I-cluster points of filters

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Abstract: In this paper, we have introduced the concept of I-cluster point of a filter on a topological space and studied its various properties. We have proved the necessary condition for a filter to have an I-cluster point. Most of the work in this paper is inspired from [2] and [23].

Keywords: Filters, nets, convergence, cluster point, ideal.

1 Introduction

The concept of convergence of a sequence of real numbers was extended to statistical convergence independently by H. Fast [4] and I. J. Schoenberg [22]. Kostyrko et. al. in [8], [9] generalized the notion of statistical convergence and introduced the concept of I–convergence of real sequences which is based on the structure of the ideal I of subsets of the set of natural numbers. Mursaleen et. al. [14] defined and studied the notion of ideal convergence in random 2–normed spaces and construct some interesting examples. Several works on I–convergence and statistical convergence have been done in [1], [3], [6], [7], [8], [9], [10], [13], [14], [15], [16], [17], [21]. The idea of I–convergences was extended from real number space to metric space by Kostyrko et. al [8] and to a normed linear space by Šalát et. al [20] in their recent works. Later the idea of I–convergence was extended to an arbitrary topological space by B. K. Lahiri and P. Das [11]. It was observed that the basic properties remained preserved in a topological space. Lahiri and Das [12] introduced the idea of I–convergence of nets in a topological space and examined how far it affects the basic properties.

Taking the idea of I and I∗–convergence of nets by Lahiri and Das in [12], Jamwal et. al introduced the concept of I–convergence of filters and studied its various properties in [6]. In [7], Jamwal et. al reintroduced the concept of I–convergence of nets in a topological space and established the equivalence of I–convergences of nets and filters on a topological space.

We recall the following definitions:

Definition 1. Let X be a non-empty set. Then a family $\mathcal{F} \subset 2^X$ is called a filter on X if

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and
(iii) $A \in \mathcal{F}, B \supset A$ implies $B \in \mathcal{F}$.

Definition 2. Let X be a non-empty set. Then a family $I \subset 2^X$ is called an ideal of X if

(i) $\emptyset \in I$, 

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Lemma 1. Let \( \text{corresponding filter} \ I \) −

Notation \( \text{Proposition 3.} \)

In this case, \( x \) _\( \text{Proposition 2.} \)_

\( \text{Proposition 4.} \) Let \( \text{Proposition 5.} \) Let \( \text{Lemma 2.} \)

\( \text{Lemma 3.} \)

\( \text{Lemma 5.} \)

\( \text{Definition 3.} \) Let \( X \) be a non-empty set. Then a filter \( F \) on \( X \) is said to be non-trivial if \( F \neq \{X\} \).

\( \text{Definition 4.} \) Let \( X \) be a non-empty set. Then an ideal \( I \) of \( X \) is said to be non-trivial if \( I \neq \{\emptyset\} \) and \( X \notin I \).

Note(i) If \( I \) is an ideal of a set \( X \), then \( F = F(I) = \{A \subset X : A \notin I\} \) is a filter on \( X \), called the filter associated with the ideal \( I \).

(ii) \( I = I(F) = \{A \subset X : A \in F\} \) is an ideal of \( X \), called the ideal associated with the filter \( F \).

(iii) A non-trivial ideal \( I \) of \( X \) is called admissible if \( I \) contains all the singleton subsets of \( X \).

We now recall some of the results on \( I \)−convergence of filters and nets in a topological space which are proved in [6], [7]. Throughout this paper, \( X = (X, \tau) \) will stand for a topological space and \( I = I(F) \) will be the ideal associated with the filter \( F \) on \( X \).

Definition 5. A filter \( F \) on \( X \) is said to be \( I \)−convergent to \( x_0 \in X \) if for each nbd \( U \) of \( x_0 \), \( \{y \in X : y \notin U\} \in I \).

In this case, \( x_0 \) is called an \( I \)−limit of \( F \) and is written as \( I \lim F = x_0 \).

Notation In case more than one filters is involved, we use the notation \( I(F) \) to denote the ideal associated with the corresponding filter \( F \).

Lemma 1. Let \( F \) and \( G \) be two filters on \( X \). Then \( F \subset G \) if and only if \( I(F) \subset I(G) \).

Proposition 1. Let \( F \) be a filter on \( X \) such that \( I \lim F = x_0 \). Then every filter \( F' \) finer than \( F \) also \( I \)−converges to \( x_0 \), where \( I = I(F) \).

Proposition 2. Let \( F \) be a filter on \( X \) such that \( I \lim F = x_0 \). Then every filter \( F' \) on \( X \) coarser than \( F \) also \( I \)−converges to \( x_0 \), where \( I = I(F) \).

