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$I-$cluster points of filters

Rohini Jamwal[1], *Renu*[2] *and Dalip Singh Jamwal*[3]

[1]Research Scholar, Department of Mathematics, University of Jammu, Jammu, India
[2]Department of Mathematics, GLDM College, Hiranagar, India
[3]Department of Mathematics, University of Jammu, Jammu, India

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Abstract: In this paper, we have introduced the concept of $I-$cluster point of a filter on a topological space and studied its various properties. We have proved the necessary condition for a filter to have an $I-$cluster point. Most of the work in this paper is inspired from [2] and [23].

Keywords: Filters, nets, convergence, cluster point, ideal.

1 Introduction

The concept of convergence of a sequence of real numbers was extended to statistical convergence independently by H. Fast [4] and I. J. Schoenberg [22]. Kostyrko et. al. in [8], [9] generalized the notion of statistical convergence and introduced the concept of $I-$convergence of real sequences which is based on the structure of the ideal I of subsets of the set of natural numbers. Mursaleen et. al. [14] defined and studied the notion of ideal convergence in random $2-$normed spaces and construct some interesting examples. Several works on $I-$convergence and statistical convergence have been done in [1], [3[, [6], [7], [8], [9], [10], [13], [14], [15], [16], [17], [21]. The idea of $I-$convergence was extended from real number space to metric space by Kostyrko et. al [8] and to a normed linear space by Šal*á*t et. al [20] in their recent works. Later the idea of $I-$convergence was extended to an arbitrary topological space by B. K. Lahiri and P. Das [11]. It was observed that the basic properties remained preserved in a topological space. Lahiri and Das [12] introduced the idea of $I-$convergence of nets in a topological space and examined how far it affects the basic properties.

Taking the idea of I and I^*-convergence of nets by Lahiri and Das in [12], Jamwal et. al introduced the concept of $I-$convergence of filters and studied its various properties in [6]. In [7], Jamwal et. al reintroduced the concept of $I-$convergence of nets in a topological space and estabilished the equivalence of $I-$convergences of nets and filters on a topological space.
We recall the following definitions:

Definition 1. *Let X be a non-empty set. Then a family $\mathscr{F} \subset 2^X$ is called a* **filter** *on X if*

(i) $\emptyset \notin \mathscr{F}$,
(ii) $A, B \in \mathscr{F}$ *implies* $A \cap B \in \mathscr{F}$ *and*
(iii) $A \in \mathscr{F}, B \supset A$ *implies* $B \in \mathscr{F}$.

Definition 2. *Let X be a non-empty set. Then a family $I \subset 2^X$ is called an* **ideal** *of X if*

(i) $\emptyset \in I$,

* Corresponding author e-mail: rohinijamwal121@gmail.com

(ii) $A,B \in I$ *implies* $A \cup B \in I$ *and*
(iii) $A \in I, B \subset A$ *implies* $B \in I$.

Definition 3. *Let X be a non-empty set. Then a filter $\mathscr{F}$ on X is said to be* **non-trivial** *if* $\mathscr{F} \neq \{X\}$.

Definition 4. *Let X be a non-empty set. Then an ideal I of X is said to be* **non-trivial** *if* $I \neq \{\emptyset\}$ *and* $X \notin I$.

Note(i) If I is an ideal of a set X, then $\mathscr{F} = \mathscr{F}(I) = \{A \subset X : X \setminus A \in I\}$ is a filter on X, called the **filter associated with the ideal** I.
(ii) $I = I(\mathscr{F}) = \{A \subset X : X \setminus A \in \mathscr{F}\}$ is an ideal of X, called the **ideal associated with the filter** $\mathscr{F}$.
(iii) A non-trivial ideal I of X is called **admissible** if I contains all the singleton subsets of X.
We now recall some of the results on $I-$convergence of filters and nets in a topological space which are proved in [6], [7]. Throughout this paper, $X = (X,\tau)$ will stand for a topological space and $I = I(\mathscr{F})$ will be the ideal associated with the filter $\mathscr{F}$ on X.

Definition 5. *A filter $\mathscr{F}$ on X is said to be $I-$***convergent** *to $x_0 \in X$ if for each nbd U of x_0, $\{y \in X : y \notin U\} \in I$.*
*In this case, x_0 is called an $I-$***limit of** *$\mathscr{F}$ and is written as $I - lim\mathscr{F} = x_0$.*

Notation In case more than one filters is involved, we use the notation $I(\mathscr{F})$ to denote the ideal associated with the corresponding filter $\mathscr{F}$.

