\mathscr{F}$ is $I-$convergent to x_0. Also, by Theorem 5, $I(L_{\mathscr{F}}) \subset I(C_{\mathscr{F}})$, where the symbols have their ususal meanings. This proves that x_0 is an $I-$cluster point of $\mathscr{F}$.

Conversely, suppose $\mathscr{F}$ is the derived filter of a net $\lambda : \mathscr{D} \to X$ such that x_0 is an $I-$cluster point of $\mathscr{F}$. Then for each nbd U of x_0, $U \notin I$.
We shall show that x_0 is a cluster point of λ. For this, let U be a nbd of x_0. Then by the given condition $U \notin I$. Now $U \notin I \Rightarrow X \setminus U \notin \mathscr{F}$. Since $\mathscr{F}$ is a derived filter, for any $d \in \mathscr{D}, \Lambda_d \nsubseteq X \setminus U$. This means that $\Lambda_d \cap U \neq \emptyset$, for every tail Λ_d and every nbd U of x_0. This shows that λ is frequently in each nbd of x_0. Hence x_0 is a cluster point of λ.

Theorem 10. *A filter $\mathscr{F}$ on X has x_0 as an $I-$cluster point if and only if every derived net of $\mathscr{F}$ has x_0 as a cluster point, where $I = I(\mathscr{F})$.*

Proof. Suppose $\mathscr{F}$ is a filter on X such that x_0 is an $I-$cluster point of $\mathscr{F}$. Let us index $\mathscr{F}$ with an index set $\mathscr{D}$ so that $\mathscr{F} = \{F_s : s \in \mathscr{D}\}$. Let us give some direction to $\mathscr{D}$ so that $c \geq d$ in $\mathscr{D}$ if and only if $F_c \subset F_d$. Let λ be a derived net of $\mathscr{F}$ so obtained. We have to show that x_0 is a cluster point of λ.

Since x_0 is an $I-$cluster point of $\mathscr{F}$, $U \notin I$, for any nbd U of x_0. This implies that $X \setminus U \notin \mathscr{F}$, for any nbd U of x_0. Since $\mathscr{F} = \{F_s : s \in \mathscr{D}\}$, $X \setminus U \neq F_s$, for any $s \in \mathscr{D}$. Let $d \in \mathscr{D}$. Then $X \setminus U \neq F_d$. Also, $\lambda(d) \in F_d$. Now for $c \geq d$ in $\mathscr{D}$, $F_c \subset F_d$ and so $\lambda(c), \lambda(d) \in F_d$. Clearly, $\lambda(c), \lambda(d) \notin X \setminus U$. Thus we conclude that $\Lambda_d \nsubseteq X \setminus U$, for every tail Λ_d of λ and every nbd U of x_0. That is, $\Lambda_d \cap U \neq \emptyset$, for every tail Λ_d of λ and every nbd U of x_0. This shows that λ is frequently in each nbd of x_0. Hence x_0 is a cluster point of λ.

Conversely, suppose every derived net λ of $\mathscr{F}$ has x_0 as a cluster point. We have to show that x_0 is an $I-$cluster point of $\mathscr{F}$. For this, let U be a nbd of x_0. Since x_0 is a cluster point of λ, for the nbd U of x_0, $\Lambda_d \subset U$, $\forall\, d \in \mathscr{D}$. Since λ is a derived net, there exists $F_d \in \mathscr{F}$, for each $d \in \mathscr{D}$ such that $\lambda(d) \in F_d$. Clearly, $\Lambda_d \in \mathscr{F}$, for each $d \in \mathscr{D}$. Since $\mathscr{F}$ is a filter on X, we must have $U \in \mathscr{F}$ ($\mathscr{F}$ is closed under superset).

Now, $U \in \mathscr{F}$, for each nbd U of x_0 implies $\mathscr{U}_{x_0} \subset \mathscr{F}$, where $\mathscr{U}_{x_0}$ is the nbd filter at x_0. Therefore, $\mathscr{F}$ converges to x_0. From Theorem 2, $\mathscr{F}$ $I-$converges to x_0. Also, by Theorem 5, $I(L_{\mathscr{F}}) \subset I(C_{\mathscr{F}})$. This proves that x_0 is an $I-$cluster point of $\mathscr{F}$.
We recall [24] that a filter $\mathscr{F}$ on a topological space X **clusters** at x_0(or, has x_0 as a **cluster point**) if each $F \in \mathscr{F}$ meets each $U \in \mathscr{U}_{x_0}$. Equivalently, $\mathscr{F}$ has x_0 as a cluster point if and only if $x_0 \in \cap\{\overline{F} : F \in \mathscr{F}\}$.

Theorem 11. *A filter $\mathscr{F}$ on X has x_0 as an $I-$cluster point if and only if it has x_0 as a cluster point.*

Proof. It follows from Theorem 10 and the fact that a filter $\mathscr{F}$ on X has x_0 as a cluster point if and only if every derived net in X has x_0 as a cluster point.

3 Equivalence of $I-$cluster points of filters and nets

We first define the $I-$cluster points of nets in X.