Proposition 3. Let \( F \) be a filter on \( X \) and \( G \) be any other filter on \( X \) finer than \( F \). Then \( I(F) = I(G) \) implies \( I(G) \)−limit of \( G = x_0 \).

Proposition 4. Let \( \tau_1 \) and \( \tau_2 \) be two topologies on \( X \) such that \( \tau_1 \) is coarser than \( \tau_2 \). Let \( F \) be a filter on \( X \) such that \( I \lim F = x_0 \) w.r.t \( \tau_2 \). Then \( I \lim F = x_0 \) w.r.t \( \tau_1 \).

Lemma 2. Let \( \mathcal{M} = \{G : G \text{ is a filter on } X\} \). Then \( F = \cap_{G \in \mathcal{M}} G \) if and only if \( I(F) = \cap_{G \in \mathcal{M}} I(G) \).

Proposition 5. Let \( \mathcal{M} \) be a collection of all those filters \( G \) on a space \( X \) which \( I(G) \)−converges to the same point \( x_0 \in X \). Then the intersection \( \mathcal{F} \) of all the filters in \( \mathcal{M} \) of \( I(F) \)−converges to \( x_0 \).

Lemma 3. If \( I_X \) is an ideal of \( X = \prod_{\alpha \in A} X_\alpha \) associated with a filter \( F \) on \( X \), then \( I_X = \cap_{\alpha \in A} I_{X_\alpha} \), where \( I_{X_\alpha} \) is an ideal of the factor space \( X_\alpha \) associated with \( p_\alpha(F) \).

Theorem 1. A filter \( F \) on \( X \) \( I \)−converges to \( x_0 \in X \) if and only if every derived net \( \lambda \) of \( F \) converges to \( x_0 \).

Theorem 2. A filter \( F \) on \( X \) \( I \)−converges to \( x_0 \in X \) if and only if \( F \) converges to \( x_0 \).
2 I—Cluster Points of Filters

We begin this section with the definition of I—cluster point of a filter with some examples.

**Definition 6.** A point \( x_0 \in X \) is called an I—cluster point of a filter \( \mathcal{F} \) on \( X \) if for each nbd \( U \) of \( x_0 \), \( \{ y \in X : y \in U \} \notin I \). In other words, \( x_0 \in X \) is called an I—cluster point of \( \mathcal{F} \) if \( U \notin I \), for each nbd \( U \) of \( x_0 \).

Equivalently, \( x_0 \) is an I—cluster point of \( \mathcal{F} \) if for each nbd \( U \) of \( x_0 \), \( \{ V \in \mathcal{P}(X) : U \cap V \neq \emptyset \} \notin I \).

**Example 1.** Let \( X = \{1, 2, 3\} \) and \( \tau \) be the discrete topology on \( X \).
Let \( \mathcal{F} = \{\{1\}, \{1, 2\}, \{1, 3\}, X\} \) be a filter on \( X \). Then \( I = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\} \) is the ideal associated with \( \mathcal{F} \).
It is easy to see that \( I \) is the only I—cluster point of \( \mathcal{F} \).

**Example 2.** Let \( \mathcal{U}_{x_0} \) be the nbd filter at a point \( x_0 \) in \( X \). Then clearly for each nbd \( U \) of \( x_0 \), \( \{ y \in X : y \in U \} \notin I \), as \( I = I(\mathcal{U}_{x_0}) \). Thus \( x_0 \) is the I—cluster point of \( \mathcal{U}_{x_0} \).

**Example 3.** Let \( \mathcal{F} \) be a filter on an indiscrete space \( X \). Then clearly, each \( x_0 \in X \) is an I—cluster point of \( \mathcal{F} \) as \( X \) is the only nbd of \( x_0 \in X \) and \( \{ y \in X : y \in X \} = X \notin I \).

**Notation** Let \( I(C_{\mathcal{F}}) \) and \( I(L_{\mathcal{F}}) \) respectively denote the set of all I—cluster points and the set of all I—limits of a filter \( \mathcal{F} \) on \( X \).

We have the following theorem establishing the relationship between I—limits and I—cluster points of a filter \( \mathcal{F} \) on \( X \).

**Theorem 3.** With usual notations, \( I(L_{\mathcal{F}}) \subset I(C_{\mathcal{F}}) \).

**Proof.** Let \( x_0 \in I(L_{\mathcal{F}}) \). Then for each nbd \( U \) of \( x_0 \), \( \{ y \in X : y \notin U \} \notin I \). That is, \( X \setminus U \in I \cdots (\ast) \). We have to show that \( x_0 \in I(C_{\mathcal{F}}) \). For this, let \( U \) be a nbd of \( x_0 \). We claim that \( \{ y \in X : y \in U \} \notin I \). That is, \( U \notin I \). Suppose \( U \in I \). From \((\ast)\), \( X \setminus U \in I \). Since \( I \) is an ideal, we have \( U \cup (X \setminus U) \in I \). That is, \( X \in I \), which contradicts the fact that \( I \) is non-trivial ideal of \( X \). Thus our supposition is wrong. Hence \( U \notin I \) and so \( x_0 \in I(C_{\mathcal{F}}) \). This proves that \( I(L_{\mathcal{F}}) \subset I(C_{\mathcal{F}}) \).