Lemma 1. *Let $\mathscr{F}$ and $\mathscr{G}$ be two filters on X. Then $\mathscr{F} \subset \mathscr{G}$ if and only if $I(\mathscr{F}) \subset I(\mathscr{G})$.*

Proposition 1. *Let $\mathscr{F}$ be a filter on X such that $I - \lim\mathscr{F} = x_0$. Then every filter $\mathscr{F}'$ finer than $\mathscr{F}$ also $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Proposition 2. *Let $\mathscr{F}$ be a filter on X such that $I - \lim\mathscr{F} = x_0$. Then every filter $\mathscr{F}'$ on X coarser than $\mathscr{F}$ also $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Proposition 3. *Let $\mathscr{F}$ be a filter on X and $\mathscr{G}$ be any other filter on X finer than $\mathscr{F}$. Then $I(\mathscr{F}) - \lim\mathscr{G} = x_0$ implies $I(\mathscr{G}) - \lim\mathscr{G} = x_0$.*

Proposition 4. *Let τ_1 and τ_2 be two topologies on X such that τ_1 is coarser than τ_2. Let $\mathscr{F}$ be a filter on X such that $I - \lim\mathscr{F} = x_0$ w.r.t τ_2. Then $I - \lim\mathscr{F} = x_0$ w.r.t τ_1.*

Lemma 2. *Let $\mathscr{M} = \{\mathscr{G} : \mathscr{G}\ is\ a\ filter\ on\ X\}$. Then $\mathscr{F} = \cap_{\mathscr{G}\in\mathscr{M}}\mathscr{G}$ if and only if $I(\mathscr{F}) = \cap_{\mathscr{G}\in\mathscr{M}} I(\mathscr{G})$.*

Proposition 5. *Let $\mathscr{M}$ be a collection of all those filters $\mathscr{G}$ on a space X which $I(\mathscr{G})-$converges to the same point $x_0 \in X$. Then the intersection $\mathscr{F}$ of all the filters in $\mathscr{M}$ $I(\mathscr{F})-$converges to x_0.*

Lemma 3. *If I_X is an ideal of $X = \prod_{\alpha\in\Lambda} X_\alpha$ associated with a filter $\mathscr{F}$ on X, then $I_X = \cap_{i=1}^{n} p_{\alpha_i}^{-1}(I_{X_{\alpha_i}})$, where $I_{X_{\alpha_i}}$ is an ideal of the factor space X_{α_i} associated with $p_{\alpha_i}(\mathscr{F})$.*

Theorem 1. *A filter $\mathscr{F}$ on X $I-$converges to $x_0 \in X$ if and only if every derived net λ of $\mathscr{F}$ converges to x_0.*

Theorem 2. *A filter $\mathscr{F}$ on X $I-$converges to $x_0 \in X$ if and only if $\mathscr{F}$ converges to x_0.*

2 $I-$Cluster Points of Filters

We begin this section with the definition of $I-$cluster point of a filter with some examples.

Definition 6. *A point $x_0 \in X$ is called an $I-$***cluster point** *of a filter $\mathscr{F}$ on X if for each nbd U of x_0, $\{y \in X : y \in U\} \notin I$. In other words, $x_0 \in X$ is called an $I-$cluster point of $\mathscr{F}$ if $U \notin I$, for each nbd U of x_0.*

Equivalently, x_0 is an $I-$cluster point of $\mathscr{F}$ if for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \subset V\} \nsubseteq I$.

Example 1. Let $X = \{1,2,3\}$ and τ be the discrete topology on X.
Let $\mathscr{F} = \{\{1\},\{1,2\},\{1,3\},X\}$ be a filter on X. Then $I = \{\emptyset,\{2\},\{3\},\{2,3\}\}$ is the ideal associated with $\mathscr{F}$.
It is easy to see that 1 is the only $I-$cluster point of $\mathscr{F}$.

Example 2. Let $\mathscr{U}_{x_0}$ be the nbd filter at a point x_0 in X. Then clearly for each nbd U of x_0, $\{y \in X : y \in U\} \notin I$, as $I = I(\mathscr{U}_{x_0})$. Thus x_0 is the $I-$cluster point of $\mathscr{U}_{x_0}$.

Example 3. Let $\mathscr{F}$ be a filter on an indiscrete space X. Then clearly, each $x_0 \in X$ is an $I-$cluster point of $\mathscr{F}$ as X is the only nbd of $x_0 \in X$ and $\{y \in X : y \in X\} = X \notin I$.

Notation Let $I(C_{\mathscr{F}})$ and $I(L_{\mathscr{F}})$ respectively denote the set of all $I-$cluster points and the set of all $I-$limits of a filter $\mathscr{F}$ on X.

