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On generators of Peiffer ideal of a pre-R-algebroid in a precrossed module and applications

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Abstract: In this study, we analyse the generators of the Peiffer ideal $[\![\mathrm{M},\mathrm{M}]\!]$ of a pre-R-algebroid M in a precrossed module $\mathscr{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ in terms of the generators of M for further using and use the outcomes to find the generators of the Peiffer ideal obtained in the coproduct construction of two crossed A-modules of R-algebroids.

Keywords: Algebroid, crossed module, Peiffer ideal, coproduct.

1 Introduction

Crossed modules which are algebraic models of 2-types were firstly introduced by Whitehead, [16,17], in his study on homotopy groups. Then for many areas of mathematics they've become an essential algebraic structure. To get a crossed module from a precrossed module we mostly use Peiffer subgroups or Peiffer ideals generated by Peiffer elements which are also named as Peiffer commutators. The term 'Peiffer element' has firstly used by R. Brown and J. Huebschmann in [8] to mark the substantial contribution of R. Peiffer in [15] and Peiffer commutator calculus has especially been developed by H. J. Baues and D. Conduché, [7]. R. Brown, P. J. Higgins and R. Sivera have used Peiffer subgroup to obtain crossed modules of groups in [9] and N. M. Shammu has used Peiffer commutators to get crossed modules of algebras in [14].

In [4], to construct a crossed module from a given precrossed module $\mathscr{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ of R-algebroids, we've introduced the Peiffer ideal $[\![\mathrm{M},\mathrm{M}]\!]$ of the pre-R-algebroid M where the procedure has given the functor $(-)^{\mathrm{cr}}$ from the category of precrossed to the category of crossed modules of R-algebroids. In [6], on the other hand, we've constructed the coproduct of given two crossed A-modules $\mathscr{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ and $\mathscr{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ of R-algebroids in two ways one of which depends basicly on dividing the free product $\mathrm{M} * \mathrm{N}$ of pre-R-algebroids M and N with the ideal $\mathbf{I}_{\mathrm{M}*\mathrm{N}}$ and the other on dividing their semidirect product $\mathrm{M} \ltimes \mathrm{N}$ with the ideal $\mathbf{I}_{\mathrm{M}\ltimes\mathrm{N}}$.

In this study, we analyse the generators of $[\![\mathrm{M},\mathrm{M}]\!]$ to determine their relations with the generators of M for further using and apply the outcomes to find the generators of the Peiffer ideals of pre-R-algebroids $\mathrm{M} * \mathrm{N}$ and $\mathrm{M} \ltimes \mathrm{N}$, in [6], to reach the unsurprising result that $\mathrm{I}_{\mathrm{M}*\mathrm{N}}$ and $\mathrm{I}_{\mathrm{M}\ltimes\mathrm{N}}$ are in fact their Peiffer ideals, respectively. Throughout the paper R is a commutative ring.

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2 Preliminaries

R-algebroids, as the basic tool of this study, were especially studied by B. Mitchell in [10,11,12], and by S. M. Amgott in [3]. G.H. Mosa has defined crossed modules of R-algebroids in [13] where he proved the equivalence of crossed modules of algebroids and special double algebroids with connections. M. Alp has constructed the pullback and pushout crossed modules of algebroids in [1] and [2], respectively. In [5], we've studied the adjointness between pullback and induced crossed modules of R-algebroids. Most of the following data can be found in [3,10,11,12] and [13].

Definition 1. *A category of which each homset has an R-module structure and of which composition is R-bilinear is called an R-*category*, and a small R-category is called an R-*algebroid*. Moreover if we omit the requirement of the existence of identities from an R-algebroid structure then the remaining structure is called a* pre-R-algebroid*.*

Remark. Given any (pre-)R-algebroid A

(1) $\mathrm{A}_0(=\mathrm{Ob}(\mathrm{A}))$ and $\mathrm{Mor}(\mathrm{A})$ are the object and the morphism sets of A, respectively, and A is said to be over A_0,
(2) $s,t:\mathrm{Mor}(\mathrm{A})\longrightarrow \mathrm{A}_0$ are the source and target functions, respectively, and any $a\in\mathrm{Mor}(\mathrm{A})$ is said to be from sa to ta which are its source and target, respectively.