**Note** The converse of the above theorem is however not true. For this, we have the following example.

**Example 4.** Let \( X = \{1, 2, 3\} \) and \( \tau = \{\emptyset, \{3\}, \{2, 3\}, X\} \). Let \( \mathcal{F} = \{\{1, 2\}, X\} \) be a filter on \( X \). Then \( I = \{\emptyset, \{3\}\} \) is the ideal associated with filter \( \mathcal{F} \). Then it is easy to see that \( I(L_{\mathcal{F}}) = \{1\} \) and \( I(C_{\mathcal{F}}) = \{1, 2\} \). So, \( I(C_{\mathcal{F}}) \notin I(L_{\mathcal{F}}) \).

We now give the necessary condition for a filter \( \mathcal{F} \) on \( X \) to have an I—cluster point.

**Theorem 4.** Let \( \mathcal{F} \) be a filter on \( X \). If \( x_0 \) is an I—cluster point of \( \mathcal{F} \), then for each nbd \( U \) of \( x_0 \), \( \{ V \in \mathcal{P}(X) : U \cap V \neq \emptyset \} \notin I \).

**Proof.** Suppose \( \mathcal{F} \) has \( x_0 \) as an I—cluster point. This means that for each nbd \( U \) of \( x_0 \), \( \{ y \in X : y \notin U \} \notin I \). That is, \( U \notin I \cdots (\ast) \). Let \( U \) be a nbd of \( x_0 \). We have to show that \( \{ V \in \mathcal{P}(X) : U \cap V \neq \emptyset \} \notin I \). We observe that \( U \in \mathcal{P}(X) \) such that \( U \cap V \neq \emptyset \) and also by \((\ast)\), \( U \notin I \). Thus it follows that \( \{ V \in \mathcal{P}(X) : U \cap V \neq \emptyset \} \notin I \).

**Remark.** The condition, for each nbd \( U \) of \( x_0 \), \( \{ V \in \mathcal{P}(X) : U \cap V \neq \emptyset \} \notin I \) is not the sufficient condition for a filter \( \mathcal{F} \) to have an I—cluster point. Consider the following example.

**Example** Let \( X = \{1, 2, 3\} \) and \( \tau = \{\emptyset, \{2\}, \{2, 3\}, X\} \) be a topology on \( X \). Let \( \mathcal{F} = \{\{1\}, \{1, 2\}, \{1, 3\}, X\} \) be a filter on \( X \).
Then \( I = I(\mathcal{F}) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\} \) is an ideal of \( X \).
We see that nbds of 2 are \( \{2\}, \{1, 2\}, \{2, 3\} \) and \( X \). We observe that for each nbd \( U \) of 2, \( \{ V \in \mathcal{P}(X) : U \cap V \neq \emptyset \} \notin I \).
But 2 is not an \( I \)–cluster point of \( \mathcal{F} \). This is because \( \{2\} \) is a nbhd of 2 and \( \{2\} \in I \).

This shows that the condition, for each nbhd \( U \) of \( x_0 \), \( \{V \in \mathcal{P}(X) : U \cap V \neq \emptyset\} \not\subseteq I \) is not the sufficient condition for a filter \( \mathcal{F} \) to have an \( I \)–cluster point.

**Proposition 6.** Let \( \mathcal{F} \) be a filter on a non-discrete space \( X \). Then \( \mathcal{F} \) has \( x_0 \) as an \( I(\mathcal{F}) \)–cluster point if and only if there is a filter \( \mathcal{G} \) on \( X \) finer than \( \mathcal{F} \) such that \( I(\mathcal{G}) \)–\( \lim \mathcal{G} = x_0 \).