We have the following theorem estabilishing the relationship between $I-$limits and $I-$cluster points of a filter $\mathscr{F}$ on X.

Theorem 3. *With usual notations, $I(L_{\mathscr{F}}) \subset I(C_{\mathscr{F}})$.*

Proof. Let $x_0 \in I(L_{\mathscr{F}})$. Then for each nbd U of x_0, $\{y \in X : y \notin U\} \in I$. That is, $X \setminus U \in I \cdots (*)$. We have to show that $x_0 \in I(C_{\mathscr{F}})$. For this, let U be a nbd of x_0. We claim that $\{y \in X : y \in U\} \notin I$. That is, $U \notin I$. Suppose $U \in I$. From $(*)$, $X \setminus U \in I$. Since I is an ideal, we have $U \cup (X \setminus U) \in I$. That is, $X \in I$, which contradicts the fact that I is non-trivial ideal of X. Thus our supposition is wrong. Hence $U \notin I$ and so $x_0 \in I(C_{\mathscr{F}})$. This proves that $I(L_{\mathscr{F}}) \subset I(C_{\mathscr{F}})$.
Note The converse of the above theorem is however not true. For this, we have the following example.

Example 4. Let $X = \{1,2,3\}$ and $\tau = \{\emptyset,\{3\},\{2,3\},X\}$.Let $\mathscr{F} = \{\{1,2\},X\}$ be a filter on X. Then $I = \{\emptyset,\{3\}\}$ is the ideal associated with filter $\mathscr{F}$. Then it is easy to see that $I(L_{\mathscr{F}}) = \{1\}$ and $I(C_{\mathscr{F}}) = \{1,2\}$. So, $I(C_{\mathscr{F}}) \nsubseteq I(L_{\mathscr{F}})$.

We now give the necessary condition for a filter $\mathscr{F}$ on X to have an $I-$cluster point.

Theorem 4. *Let $\mathscr{F}$ be a filter on X. If x_0 is an $I-$cluster point of $\mathscr{F}$, then for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I$.*

Proof. Suppose $\mathscr{F}$ has x_0 as an $I-$cluster point. This means that for each nbd U of x_0, $\{y \in X : y \in U\} \notin I$. That is, $U \notin I \cdots (*)$. Let U be a nbd of x_0. We have to show that $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I$. We observe that $U \in \mathscr{P}(X)$ such that $U \cap U \neq \emptyset$ and also by $(*)$, $U \notin I$. Thus it follows that $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I$.

Remark. The condition, for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I$ is not the sufficient condition for a filter $\mathscr{F}$ to have an $I-$cluster point. Consider the following example.

Example Let $X = \{1,2,3\}$ and $\tau = \{\emptyset,\{2\},\{2,3\},X\}$ be a topology on X. Let $\mathscr{F} = \{\{1\},\{1,2\},\{1,3\},X\}$ be a filter on X.
Then $I = I(\mathscr{F}) = \{\emptyset,\{2\},\{3\},\{2,3\}\}$ is an ideal of X.
We see that nbds of 2 are $\{2\},\{1,2\},\{2,3\}$ and X. We observe that for each nbd U of 2, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I$.

But 2 is not an $I-$cluster point of $\mathscr{F}$. This is because $\{2\}$ is a nbd of 2 and $\{2\} \in I$.

This shows that the condition, for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I$ is not the sufficient condition for a filter $\mathscr{F}$ to have an $I-$cluster point.

Proposition 6. *Let $\mathscr{F}$ be a filter on a non-discrete space X. Then $\mathscr{F}$ has x_0 as an $I(\mathscr{F})-$cluster point if and only if there is a filter $\mathscr{G}$ on X finer than $\mathscr{F}$ such that $I(\mathscr{G}) - \lim \mathscr{G} = x_0$.*