Remark. Throughout the paper $a\in\mathrm{A}$ will mean that a is a morphism of A and the composition of any $a,b\in\mathrm{A}$ with $ta=sb$ will be denoted by ab. The identity morphism on any $x\in\mathrm{A}_0$ will be denoted by 1_x and the zero morphism of each *homset* $\mathrm{A}(x,y)$, the set of all morphisms from x to y, will be denoted by $0_{\mathrm{A}(x,y)}$ or only by 0 if there is no ambiguity.

Definition 2. *An R-linear functor between two R-categories is called an R-*functor *and an R-functor between two R-algebroids is called an R-*algebroid morphism*. Moreover an assignment between two pre-R-algebroids which has the same conditions as an R-algebroid morphism except for the identity preservation axiom is called a* pre-R-algebroid morphism.

Note that an R-algebroid is a pre-R-algebroid and an R-algebroid morphism is a pre-R-algebroid morphism.

Definition 3. *A* two-sided ideal *of a pre-R-algebroid A is a family I of R-submodules of A such that $a'a,aa''\in\mathrm{I}$ for all $a',a''\in\mathrm{A}$ and $a\in\mathrm{I}$ with $ta'=sa$ and $ta=sa''$.*

Definition 4. *Let A and M be two pre-R-algebroids with $\mathrm{A}_0=\mathrm{M}_0$. A family of maps defined for all $x,y,z\in\mathrm{A}_0$ as*

$$\begin{array}{ccc}\mathrm{M}(x,y)\times\mathrm{A}(y,z) & \longrightarrow & \mathrm{M}(x,z)\\ (m,a) & \longmapsto & m^a\end{array}$$

is called a right action *of A on M if for all $r\in R$, $a,a',a_1,a_2\in\mathrm{A}$, $m,m',m_1,m_2\in\mathrm{M}$ with compatible sources and targets the conditions*

$$\begin{array}{lll}1.\ (m^a)^{a'}=m^{aa'}, & 3.\ (m'm)^a=m'm^a, & 5.\ (r\cdot m)^a=r\cdot m^a=m^{r\cdot a}\\ 2.\ m^{a_1+a_2}=m^{a_1}+m^{a_2}, & 4.\ (m_1+m_2)^a=m_1^a+m_2^a, & \end{array}$$

and the condition $m^{1_{tm}}=m$ whenever 1_{tm} exists are satisfied.

A left action *of A on M is defined similarly. Moreover if A has a right and a left action on M and if the condition*

$$({}^a m)^{a'}={}^a\left(m^{a'}\right)$$

is satisfied for all $m\in\mathrm{M}$, $a,a'\in\mathrm{A}$ with $ta=sm$, $tm=sa'$ then A is said to have an associative action *on M or to act on M* associatively*. Furthermore, if A is an R-algebroid and has an* associative action *on M then M is called an* A-module.

Definition 5. *Let* A *be an* R*-algebroid,* M *be a pre-*R*-algebroid,* $\mathrm{A}_0 = \mathrm{M}_0$ *and* A *have an associative action on* M*. A pre-*R*-algebroid morphism* $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ *is called a* crossed (A-)module of R-algebroids *if the conditions*

$$\begin{aligned}&\mathrm{CM1)}\ \mu\left({}^{a}m\right) = a\left(\mu m\right)\ \ \text{and}\ \ \mu\left(m^{a'}\right) = \left(\mu m\right)a'\\ &\mathrm{CM2)}\ m^{\mu m'} = mm' = {}^{\mu m}m'\end{aligned}$$

are satisfied for all $a, a' \in \mathrm{A}$ *and* $m, m' \in \mathrm{M}$ *with* $ta = sm$, $tm = sa' = sm'$. $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ *is called a* precrossed (A-)module of R-algebroids *if it satisfies* CM1*. So a crossed module is a precrossed module satisfying* CM2*.*

Note from the definition that if $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ is a precrossed module then it's identity on A_0.

Definition 6. *Given two (pre)crossed modules* $\mathscr{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ *and* $\mathscr{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{B})$ *of* R*-algebroids, a pre-*R*-algebroid morphism* $f : \mathrm{M} \longrightarrow \mathrm{N}$ *and an* R*-algebroid morphism* $g : \mathrm{A} \longrightarrow \mathrm{B}$ *if the conditions*

(1) $f\left({}^{a}m\right) = {}^{ga}\left(fm\right)$ *and* $f\left(m^{a'}\right) = \left(fm\right)^{ga'}$,
(2) $\eta f = g\mu$.

are satisfied for all $a, a' \in \mathrm{A}$, $m \in \mathrm{M}$ *with* $ta = sm$, $tm = sa'$ *then the pair* $(f, g) : \mathscr{M} \longrightarrow \mathscr{N}$ *is called a* (pre)crossed module morphism.