**Proof.** Suppose \( x_0 \) is an \( I(\mathcal{F}) \)–cluster point of \( \mathcal{F} \). Then for each nbhd \( U \) of \( x_0 \), \( \{V \in \mathcal{P}(X) : U \cap V \neq \emptyset\} \not\subseteq I(\mathcal{F}) \). Since \( X \setminus \{x\} \in I(\mathcal{F}) \), \( \forall F \in \mathcal{F} \), we find that \( \mathcal{F} = \{U \subseteq X \setminus \{x\} : U \subseteq \mathcal{P}(X)\} \). Let \( \mathcal{G} = \{U \cap F : U \subseteq \mathcal{P}(X) \} \). Then clearly, \( \mathcal{G} \) is a non-empty family of non-empty subsets of \( X \) which is closed under finite intersection and so a filter base for some filter, say \( \mathcal{H} \) on \( X \). If \( \mathcal{G} \) is a seq.\( \mathcal{G} \) union of \( \mathcal{F} \) and \( \mathcal{F} \subseteq \mathcal{G} \), then \( \mathcal{G} \) is a seq.\( \mathcal{G} \) union of \( \mathcal{F} \) and \( \mathcal{F} \subseteq \mathcal{G} \). This implies that \( \mathcal{F} \subseteq \mathcal{G} \). Therefore \( \mathcal{G} \) is finer than \( \mathcal{F} \). By Lemma 1, \( I(\mathcal{F}) \subset I(\mathcal{G}) \). We shall show that \( I(\mathcal{G}) \)–\( \lim \mathcal{G} = x_0 \). For this, we need to prove that for each nbhd \( U \) of \( x_0 \), \( \{V \in \mathcal{P}(X) : U \cap V \neq \emptyset\} \not\subseteq I(\mathcal{G}) \). So, let \( U \) be a nbhd of \( x_0 \) and \( V \in \mathcal{P}(X) \) such that \( U \cap V = \emptyset \). Now \( U \cap V = \emptyset \)

Conversely, suppose there is a filter \( \mathcal{G} \) on finer \( \mathcal{F} \) such that \( I(\mathcal{G}) \)–\( \lim \mathcal{G} = x_0 \). We have to show that \( x_0 \) is the \( I(\mathcal{F}) \)–cluster point of \( \mathcal{F} \). For this, let \( U \) be a nbhd of \( x_0 \). We claim that \( \{y \in X : y \in U \} \notin I(\mathcal{F}) \). That is, \( U \notin I(\mathcal{F}) \).

**Remark.** (a) The above Proposition 6 need not be true if \( X \) has the discrete topology. Consider the example:

**The above Remark 2** motivated us to have the following proposition:

**Proposition 7.** Let \( \mathcal{F} \) be a filter on \( X \) such that \( \mathcal{F} \) has \( x_0 \in X \) as an \( I \)–cluster point. Then every filter \( \mathcal{F}' \) finer than \( \mathcal{F} \) also has \( x_0 \) as an \( I \)–cluster point, where \( I = I(\mathcal{F}) \).

**Proof.** Suppose \( \mathcal{F} \) is a filter on \( X \) such that \( \mathcal{F} \) has \( x_0 \) as an \( I \)–cluster point of \( \mathcal{F} \). Then for each nbhd \( U \) of \( x_0 \), \( U \notin I \). Let \( \mathcal{F}' \) be an arbitrary filter on \( X \) such that \( \mathcal{F}' \supseteq \mathcal{F} \). We shall show that \( I \)–cluster point of \( \mathcal{F}' = x_0 \), where \( I = I(\mathcal{F}) \). For this, let \( U \) be a nbhd of \( x_0 \). Then clearly by (*), \( U \notin I \). Hence the proof.

**Remark.** Let \( \mathcal{F} \) be a filter on \( X \) and \( \mathcal{F}' \) be a filter on \( X \) finer than \( \mathcal{F} \). Then \( I(\mathcal{F}) \)–cluster point of \( \mathcal{F} = x_0 \) need not imply that \( I(\mathcal{F}') \)–cluster point of \( \mathcal{F}' = x_0 \). Consider the example in Remark 2.

We can see that 1 and 2 are \( I(\mathcal{F}) \)–cluster points of \( \mathcal{F} \). But 2 is not an \( I(\mathcal{F}') \)–cluster point of \( \mathcal{F}' \).
Proposition 8. Let $\mathscr{F}$ be a filter on $X$ such that $\mathscr{F}$ has $x_0 \in X$ as an $I-$cluster point. Then every filter $\mathscr{F}'$ coarser than $\mathscr{F}$ also has $x_0$ as an $I-$cluster point, where $I = I(\mathscr{F})$.

Proof. Suppose $\mathscr{F}$ is a filter on $X$ such that $x_0$ is an $I-$cluster point of $\mathscr{F}$. Then for each nbd $U$ of $x_0$, $\{y \in X : y \in U\} \notin I \cdots (\ast)$. Let $\mathscr{F}'$ be an arbitrary filter on $X$ such that $\mathscr{F}' \subset \mathscr{F}$. We shall show that $I-$cluster point of $\mathscr{F}' = x_0$, where $I = I(\mathscr{F})$. For this, let $U$ be a nbd of $x_0$. We claim that $\{y \in X : y \in U\} \notin I$. But it follows clearly by $(\ast)$. Hence the proof.