Proof. Suppose x_0 is an $I(\mathscr{F})-$cluster point of $\mathscr{F}$. Then for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I(\mathscr{F})$. Since $X \setminus F \in I(\mathscr{F})$, $\forall\, F \in \mathscr{F}$, we find that $U \cap F \neq \emptyset, \forall\, U \in \mathscr{U}_{x_0}$ and $F \in \mathscr{F}$. Let $\mathscr{B} = \{U \cap F : U \in \mathscr{U}_{x_0} \textit{ and } F \in \mathscr{F}\}$. Then clearly, $\mathscr{B}$ is a non-empty family of non-empty subsets of X which is closed under finite intersection and so a filter base for some filter, say $\mathscr{G}$ on X. If $G \in \mathscr{F}$, then $G \supset U \cap G$ and so $G \in \mathscr{G}$. This implies that $\mathscr{F} \subset \mathscr{G}$. Therefore $\mathscr{G}$ is finer than $\mathscr{F}$. By Lemma 1, $I(\mathscr{F}) \subset I(\mathscr{G})$. We shall show that $I(\mathscr{G}) - \lim \mathscr{G} = x_0$. For this, we need to prove that for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I(\mathscr{G})$. So, let U be a nbd of x_0 and $V \in \mathscr{P}(X)$ such that $U \cap V = \emptyset$. Now $U \cap V = \emptyset$
$\Rightarrow U \subset X \setminus V$
$\Rightarrow U \cap (X \setminus V) \neq \emptyset$.
Also, $U \cap U \subset U \cap (X \setminus V)$. That is, $U \subset U \cap (X \setminus V)$. Now, $U \cap F \subset U$, for all $F \in \mathscr{F}$ and $U \subset U \cap (X \setminus V)$ implies that $U \cap F \subset U \cap (X \setminus V)$, for all $F \in \mathscr{F}$. Also, $U \cap (X \setminus V) \subset X \setminus V$. Thus we have $U \cap F \subset X \setminus V$, for all $F \in \mathscr{F}$. Since $\mathscr{B}$ is a base for $\mathscr{G}$, $X \setminus V \in \mathscr{G}$. This implies that $V \in I(\mathscr{G})$. Therefore, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I(\mathscr{G})$.

Conversely, suppose there is a filter $\mathscr{G}$ on X finer than $\mathscr{F}$ such that $I(\mathscr{G}) - \lim \mathscr{G} = x_0$. We have to show that x_0 is the $I(\mathscr{F})-$cluster point of $\mathscr{F}$. For this, let U be a nbd of x_0. We claim that $\{y \in X : y \in U\} \notin I(\mathscr{F})$. That is, $U \notin I(\mathscr{F})$. Suppose $U \in I(\mathscr{F})$. Since $I(\mathscr{F}) \subset I(\mathscr{G})$, we find that $U \in I(\mathscr{G})$. Since $I(\mathscr{G}) - \lim \mathscr{G} = x_0$, $U \in I(\mathscr{G})$ implies $U \cap U = \emptyset$, which is not possible. Therefore, $U \notin I(\mathscr{F})$. Thus $\{y \in X : y \in U\} \notin I(\mathscr{F})$. Therefore, x_0 is an $I(\mathscr{F})-$cluster point of $\mathscr{F}$.

Remark. (a) The above Proposition 6 need not be true if X has the discrete topology. Consider the example:
Let $X = \{1,2,3\}$ and τ be the discrete topology on X. Let $\mathscr{F} = \{\{1,2\}, X\}$ be a filter on X. Then $I(\mathscr{F}) = \{\phi, \{3\}\}$. Let $\mathscr{G} = \{\{1\}, \{1,2\}, \{1,3\}, X\}$ be a filter on X finer than $\mathscr{F}$. Then $I(\mathscr{G}) = \{\phi, \{2\}, \{3\}, \{2,3\}\}$. We can easily see that
$I(\mathscr{F})-$cluster points of $\mathscr{F} = 1,2$.
$I(\mathscr{F})-$limit of $\mathscr{G} =$ nil.
$I(\mathscr{G})-$limit of $\mathscr{G} = 1$ and
$I(\mathscr{G})-$cluster points of $\mathscr{F} = 1$.
We observe that 1 and 2 are $I(\mathscr{F})-$cluster points of $\mathscr{F}$ but 2 is not the $I(\mathscr{G})-$limit of $\mathscr{G}$.
(b) The above Proposition 6 is again not true if we take both the ideals to be $I(\mathscr{F})$. From the above example, we can see that 1 and 2 are $I(\mathscr{F})-$cluster points of $\mathscr{F}$ but there is no $I(\mathscr{F})-$limit of $\mathscr{G}$.

The above Remark 2 motivated us to have the following proposition:

Proposition 7. *Let $\mathscr{F}$ be a filter on X such that $\mathscr{F}$ has $x_0 \in X$ as an $I-$cluster point. Then every filter $\mathscr{F}'$ finer than $\mathscr{F}$ also has x_0 as an $I-$cluster point, where $I = I(\mathscr{F})$.*

Proof. Suppose $\mathscr{F}$ is a filter on X such that x_0 is an $I-$cluster point of $\mathscr{F}$. Then for each nbd U of x_0, $U \notin I \cdots (*)$. Let $\mathscr{F}'$ be an arbitrary filter on X such that $\mathscr{F}' \supset \mathscr{F}$. We shall show that $I-$cluster point of $\mathscr{F}' = x_0$, where $I = I(\mathscr{F})$. For this, let U be a nbd of x_0. Then clearly by $(*)$, $U \notin I$. Hence the proof.