Example 1. Let A be an R-algebroid and I be a two-sided ideal of A. Then the inclusion morphism $i : \mathrm{I} \longrightarrow \mathrm{A}$ is a crossed module where A acts on I as ${}^{a}b = ab$ and $b^{a'} = ba'$ for all $a, a' \in \mathrm{A}$, $b \in \mathrm{I}$ with $ta = sb$, $tb = sa'$.

Proposition 1.

(1) *If* $f : \mathrm{A} \longrightarrow \mathrm{B}$ *and* $f' : \mathrm{B} \longrightarrow \mathrm{C}$ *are (pre-)*R*-algebroid morphisms then their composition* $f'f : \mathrm{A} \longrightarrow \mathrm{C}$ *defined as* $(f'f)(x) = f'(fx)$ *on* A_0 *and* $(f'f)(a) = f'(fa)$ *on* $\mathrm{Mor}(\mathrm{A})$ *is a (pre-)*R*-algebroid morphism,*
(2) *If* $(f, g) : \mathscr{M} \longrightarrow \mathscr{N}$ *and* $(f', g') : \mathscr{N} \longrightarrow \mathscr{K}$ *are (pre)crossed module morphisms then* $(f'f, g'g) : \mathscr{M} \longrightarrow \mathscr{K}$ *is a (pre)crossed module morphism.*

Remark. All precrossed modules of R-algebroids with their morphisms form a category denoted by $\mathrm{PXAlg}(R)$. Similarly all crossed modules of R-algebroids form the category $\mathrm{XAlg}(R)$. Moreover for a fixed R-algebroid A all precrossed and crossed A-modules of R-algebroids with the identity morphism on A form the subcategory $\mathrm{PXAlg}(R)/\mathrm{A}$ of $\mathrm{PXAlg}(R)$ and the subcategory $\mathrm{XAlg}(R)/\mathrm{A}$ of $\mathrm{XAlg}(R)$, respectively. It can also clearly be seen that $\mathrm{XAlg}(R)$ and $\mathrm{XAlg}(R)/\mathrm{A}$ are full subcategories of $\mathrm{PXAlg}(R)$ and $\mathrm{PXAlg}(R)/\mathrm{A}$, respectively.

3 Generators of Peiffer ideal of a pre-R-algebroid in a precrossed module

In [4], given a precrossed module $\mathscr{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$, to get a crossed module we've divided the pre-R-algebroid M with its Peiffer ideal $[\![\mathrm{M}, \mathrm{M}]\!]$ generated by all Peiffer commutators which are of the forms $[\![m, m']\!]_1 = m^{\mu m'} - mm'$ and $[\![m, m']\!]_2 = {}^{\mu m}m' - mm'$. Then we've obtained the crossed module $\mathscr{M}^{\mathrm{cr}} = (\mu^{\mathrm{cr}} : \mathrm{M}^{\mathrm{cr}} \longrightarrow \mathrm{A})$ where $\mathrm{M}^{\mathrm{cr}} = \frac{\mathrm{M}}{[\![\mathrm{M},\mathrm{M}]\!]}$ and μ^{cr} is defined as $\mu^{\mathrm{cr}}\overline{m} = \mu m$ for all $\overline{m} \in \mathrm{M}^{\mathrm{cr}}$, with $\overline{m}$ being the coset of $m \in \mathrm{M}$, and this procedure has given the functor $(-)^{\mathrm{cr}} : \mathrm{PXAlg}(R) \longrightarrow \mathrm{XAlg}(R)$ assigning $\mathscr{M}^{\mathrm{cr}}$ to each precrossed module $\mathscr{M}$ and $(f, g)^{\mathrm{cr}} = (f^{\mathrm{cr}}, g)$ to each precrossed module morphisms (f, g) where $f^{\mathrm{cr}}\overline{m} = \overline{fm}$ for all $m \in \mathrm{M}$.

In some cases, however, we may need to simplify and minimize the generator family of Peiffer ideal. This is because dealing with the whole family of Peiffer commutators may sometimes be challenging. In this context, we aimed to prove in this section that if V is a family of subsets of M which generates M as a pre-R-algebroid and is closed under the actions of R and A then the Peiffer ideal $[\![\mathrm{M}, \mathrm{M}]\!]$ is generated by the family of all Peiffer commutators formed by elements of V.