Proposition 9. Let $\tau_1$ and $\tau_2$ be two topologies on $X$ such that $\tau_1$ is coarser than $\tau_2$. Let $\mathscr{F}$ be a filter on $X$ such that $x_0$ is an $I-$cluster point of $\mathscr{F}$ w.r.t $\tau_2$. Then $x_0$ is also an $I-$cluster point of $\mathscr{F}$ w.r.t $\tau_1$.

Proof. Let $U$ be a nbd of $x_0$ w.r.t $\tau_1$. Since $\tau_1 \subset \tau_2$, $U$ is also a nbd of $x_0$ w.r.t $\tau_2$. But $x_0$ is an $I-$cluster point of $\mathscr{F}$ w.r.t $\tau_2$. Thus for above nbd $U$ of $x_0$, $U \notin I$. Hence $x_0$ is also an $I-$cluster point of $\mathscr{F}$ w.r.t $\tau_1$.

Remark. The converse of above proposition need not be true. That is, if $\tau_1$ and $\tau_2$ are two topologies on $X$ such that $\tau_1$ is coarser than $\tau_2$ and $x_0$ is an $I-$cluster point of $\mathscr{F}$ w.r.t $\tau_1$, then $x_0$ need not be an $I-$cluster point of $\mathscr{F}$ w.r.t $\tau_2$. Consider the following example:

Let $X = \{1,2,3\}$. Suppose $\tau_2$ is the discrete topology on $X$ and $\tau_1 = \{\emptyset,\{2\},X\}$. Then $\tau_1 \subset \tau_2$. Let $\mathscr{F} = \{\{1\},\{1,2\},\{1,3\},X\}$. Let $\mathcal{F}$ be a filter on $X$. Then $I(\mathcal{F}) = \{\emptyset,\{2\},\{3\},\{2,3\}\}$ is the ideal associated with $\mathcal{F}$. It is easy to see that 1 and 3 are the $I-$cluster points of $\mathcal{F}$ w.r.t $\tau_1$. But 3 is not an $I-$cluster point of $\mathcal{F}$ w.r.t $\tau_2$.

Proposition 10. Let $\mathscr{M}$ be a collection of all those filters $\mathcal{G}$ on a space $X$ which have $x_0 \in X$ as an $I(\mathcal{G})-$cluster point. Then the intersection $\mathscr{F}$ of all the filters in $\mathscr{M}$ also has $x_0$ as an $I(\mathcal{F})-$cluster point.

Proof. Here $\mathscr{M} = \{\mathcal{G} : \mathcal{G}$ is a filter on $X$ such that $I(\mathcal{G})-$cluster point of $\mathcal{G} = x_0\}$. Let $\mathscr{F} = \bigcap\{\mathcal{G} : \mathcal{G} \in \mathscr{M}\}$. We shall show that $x_0$ is an $I(\mathcal{F})-$cluster point of $\mathcal{F}$. For this, let $U$ be a nbd of $x_0$ w.r.t $\mathcal{F}$. Then $\mathcal{U}$ is a nbd of $x_0$ w.r.t all $\mathcal{G} \in \mathscr{M}$. Since $x_0$ is an $I(\mathcal{G})-$cluster point of $\mathcal{G}$, $\forall \mathcal{G} \in \mathscr{M}$, it follows that $\{y \in X : y \in U\} \notin I(\mathcal{G})$, $\forall \mathcal{G} \in \mathscr{M}$ and so $\{y \in X : y \in U\} \notin \bigcap\mathcal{G} \in \mathcal{M} I(\mathcal{G})$. By Lemma 1, $\{y \in X : y \in U\} \notin I(\mathcal{F})$. Hence $x_0$ is also an $I(\mathcal{F})-$cluster point of $\mathcal{F}$.

In view of Remark 2, we have the following proposition:

Proposition 11. Let $\mathcal{F}$ be a filter on $X$ and $\mathcal{G}$ be a filter on $X$ finer than $\mathcal{F}$. Then $\mathcal{F}$ has $x_0$ as an $I(\mathcal{G})-$cluster point if and only if $I(\mathcal{F}) - \operatorname{lim}\mathcal{G} = x_0$.