Remark. Let $\mathscr{F}$ be a filter on X and $\mathscr{F}'$ be a filter on X finer than $\mathscr{F}$. Then $I(\mathscr{F})-$cluster point of $\mathscr{F} = x_0$ need not imply that $I(\mathscr{F}')-$cluster point of $\mathscr{F}' = x_0$. Consider the example in Remark 2.

We can see that 1 and 2 are $I(\mathscr{F})-$cluster points of $\mathscr{F}$. But 2 is not an $I(\mathscr{F}')-$cluster point of $\mathscr{F}'$.

Proposition 8. *Let $\mathscr{F}$ be a filter on X such that $\mathscr{F}$ has $x_0 \in X$ as an $I-$cluster point. Then every filter $\mathscr{F}'$ coarser than $\mathscr{F}$ also has x_0 as an $I-$cluster point, where $I = I(\mathscr{F})$.*

Proof. Suppose $\mathscr{F}$ is a filter on X such that x_0 is an $I-$cluster point of $\mathscr{F}$. Then for each nbd U of x_0, $\{y \in X : y \in U\} \notin I \cdots (*)$. Let $\mathscr{F}'$ be an arbitrary filter on X such that $\mathscr{F}' \subset \mathscr{F}$. We shall show that $I-$cluster point of $\mathscr{F}' = x_0$, where $I = I(\mathscr{F})$. For this, let U be a nbd of x_0. We claim that $\{y \in X : y \in U\} \notin I$. But it follows clearly by $(*)$. Hence the proof.

Proposition 9. *Let τ_1 and τ_2 be two topologies on X such that τ_1 is coarser than τ_2. Let $\mathscr{F}$ be a filter on X such that x_0 is an $I-$cluster point of $\mathscr{F}$ w.r.t τ_2. Then x_0 is also an $I-$cluster point of $\mathscr{F}$ w.r.t τ_1.*

Proof. Let U be a nbd of x_0 w.r.t τ_1. Since $\tau_1 \subset \tau_2$, U is also a nbd of x_0 w.r.t τ_2. But x_0 is an $I-$cluster point of $\mathscr{F}$ w.r.t τ_2. Thus for above nbd U of x_0, $U \notin I$. Hence x_0 is also an $I-$cluster point of $\mathscr{F}$ w.r.t τ_1.

Remark. The converse of above proposition need not be true. That is, if τ_1 and τ_2 are two topologies on X such that τ_1 is coarser than τ_2 and x_0 is an $I-$cluster point of $\mathscr{F}$ w.r.t τ_1, then x_0 need not be an $I-$cluster point of $\mathscr{F}$ w.r.t τ_2. Consider the following example:
Let $X = \{1,2,3\}$. Suppose τ_2 is the discrete topology on X and $\tau_1 = \{\emptyset, \{2\}, X\}$. Then $\tau_1 \subset \tau_2$. Let $\mathscr{F} = \{\{1\},\{1,2\},\{1,3\},X\}$ be a filter on X. Then $I(\mathscr{F}) = \{\phi, \{2\}, \{3\}, \{2,3\}\}$ is the ideal associated with $\mathscr{F}$. It is easy to see that 1 and 3 are the $I-$cluster points of $\mathscr{F}$ w.r.t τ_1. But 3 is not an $I-$cluster point of $\mathscr{F}$ w.r.t τ_2.

Proposition 10. *Let $\mathscr{M}$ be a collection of all those filters $\mathscr{G}$ on a space X which have $x_0 \in X$ as an $I(\mathscr{G})-$cluster point. Then the intersection $\mathscr{F}$ of all the filters in $\mathscr{M}$ also has x_0 as an $I(\mathscr{F})-$cluster point.*

Proof. Here $\mathscr{M} = \{\mathscr{G} : \mathscr{G}\ is\ a\ filter\ on\ X\ such\ that\ I(\mathscr{G}) - \text{cluster point of } \mathscr{G} = x_0\}$. Let $\mathscr{F} = \bigcap\{\mathscr{G} : \mathscr{G} \in \mathscr{M}\}$. We shall show that x_0 is an $I(\mathscr{F})-$cluster point of $\mathscr{F}$. For this, let U be a nbd of x_0(w.r.t $\mathscr{F}$). Then U is a nbd of x_0(w.r.t all $\mathscr{G} \in \mathscr{M}$). Since x_0 is an $I(\mathscr{G})-$cluster point of $\mathscr{G}$, $\forall\, \mathscr{G} \in \mathscr{M}$, it follows that $\{y \in X : y \in U\} \notin I(\mathscr{G})$, $\forall\, \mathscr{G} \in \mathscr{M}$ and so $\{y \in X : y \in U\} \notin \cap_{\mathscr{G} \in \mathscr{M}} I(\mathscr{G})$. By Lemma 1, $\{y \in X : y \in U\} \notin I(\mathscr{F})$. Hence x_0 is also an $I(\mathscr{F})-$cluster point of $\mathscr{F}$.
In view of Remark 2, we have the following proposition:

Proposition 11. *Let $\mathscr{F}$ be a filter on X and $\mathscr{G}$ be a filter on X finer than $\mathscr{F}$. Then $\mathscr{F}$ has x_0 as an $I(\mathscr{G})-$cluster point if and only if $I(\mathscr{G}) - \lim \mathscr{G} = x_0$.*

Proof. Let $\mathscr{F}$ be a filter on X and $\mathscr{G}$ be a filter on X finer than $\mathscr{F}$ such that x_0 is an $I(\mathscr{G})-$cluster point of $\mathscr{F}$. Let $\mathscr{B}$ be a base for $\mathscr{G}$. Then $\mathscr{G} = \{G \subset X : B \subset G, \text{ for some } B \in \mathscr{B}\}$. We shall show that $I(\mathscr{G}) - \lim \mathscr{G} = x_0$. For this, let U be a nbd of x_0. We shall show that $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I(\mathscr{G})$. So, let $V \in \mathscr{P}(X)$ such that $U \cap V = \emptyset$. Now, $U \cap V = \emptyset$ implies $U \subset X \setminus V$ which further implies that $U \cap U \subset U \cap (X \setminus V)$. That is, $U \subset U \cap (X \setminus V)$. Also, $U \cap G \subset U$, for all $G \in \mathscr{G}$. Thus $U \cap G \subset U \cap (X \setminus V)$, for all $G \in \mathscr{G}$. But $U \cap (X \setminus V) \subset X \setminus V$. Thus $U \cap G \subset X \setminus V$, for all $G \in \mathscr{G}$. Since $\mathscr{B}$ is a base for $\mathscr{G}$, $U \cap G \in \mathscr{B}$ and so $X \setminus V \in \mathscr{G}$. Hence $V \in I(\mathscr{G})$. This proves that $I(\mathscr{G}) - \lim \mathscr{G} = x_0$.

Conversely, suppose $I(\mathscr{G}) - \lim \mathscr{G} = x_0$. By using Theorem 3, we find that x_0 is also an $I(\mathscr{G})-$cluster point of $\mathscr{G}$. Since $\mathscr{F}$ is coarser than $\mathscr{G}$, by Proposition 8, it follows that x_0 is also $I(\mathscr{G})-$cluster point of $\mathscr{F}$.

Theorem 5. *Let $f : X \to Y$ be a surjective map. Let $\mathscr{F}$ be a filter on X. Then $f : X \to Y$ is continuous at $x_0 \in X$ if and only if whenever x_0 is an I_X-cluster point of $\mathscr{F}$, then $f(x_0)$ is an I_Y-cluster point of $f(\mathscr{F})$, where $I_X = I_X(\mathscr{F})$ is the ideal associated with $\mathscr{F}$ and $I_Y = I_Y(f(\mathscr{F}))$ is the ideal associated with the filter $f(\mathscr{F})$ on Y.*

Proof. First suppose that the surjection $f : X \to Y$ is continuous at x_0 in X. Let x_0 be an I_X-cluster point of $\mathscr{F}$ in X. We have to show that $f(x_0)$ is an I_Y-cluster point of $f(\mathscr{F})$.

For this, let V be a nbd of $f(x_0)$ in Y. Since f is continuous at x_0, for above nbd V of $f(x_0)$ in Y, there is a nbd U of x_0 in

X such that $f(U)\subset V$. Since x_0 is an I_X-cluster point of $\mathscr{F}$, for above nbd U of x_0 in X, $U\notin I_X$ and so $X\setminus U\notin\mathscr{F}\cdots(1)$.
We claim that $V\notin I_Y$. Then $Y\setminus V\in f(\mathscr{F})\cdots(2)$.
Now $f(U)\subset V$ implies $Y\setminus V\subset Y\setminus f(U)\cdots(3)$.