The group version of this notion can be found in [9]. It's proved there that if $\mu : \mathrm{M} \longrightarrow \mathrm{P}$ is a precrossed module of groups and V is a subset of M which generates M as a group and is P-invariant then $[\![\mathrm{M},\mathrm{M}]\!]$ is the normal closure in M of the set $\{[\![a,b]\!] \,|\, a,b \in \mathrm{V}\}$ of Peiffer commutators. Here, to prove theorem 1 we use a similar way to that used in [9]. Before going on, we need to prove the following lemma.

Lemma 1. *If $\mathscr{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ is a precrossed module of R-algebroids then*

(1) $\left(0_{\mathrm{A}(x,sa)}\right)^a = 0_{\mathrm{A}(x,ta)}$ *and* ${}^{a'}\left(0_{\mathrm{A}(ta',y)}\right) = 0_{\mathrm{A}(sa',y)}$
(2) $(-m)^a = -m^a$ *and* ${}^{a'}(-m) = -{}^{a'}m.$

for all $x,y \in \mathrm{A}_0$, $m \in \mathrm{M}$ and $a,a' \in \mathrm{A}$ with $tm = sa$, $ta' = sm$.

Proof.

(1) $\left(0_{\mathrm{A}(x,sa)}\right)^a + 0_{\mathrm{A}(x,ta)} = \left(0_{\mathrm{A}(x,sa)}\right)^a = \left(0_{\mathrm{A}(x,sa)} + 0_{\mathrm{A}(x,sa)}\right)^a = \left(0_{\mathrm{A}(x,sa)}\right)^a + \left(0_{\mathrm{A}(x,sa)}\right)^a$ resulting in that $\left(0_{\mathrm{A}(x,sa)}\right)^a = 0_{\mathrm{A}(x,ta)}$. A similar calculation gives the equality ${}^{a'}\left(0_{\mathrm{A}(ta',y)}\right) = 0_{\mathrm{A}(sa',y)}$ as required.
(2) $0_{\mathrm{A}(sm,ta)} = \left(0_{\mathrm{A}(sm,tm)}\right)^a = (m+(-m))^a = m^a + (-m)^a$ resulting in that $(-m)^a = -m^a$, and similarly ${}^{a'}(-m) = -{}^{a'}m$.

In fact, for the equations in lemma 1 we don't need to be given a precrossed module. The equations $\left(0_{\mathrm{A}(x,sa)}\right)^a = 0_{\mathrm{A}(x,ta)}$ and $(-m)^a = -m^a$ are satisfied whenever A has a right action on M and the equations ${}^{a'}\left(0_{\mathrm{A}(ta',y)}\right) = 0_{\mathrm{A}(sa',y)}$ and ${}^{a'}(-m) = -{}^{a'}m$ are satisfied whenever A has a left action on M.

Theorem 1. *Let $\mathscr{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ be a precrossed module of R-algebroids and $\mathrm{V} = \{\mathrm{V}(x,y) \subseteq \mathrm{M}(x,y) : x,y \in \mathrm{A}_0\}$ be a family of subsets of M which generates M as a pre-R-algebroid and is closed under the actions of R and A. The Peiffer ideal $[\![\mathrm{M},\mathrm{M}]\!]$ of M is the ideal generated by the family $[\![\mathrm{V},\mathrm{V}]\!]_{\mathrm{g}} = \left\{ [\![\mathrm{V},\mathrm{V}]\!]_{\mathrm{g}}(x,y) \,\middle|\, x,y \in \mathrm{A}_0 \right\}$ of Peiffer commutators where*

$$[\![\mathrm{V},\mathrm{V}]\!]_{\mathrm{g}}(x,y) = \left\{ [\![v,v']\!]_1, [\![v,v']\!]_2 : v,v' \in \mathrm{V},\ x = sv,\ tv' = y \right\}$$

for all $x,y \in \mathrm{A}_0$.