Proof. Let $\mathcal{F}$ be a filter on $X$ and $\mathcal{G}$ be a filter on $X$ finer than $\mathcal{F}$ such that $x_0$ is an $I(\mathcal{G})-$cluster point of $\mathcal{F}$. Let $\mathcal{B}$ be a base for $\mathcal{G}$. Then $\mathcal{B} = \{G \subset X : B \subset G$, for some $B \subset \mathcal{F}\}$. We shall show that $I(\mathcal{G}) - \operatorname{lim}\mathcal{G} = x_0$. For this, let $U$ be a nbd of $x_0$. We shall show that $\{V \in \mathcal{P}(X) : U \cap V = \emptyset\} \subset I(\mathcal{G})$. So, let $V \in \mathcal{P}(X)$ such that $U \cap V = \emptyset$. Now, $U \cap V = \emptyset$ implies $U \subset X \setminus V$ which further implies that $U \cap U \subset U \cap (X \setminus V)$. That is, $U \subset U \cap (X \setminus V)$. Also, $U \supset G \subset U$, for all $G \in \mathcal{G}$. Thus $U \cap G \subset U \cap (X \setminus V)$, for all $G \in \mathcal{G}$. But $U \cap (X \setminus V) \subset X \setminus V$. Thus $U \cap G \subset X \setminus V$, for all $G \in \mathcal{G}$. Since $\mathcal{B}$ is a base for $\mathcal{G}$, $U \cap G \in \mathcal{B}$ and so $X \setminus V \in \mathcal{G}$. Hence $V \in I(\mathcal{G})$. This proves that $I(\mathcal{G}) - \operatorname{lim}\mathcal{G} = x_0$.

Conversely, suppose $I(\mathcal{G}) - \operatorname{lim}\mathcal{G} = x_0$. By using Theorem 3, we find that $x_0$ is also an $I(\mathcal{G})-$cluster point of $\mathcal{G}$. Since $\mathcal{F}$ is coarser than $\mathcal{G}$, by Proposition 8, it follows that $x_0$ is also $I(\mathcal{F})-$cluster point of $\mathcal{F}$.

Theorem 5. Let $f : X \to Y$ be a surjective map. Let $\mathcal{F}$ be a filter on $X$. Then $f : X \to Y$ is continuous at $x_0 \in X$ if and only if whenever $x_0$ is an $I_X-$cluster point of $\mathcal{F}$, then $f(x_0)$ is an $I_Y-$cluster point of $f(\mathcal{F})$, where $I_X = I_X(\mathcal{F})$ is the ideal associated with $\mathcal{F}$ and $I_Y = I_Y(f(\mathcal{F}))$ is the ideal associated with the filter $f(\mathcal{F})$ on $Y$.

Proof. First suppose that the surjection $f : X \to Y$ is continuous at $x_0$ in $X$. Let $x_0$ be an $I_X-$cluster point of $\mathcal{F}$ in $X$. We have to show that $f(x_0)$ is an $I_Y-$cluster point of $f(\mathcal{F})$.

For this, let $V$ be a nbd of $f(x_0)$ in $Y$. Since $f$ is continuous at $x_0$, for above nbd $V$ of $f(x_0)$ in $Y$, there is a nbd $U$ of $x_0$ in
X such that \( f(U) \subseteq V \). Since \( x_0 \) is an \( I_X \)-cluster point of \( \mathcal{F} \), for above nbd \( U \) of \( x_0 \) in \( X \), \( U \notin I_X \) and so \( X \setminus U \notin \mathcal{F} \). (1).

We claim that \( V \notin I_Y \). Then \( Y \setminus V \in f(\mathcal{F}) \). (2).

Now \( f(U) \subseteq V \) implies \( Y \setminus V \subseteq Y \setminus f(U) \). (3).

Since \( f(\mathcal{F}) \) is a filter on \( Y \), from (2) and (3), we get \( Y \setminus f(U) \in f(\mathcal{F}) \). (4). From (1), \( X \setminus U \notin \mathcal{F} \) implies \( f(X \setminus U) \notin f(\mathcal{F}) \). But \( f(X \setminus U) \supset f(X) \setminus f(U) = Y \setminus f(U) \). That is, \( Y \setminus f(U) \subseteq f(X \setminus U) \). Now \( f(X \setminus U) \notin f(\mathcal{F}) \) and \( Y \setminus f(U) \subseteq f(X \setminus U) \) implies that \( Y \setminus f(U) \notin f(\mathcal{F}) \), which contradicts (4). Thus our supposition is wrong.

Hence \( f(x_0) \) is an \( I_Y \)-cluster point of \( Y \).

Conversely, suppose \( f : X \to Y \) is a surjection such that the given condition holds. We have to show that \( f \) is continuous at \( x_0 \). Suppose not. This means that there is a nbd \( V \) of \( f(x_0) \) in \( Y \) such that \( f^{-1}(V) \) is not a nbd of \( x_0 \).