Since $f(\mathscr{F})$ is a filter on Y, from (2) and (3), we get $Y\setminus f(U)\in f(\mathscr{F})\cdots(4)$. From (1), $X\setminus U\notin\mathscr{F}$ implies $f(X\setminus U)\notin f(\mathscr{F})$. But $f(X\setminus U)\supset f(X)\setminus f(U)=Y\setminus f(U)$. That is, $Y\setminus f(U)\subset f(X\setminus U)$. Now $f(X\setminus U)\notin f(\mathscr{F})$ and $Y\setminus f(U)\subset f(X\setminus U)$ implies that $Y\setminus f(U)\notin f(\mathscr{F})$, which contradicts (4). Thus our supposition is wrong.

Hence $f(x_0)$ is an I_Y-cluster point of Y.

Conversely, suppose $f:X\to Y$ is a surjection such that the given condition holds. We have to show that f is continuous at x_0. Suppose not. This means that there is a nbd V of $f(x_0)$ in Y such that $f^{-1}(V)$ is not a nbd of x_0.

Let $\mathscr{F}=\{U\setminus f^{-1}(V):U$ is a nbd of x_0 in $X\}\cdots(5)$. Then clearly, $\mathscr{F}$ is a filter on X. We claim that x_0 is an I_X-cluster point of $\mathscr{F}$. For this, let T be a nbd of x_0. We shall show that $T\notin I_X$.

Suppose the contrary $T\in I_X$. Then $X\setminus T\in\mathscr{F}$. Also by (5), $T\setminus f^{-1}(V)\in\mathscr{F}$. Since $\mathscr{F}$ is a filter on X, we have $\emptyset=(X\setminus T)\cap(T\setminus f^{-1}(V))\in\mathscr{F}$, i.e., $\emptyset\in\mathscr{F}$, which is not possible. Thus $T\notin I_X$. Therefore, x_0 is an I_X-cluster point of $\mathscr{F}$. By the given condition, $f(x_0)$ is an I_Y-cluster point of $f(\mathscr{F})$. So $V\notin I_Y$ i.e., $Y\setminus V\notin f(\mathscr{F})$. Now $Y\setminus V\notin f(\mathscr{F})$ implies $f^{-1}(Y\setminus V)\notin\mathscr{F}$. This further implies that $X\setminus f^{-1}(V)\notin\mathscr{F}$, which contradicts (5). Thus our supposition is wrong.

Hence f is continuous at x_0.

Remark. The above Theorem 5 holds even if f is not surjective. In that case, we shall assume $f(\mathscr{F})$ to be a filter on Y generated by the filter base $\{f(F):F\in\mathscr{F}\}$.

Theorem 6. *A filter $\mathscr{F}$ on $X=\prod_{\alpha\in\Lambda}X_\alpha$ has x as an I_X-cluster point if and only if $p_\alpha(\mathscr{F})$ has $p_\alpha(x)$ as an $I_{X_\alpha}-$cluster point, $\forall\alpha\in\Lambda$, where $I_X=I_X(\mathscr{F})$ and $I_{X_\alpha}=I_{X_\alpha}(p_\alpha\mathscr{F})$.*

Proof. Suppose $\mathscr{F}$ has x as an I_X-cluster point in $X=\prod_{\alpha\in\Lambda}X_\alpha$. Since each projection $p_\alpha:X\to X_\alpha$ is continuous at x in X, by above Theorem 5, we find that $p_\alpha(x)$ is an $I_{X_\alpha}-$cluster point of $p_\alpha(\mathscr{F})$ in $X_\alpha,\forall\ \alpha$.

Conversely, suppose $p_\alpha(x)$ is an $I_{X_\alpha}-$cluster point of $p_\alpha(\mathscr{F})$ in $X_\alpha,\forall\ \alpha$. We have to show that x is an I_X-cluster point of $\mathscr{F}$ in X. For this, let $U=\cap_{i=1}^{n}p_{\alpha_i}^{-1}U(\alpha_i)$ be a basic nbd of x. This means that U_{α_i} is a nbd of $x_{\alpha_i}=p_{\alpha_i}(x)$, for $i=1,2,\ldots,n$ in X_{α_i}. We claim that $U\notin I_X$. Since $p_\alpha(x)$ is an $I_{X_{\alpha_i}}-$cluster point of $p_\alpha(\mathscr{F})$ in X_{α_i}, we have $U_{\alpha_i}\notin I_{X_{\alpha_i}},\ \forall\ i=1,2,\ldots,n$. This further implies that $p_{\alpha_i}^{-1}(U_{\alpha_i})\notin p_{\alpha_i}^{-1}(I_{X_{\alpha_i}}),\ \forall\ i=1,2,\ldots,n$. Clearly, $\cap_{i=1}^{n}p_{\alpha_i}^{-1}(U_{\alpha_i})\notin\cap_{i=1}^{n}p_{\alpha_i}^{-1}(I_{X_{\alpha_i}})=I_X$, by Lemma 3. That is, $U\notin I_X$.