Proof. Let $[\![\mathrm{V},\mathrm{V}]\!] = \{[\![\mathrm{V},\mathrm{V}]\!](x,y) : x,y \in \mathrm{A}_0\}$ be the two-sided ideal generated by $[\![\mathrm{V},\mathrm{V}]\!]_{\mathrm{g}}$ and let $[\![\mathrm{M},\mathrm{M}]\!]_{\mathrm{g}} = \left\{ [\![\mathrm{M},\mathrm{M}]\!]_{\mathrm{g}}(x,y) \,\middle|\, : x,y \in \mathrm{A}_0 \right\}$ be such that $[\![\mathrm{M},\mathrm{M}]\!]_{\mathrm{g}}(x,y) = \{[\![m,m']\!]_1, [\![m,m']\!]_2 : m,m' \in \mathrm{M},\ x = sm,\ tm' = y\}$. Note firstly that $[\![\mathrm{V},\mathrm{V}]\!](x,y) \subseteq [\![\mathrm{M},\mathrm{M}]\!](x,y) \subseteq \mathrm{Ker}\mu(x,y)$ since $[\![\mathrm{V},\mathrm{V}]\!]$ and $[\![\mathrm{M},\mathrm{M}]\!]$ are generated by $[\![\mathrm{V},\mathrm{V}]\!]_{\mathrm{g}}$ and $[\![\mathrm{M},\mathrm{M}]\!]_{\mathrm{g}}$, respectively, and since $[\![\mathrm{V},\mathrm{V}]\!]_{\mathrm{g}}(x,y) \subseteq [\![\mathrm{M},\mathrm{M}]\!]_{\mathrm{g}}(x,y)$ for all $x,y \in \mathrm{A}_0$. Secondly note that $[\![\mathrm{V},\mathrm{V}]\!]_{\mathrm{g}}$ and $[\![\mathrm{V},\mathrm{V}]\!]$ are closed under the actions of R and A since V is so. So each quotient group $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y) = \frac{\mathrm{M}(x,y)}{[\![\mathrm{V},\mathrm{V}]\!](x,y)}$ becomes an R-module with the addition and R-action defined, respectively, as $\overline{m_1} + \overline{m_2} = \overline{m_1 + m_2}$ and $r \cdot \overline{m} = \overline{r \cdot m}$ on cosets, and the family $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]} = \left\{ \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y) : x,y \in \mathrm{A}_0 \right\}$ of R-modules becomes a pre-R-algebroid and an A-module with the composition and A-action defined, respectively, as $\overline{m}\overline{m'} = \overline{mm'}$ and ${}^a\overline{m} = \overline{{}^a m}$, $\overline{m}^{a'} = \overline{m^{a'}}$ on cosets having compatible sources and targets with each other and with $a,a' \in \mathrm{A}$.

Moreover, since $[\![\mathrm{V},\mathrm{V}]\!](x,y) \subseteq \mathrm{Ker}\mu(x,y)$ for all $x,y \in \mathrm{A}_0$, μ reduces a pre-R-algebroid morphism $\overline{\mu} : \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]} \longrightarrow \mathrm{A}$ defined as $\overline{\mu}\,\overline{m} = \mu m$, and $\overline{\mu}$ is a precrossed module since μ is so. Now, let's see that $\overline{\mu}$ is a crossed module in three steps.

(I) Define the family $\overline{\mathrm{V}} = \left\{ \overline{\mathrm{V}(x,y)} : x,y \in \mathrm{A}_0 \right\}$ where

$$\overline{\mathrm{V}(x,y)} = \{\overline{v} = v + [\![\mathrm{V},\mathrm{V}]\!](x,y) : v \in \mathrm{V}(x,y)\}$$

for all $x,y \in \mathrm{A}_0$. Obviously $\overline{\mathrm{V}}$ is a generator family of the pre-R-algebroid $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$ and for all $w,x,y,z \in \mathrm{A}_0$, $v' \in \mathrm{V}(w,x)$, $v \in \mathrm{V}(x,y)$ and $v'' \in \mathrm{V}(y,z)$

$${}^{\overline{\mu}\overline{v'}}\overline{v} = {}^{\mu v'}\overline{v} = \overline{{}^{\mu v'}v} = \overline{v'v} = \overline{v'}\,\overline{v}$$

and

$$\overline{v}^{\overline{\mu}\overline{v''}} = \overline{v}^{\mu v''} = \overline{v^{\mu v''}} = \overline{vv''} = \overline{v}\overline{v''}$$

since $[\![v', v]\!]_2 = {}^{\mu v'}v - v'v \in [\![\mathrm{V},\mathrm{V}]\!](w,y)$ and $[\![v, v'']\!]_1 = v^{\mu v''} - vv'' \in [\![\mathrm{V},\mathrm{V}]\!](x,z)$.