Let \( \mathcal{F} = \{ U \setminus f^{-1}(V) : U \text{ is a nbd of } x_0 \text{ in } X \} \). Then clearly, \( \mathcal{F} \) is a filter on \( X \). We claim that \( x_0 \) is an \( I_X \)-cluster point of \( \mathcal{F} \). For this, let \( U \) be a nbd of \( x_0 \). We shall show that \( T \notin I_X \).

Suppose the contrary \( T \in I_X \). Then \( X \setminus T \notin \mathcal{F} \). Also by (5), \( T \setminus f^{-1}(V) \notin \mathcal{F} \). Since \( \mathcal{F} \) is a filter on \( X \), we have \( \emptyset = (X \setminus T) \cap (T \setminus f^{-1}(V)) \in \mathcal{F} \), i.e., \( \emptyset \notin \mathcal{F} \), which is not possible. Thus \( T \notin I_X \). Therefore, \( x_0 \) is an \( I_X \)-cluster point of \( \mathcal{F} \). By the given condition, \( f(x_0) \) is an \( I_Y \)-cluster point of \( f(\mathcal{F}) \). So \( V \notin I_Y \), i.e., \( Y \setminus V \notin f(\mathcal{F}) \). Now \( Y \setminus V \notin f(\mathcal{F}) \) implies \( f^{-1}(Y \setminus V) \notin \mathcal{F} \). This further implies that \( X \setminus f^{-1}(V) \notin \mathcal{F} \), which contradicts (5). Thus our supposition is wrong.

Hence \( f \) is continuous at \( x_0 \).

**Remark.** The above Theorem 5 holds even if \( f \) is not surjective. In that case, we shall assume \( f(\mathcal{F}) \) to be a filter on \( Y \) generated by the filter base \( \{ f(F) : F \in \mathcal{F} \} \).

**Theorem 6.** A filter \( \mathcal{F} \) on \( X = \prod_{\alpha \in A} X_\alpha \) has \( x \) as an \( I_X \)-cluster point if and only if \( p_\alpha(\mathcal{F}) \) has \( p_\alpha(x) \) as an \( I_{X_\alpha} \)-cluster point, \( \forall \alpha \in A \), where \( I_X = I_X(\mathcal{F}) \) and \( I_{X_\alpha} = I_{X_\alpha}(p_\alpha F) \).

**Proof.** Suppose \( \mathcal{F} \) has \( x \) as an \( I_X \)-cluster point in \( X = \prod_{\alpha \in A} X_\alpha \). Since each projection \( p_\alpha : X \to X_\alpha \) is continuous at \( x \) in \( X \), by above Theorem 5, we find that \( p_\alpha(x) \) is an \( I_{X_\alpha} \)-cluster point of \( p_\alpha(\mathcal{F}) \) in \( X_\alpha \), \( \forall \alpha \).

Conversely, suppose \( p_\alpha(x) \) is an \( I_{X_\alpha} \)-cluster point of \( p_\alpha(\mathcal{F}) \) in \( X_\alpha \), \( \forall \alpha \). We have to show that \( x \) is an \( I_X \)-cluster point of \( \mathcal{F} \) in \( X \). For this, let \( U = \cap_{i=1}^n p_\alpha^{-1}(U_\alpha) \) be a basic nbd of \( x \). This means that \( U_\alpha \) is a nbd of \( x_\alpha = p_\alpha(x) \), for \( i = 1,2,\ldots,n \) in \( X_\alpha \). We claim that \( U \notin I_X \). Since \( p_\alpha(x) \) is an \( I_{X_\alpha} \)-cluster point of \( p_\alpha(\mathcal{F}) \) in \( X_\alpha \), we have \( U_\alpha \notin I_{X_\alpha} \), \( \forall i = 1,2,\ldots,n \). This further implies that \( p_\alpha^{-1}(U_\alpha) \notin p_\alpha^{-1}(I_{X_\alpha}) \), \( \forall i = 1,2,\ldots,n \). Clearly, \( \cap_{i=1}^n p_\alpha^{-1}(U_\alpha) \notin \cap_{i=1}^n p_\alpha^{-1}(I_{X_\alpha}) = I_X \), by Lemma 3. That is, \( U \notin I_X \).

This proves that \( x \) is an \( I_X \)-cluster point of \( \mathcal{F} \) in \( X \).

**Theorem 7.** Let \( X \) be a Lindelöf space such that every filter on \( X \) has an \( I \)-cluster point, where \( I \) is an admissible ideal of \( X \). Then \( X \) is compact.

**Proof.** Let \( X \) be a Lindelöf space such that every filter on \( X \) has an \( I \)-cluster point, where \( I \) is an admissible ideal of \( X \). We have to show that \( X \) is compact. For this, let \( \mathcal{U} = \{ U_\alpha : \alpha \in A \} \) be an open cover of \( X \), where \( A \) is an index set.