This proves that x is an I_X-cluster point of $\mathscr{F}$ in X.

Theorem 7. *Let X be a Lindelöf space such that every filter on X has an $I-$cluster point, where I is an admissible ideal of X. Then X is compact.*

Proof. Let X be a *Lindelöf* space such that every filter on X has an $I-$cluster point, where I is an admissible ideal of X. We have to show that X is compact. For this, let $\mathfrak{U}=\{U_\alpha:\alpha\in\Lambda\}$ be an open cover of X, where Λ is an index set. Since X is *Lindelöf*, the above open cover $\mathfrak{U}$ of X has a countable subcover, say $\mathfrak{U}'=\{U_1,U_2,\cdots,U_n,\cdots\}$. Proceeding inductively, let $V_1=U_1$ and for each $m>1$, let V_m be the first member of $\mathfrak{U}'$ which is not covered by $V_1\cup V_2\cup\cdots\cup V_{m-1}$.

After some finite number of steps, the set of above $V_i's$ selected becomes a required finite subcover. Otherwise, we can choose a point $v_n \in V_n$, for each positive integer n such that $v_n \notin V_r$, for $r < n \cdots (*)$. Consider a net $\lambda = (v_n)_{n\in\mathbb{N}}$. Let $\mathscr{F}$ be the derived filter of λ. That is, $\mathscr{F} = \{F \subset X : \lambda$ is eventually in $F\}$. By λ eventually in F, we mean that some tail $\Lambda_m = \{\lambda(n) = v_n : n \geq m$ in $\mathbb{N}\}$ of λ is contained in F. Let x_0 be an $I-$cluster point of $\mathscr{F}$. Then $x_0 \in V_p$, for some p. By definition of $I-$cluster point of $\mathscr{F}$, in particular for V_p, $\{y \in X : y \in V_p\} \notin I$. Since I is an admissible ideal, $\{y \in X : y \in V_p\}$ must be infinite subset of X. So, there exists some $n > p$ such that $v_n \in \{y \in X : y \in V_p\}$. That is, there exists some $n > p$ such that $v_n \in V_p$, which contradicts $(*)$. Thus the above set of $V_i's$ form the required finite subcover. Hence X is compact.

Theorem 8. *A topological space X is compact if and only if every filter on X has an $I-$cluster point.*

Proof. First suppose X is compact. Let $\mathscr{F}$ be a filter on X. Consider a family $\{\overline{F} : F \in \mathscr{F}\}$ of closed subsets of X. Since X is compact, the family $\{\overline{F} : F \in \mathscr{F}\}$ has finite intersection property. That is, $\bigcap\{\overline{F} : F \in \mathscr{F}\} \neq \emptyset$. Let $x_0 \in \bigcap\{\overline{F} : F \in \mathscr{F}\}$. Then for each nbd U of x_0, $U \cap F \neq \emptyset, \forall\, F \in \mathscr{F}$. We claim that $U \notin I$. Suppose that $U \in I$. Then $U \cap F \neq \emptyset, \forall\, F \in \mathscr{F}$ and for each nbd U of x_0 would contradict the fact that $I = I(\mathscr{F})$. This proves that x_0 is an $I-$cluster point of $\mathscr{F}$.

Conversely, suppose that every filter on X has an $I-$cluster point. We have to show that X is compact. Suppose X is not compact and let $\mathfrak{U}$ be an open cover of X with no finite subcover. Let $\mathscr{B} = \{X \setminus \cup_{i=1}^{n} U_i : U_i \in \mathfrak{U},$ for $i = 1, 2, \ldots, n\}$. Then clearly, $\mathscr{B}$ is a non-empty family of non-empty subsets of X which is closed under finite intersection and so a filter base for some filter, say $\mathscr{F}$ on X. By the given condition, $\mathscr{F}$ has an $I(\mathscr{F})-$cluster point, say x_0. This means that for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V \neq \emptyset\} \nsubseteq I(\mathscr{F}) \cdots (*)$. Let $U \in \mathfrak{U}$ such that $x_0 \in U$. Now, $U \in \mathfrak{U}$ implies that $X \setminus U \in \mathscr{B}$ and so $X \setminus U \in \mathscr{F}$. Now, $X \setminus U \in \mathscr{F}$ implies $U \in I(\mathscr{F})$. Finally, $x_0 \in U$ and $U \in I(\mathscr{F})$ implies that $\{x_0\} \in I(\mathscr{F})$, which contradicts $(*)$ with $V = \{x_0\}$. Thus our supposition is wrong. Hence X is compact.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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