Definition 9. *Let I be a non-trivial ideal of subsets of X. Let $\lambda : \mathscr{D} \to X$ be a net in X, where $\mathscr{D}$ is a directed set. Then a point $x_0 \in X$ is said to be an $I-$***cluster point** *of the net λ if for each nbd U of x_0, $\{\lambda(c) \in X : \lambda(c) \in U\} \notin I$.*

Theorem 12. *A filter $\mathscr{F}$ on X has x_0 as an $I-$cluster point if and only if every derived net λ of $\mathscr{F}$ has x_0 as an $I-$cluster point, where $I = I(\mathscr{F})$.*

Proof. Suppose a filter $\mathscr{F}$ on X has x_0 as an $I-$cluster point. Then for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I \cdots (*)$.
Let us index $\mathscr{F}$ with an index set $\mathscr{D}$ so that $\mathscr{F} = \{F_s : s \in \mathscr{D}\}$. Let us give some direction to $\mathscr{D}$ such that $c \geq d$ in $\mathscr{D}$ if and only if $F_c \subset F_d$. Let $\lambda : \mathscr{D} \to X$ be the derived net of $\mathscr{F}$ so obtained. This means that $\lambda(s) \in F_s$, for $s \in \mathscr{D}$. We have to show that x_0 is an $I-$cluster point of λ.

For this, let U be a nbd of x_0. We claim that $\{\lambda(c) \in X : \lambda(c) \in U\} \notin I$. So, let $\lambda(c) \in X$ such that $\lambda(c) \in U$. Now $\lambda(c) \in U \Rightarrow \{\lambda(c)\} \subset U$ and so $\{\lambda(c)\} \cap U \neq \emptyset$. By $(*)$, $\{\lambda(c)\} \notin I$. Therefore, x_0 is an $I-$cluster point of λ.

Conversely, suppose $\mathscr{F}$ is a filter on X such that every derived net λ of $\mathscr{F}$ has x_0 as an $I-$cluster point. Then for each nbd U of x_0, $\{\lambda(c) \in X : \lambda(c) \in U\} \notin I \cdots (**)$.
We have to show that x_0 is an $I-$cluster point of $\mathscr{F}$. For this, let U be a nbd of x_0. We claim that $U \notin I$. Suppose the contrary $U \in I$. From the given condition $(**)$, $\lambda(c) \in U \Rightarrow \{\lambda(c)\} \notin I$. Since $\mathscr{F}$ is a derived filter, $\lambda(c) \in F_c$, where $F_c \in \mathscr{F}$. Thus $\lambda(c) \in U \cap F_c \Rightarrow \{\lambda(c)\} \subset U \cap F_c$. Now, $U \cap F_c \subset U$, $U \in I$ and I is an ideal implies that $U \cap F_c \in I$ and so $\{\lambda(c)\} \in I$, a contradiction. Thus our supposition is wrong. Hence $U \notin I$. This proves that x_0 is an $I-$cluster point of $\mathscr{F}$.

Theorem 13. *Let $\lambda : \mathscr{D} \to X$ be a net in X and $\mathscr{F}$ be the derived filter of λ. Then x_0 is an $I-$cluster point of λ if and only if the derived filter $\mathscr{F}$ of λ has x_0 as an $I-$cluster point, where $I = I(\mathscr{F})$.*

Proof. Suppose x_0 is an $I-$cluster point of a net $\lambda : \mathscr{D} \to X$. Then for each nbd U of x_0, $\{\lambda(c) \in X : \lambda(c) \in U\} \notin I \cdots (*)$.

Let $\mathscr{F}$ be the derived filter of λ. Then $\mathscr{F} = \{F \subset X : \text{some tail of } \lambda \text{ is contained in } F\}$. We have to show that x_0 is an $I-$cluster point of $\mathscr{F}$. For this, let U be a nbd of x_0. We claim that $U \notin I$. Suppose the contrary $U \in I$. Then $X \setminus U \in \mathscr{F}$. Since $\mathscr{F}$ is a derived filter, there exists $d \in \mathscr{D}$ such that $\Lambda_d \subset X \setminus U$, where $\Lambda_d = \{\lambda(c) : c \geq d \text{ in } \mathscr{D}\}$ is the tail of λ. This means that $U \cap \Lambda_d = \emptyset$. Since $\Lambda_d = \{\lambda(c) : c \geq d \text{ in } \mathscr{D}\}$, ther is some $t < d$ in $\mathscr{D}$ such that $\lambda(t) \in U$ but $\lambda(t) \notin \Lambda_d$. Now $\lambda(t) \in U \Rightarrow \{\lambda(t)\} \subset U$. Also, $U \in I$ and I is an ideal implies that $\{\lambda(t)\} \in I$, which contradicts $(*)$. Therefore, x_0 is an $I-$cluster point of $\mathscr{F}$.

Conversely, suppose $\lambda : \mathscr{D} \to X$ is a net in X and $\mathscr{F}$ be the derived filter of λ such that $\mathscr{F}$ has x_0 as an $I-$cluster point. Then for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I \cdots (**)$.

We have to show that x_0 is an $I-$cluster point of λ. For this, let U be a nbd of x_0. We claim that $\{\lambda(c) \in X : \lambda(c) \in U\} \notin I$. So let $\lambda(c) \in X$ such that $\lambda(c) \in U$. Now $\lambda(c) \in U \Rightarrow \{\lambda(c)\} \subset U$. Therefore, $\{\lambda(c)\} \cap U \neq \emptyset$ and so by $(**)$, $\{\lambda(c)\} \notin I$. Therefore, x_0 is an $I-$cluster point of λ.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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