Since \( X \) is Lindelöf, the above open cover \( \mathcal{U} \) of \( X \) has a countable subcover, say \( \mathcal{U}' = \{ U_1, U_2, \ldots, U_n, \ldots \} \). Proceeding inductively, let \( V_1 = U_1 \) and for each \( m > 1 \), let \( V_m \) be the first member of \( \mathcal{U}' \) which is not covered by \( V_1 \cup V_2 \cup \cdots \cup V_{m-1} \).
After some finite number of steps, the set of above \( V_i \)'s selected becomes a required finite subcover. Otherwise, we can choose a point \( v_n \in V_n \), for each positive integer \( n \) such that \( v_n \notin V_r \), for \( r < n \cdot (\ast) \). Consider a net \( \lambda = (v_n)_{n \in \mathbb{N}} \). Let \( \mathcal{F} \) be the derived filter of \( \lambda \). That is, \( \mathcal{F} = \{ F \subseteq X : \lambda \text{ is eventually in } F \} \). By \( \lambda \) eventually in \( F \), we mean that some tail \( \Lambda_m = \{ \lambda(n) = v_n : n \geq m \in \mathbb{N} \} \) of \( \lambda \) is contained in \( F \). Let \( x_0 \) be an \( I \)–cluster point of \( \mathcal{F} \). Then \( x_0 \in V_p \), for some \( p \). By definition of \( I \)–cluster point of \( \mathcal{F} \), in particular for \( V_p \), \( \{ y \in X : y \in V_p \} \notin I \). Since \( I \) is an admissible ideal, \( \{ y \in X : y \in V_p \} \) must be infinite subset of \( X \). So, there exists some \( n > p \) such that \( v_n \in \{ y \in X : y \in V_p \} \). That is, there exists some \( n > p \) such that \( v_n \in V_p \), which contradicts \( (\ast) \). Thus the above set of \( V_i \)'s form the required finite subcover. Hence \( X \) is compact.

**Theorem 8.** A topological space \( X \) is compact if and only if every filter on \( X \) has an \( I \)–cluster point.

**Proof.** First suppose \( X \) is compact. Let \( \mathcal{F} \) be a filter on \( X \). Consider a family \( \{ \mathcal{F} : F \in \mathcal{F} \} \) of closed subsets of \( X \). Since \( X \) is compact, the family \( \{ \mathcal{F} : F \in \mathcal{F} \} \) has finite intersection property. That is, \( \cap \{ \mathcal{F} : F \in \mathcal{F} \} \neq \emptyset \). Let \( x_0 \in \cap \{ \mathcal{F} : F \in \mathcal{F} \} \). Then for each nbd \( U \) of \( x_0 \), \( U \cap F \neq \emptyset \), \( \forall F \in \mathcal{F} \). We claim that \( U \notin I \). Suppose that \( U \in I \). Then \( U \cap F \neq \emptyset \), \( \forall F \in \mathcal{F} \) and for each nbd \( U \) of \( x_0 \) would contradict the fact that \( I = I(\mathcal{F}) \). This proves that \( x_0 \) is an \( I \)–cluster point of \( \mathcal{F} \).

Conversely, suppose that every filter on \( X \) has an \( I \)–cluster point. We have to show that \( X \) is compact. Suppose \( X \) is not compact and let \( \mathcal{U} \) be an open cover of \( X \) with no finite subcover. Let \( \mathcal{B} = \{ X \setminus \bigcup_{i=1}^{n} U_i : U_i \in \mathcal{U} \} \) for \( i = 1, 2, \ldots, n \). Then clearly, \( \mathcal{B} \) is a non-empty family of non-empty subsets of \( X \) which is closed under finite intersection and so a filter base for some filter, say \( \mathcal{F} \) on \( X \). By the given condition, \( \mathcal{F} \) has an \( I(\mathcal{F}) \)–cluster point, say \( x_0 \). This means that for each nbd \( U \) of \( x_0 \), \( \{ V \in \mathcal{P}(X) : U \cap V \neq \emptyset \} \notin I(\mathcal{F}) \). Let \( U \in \mathcal{U} \) such that \( x_0 \in U \). Now, \( U \in \mathcal{U} \) implies that \( X \setminus U \in \mathcal{B} \) and so \( X \setminus U \notin \mathcal{F} \). Now, \( X \setminus U \in \mathcal{F} \) implies \( U \in I(\mathcal{F}) \). Finally, \( x_0 \in U \) and \( U \in I(\mathcal{F}) \) implies that \( \{ x_0 \} \in I(\mathcal{F}) \), which contradicts \( (\ast) \) with \( V = \{ x_0 \} \). Thus our supposition is wrong. Hence \( X \) is compact.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

**References**