(II) For each $x,y \in \mathrm{A}_0$ fix a $v_{xy} \in \mathrm{V}(x,y)$ and define $\mathrm{P}(x,y)$ as the set of all $\overline{m} \in \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$ satisfying for all $w,x,y,z \in \mathrm{A}_0$ the equalities ${}^{\overline{\mu}\,\overline{v_{wx}}}\overline{m} = \overline{v_{wx}}\,\overline{m}$ and $\overline{m}^{\overline{\mu}\,\overline{v_{yz}}} = \overline{m}\,\overline{v_{yz}}$.

(1) Clearly $\overline{\mathrm{V}(x,y)} \subseteq \mathrm{P}(x,y) \subseteq \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$ for all $x,y \in \mathrm{A}_0$.

(2) For all $w,x,y,z \in \mathrm{A}_0$ and $\overline{m}, \overline{m_1}, \overline{m_2} \in \mathrm{P}(x,y)$

$$\begin{aligned} {}^{\overline{\mu}\,\overline{v_{wx}}}\left(\overline{m_1}+\overline{m_2}\right) &= {}^{\overline{\mu}\,\overline{v_{wx}}}\,\overline{m_1} + {}^{\overline{\mu}\,\overline{v_{wx}}}\,\overline{m_2} = \overline{v_{wx}}\,\overline{m_1} + \overline{v_{wx}}\,\overline{m_2} = \overline{v_{wx}}\left(\overline{m_1}+\overline{m_2}\right), \\ {}^{\overline{\mu}\,\overline{v_{wx}}}\left(-\overline{m}\right) &= -\left({}^{\overline{\mu}\,\overline{v_{wx}}}\overline{m}\right) = -\left(\overline{v_{wx}}\,\overline{m}\right) = \overline{v_{wx}}\left(-\overline{m}\right) \end{aligned}$$

thanks to lemma 1 and the clear equalities $-\left(\overline{v_{wx}}\,\overline{m}\right) = \overline{v_{wx}}\left(-\overline{m}\right)$ and $-\overline{m} = \overline{-m}$ in the pre-R-algebroid $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$. Similarly it can be shown that $\left(\overline{m_1}+\overline{m_2}\right)^{\overline{\mu}\,\overline{v_{yz}}} = \left(\overline{m_1}+\overline{m_2}\right)\overline{v_{yz}}$ and $\left(-\overline{m}\right)^{\overline{\mu}\,\overline{v_{yz}}} = \left(-\overline{m}\right)\overline{v_{yz}}$. Thus $\overline{m_1}+\overline{m_2} \in \mathrm{P}(x,y)$ and $-\overline{m} \in \mathrm{P}(x,y)$ for all $\overline{m}, \overline{m_1}, \overline{m_2} \in \mathrm{P}(x,y)$ giving rise to $\mathrm{P}(x,y)$ to be a subgroup of $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$. Obviously $\mathrm{P}(x,y)$ is abelian since $\mathrm{M}(x,y)$ and $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$ is so.

(3) For all $w,x,y,z \in \mathrm{A}_0$, $\overline{m} \in \mathrm{P}(x,y)$ and $r \in \mathrm{R}$

$${}^{\overline{\mu}\,\overline{m_{wx}}}\left(r\cdot\overline{m}\right) = r\cdot\left({}^{\overline{\mu}\,\overline{m_{wx}}}\overline{m}\right) = r\cdot\left(\overline{m_{wx}}\,\overline{m}\right) = \overline{m_{wx}}\left(r\cdot\overline{m}\right)$$

and similarly $\left(r\cdot\overline{m}\right)^{\overline{\mu}\,\overline{m_{yz}}} = \left(r\cdot\overline{m}\right)\overline{m_{yz}}$. So $r\cdot\overline{m} \in \mathrm{P}(x,y)$ for all $\overline{m} \in \mathrm{P}(x,y)$ and $r \in \mathrm{R}$, and this causes $\mathrm{P}(x,y)$ to be an R-submodule of $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$.

(4) For all $w,x,y,z,t \in \mathrm{A}_0$, $\overline{m} \in \mathrm{P}(x,y)$ and $\overline{m''} \in \mathrm{P}(y,z)$

$${}^{\overline{\mu}\,\overline{m_{wx}}}\left(\overline{m}\overline{m''}\right) = \left({}^{\overline{\mu}\,\overline{m_{wx}}}\overline{m}\right)\overline{m''} = \left(\overline{m_{wx}}\,\overline{m}\right)\overline{m''} = \overline{m_{wx}}\left(\overline{m}\overline{m''}\right)$$

and similarly $\left(\overline{m}\overline{m''}\right)^{\overline{\mu}\,\overline{m_{zt}}} = \left(\overline{m}\overline{m''}\right)\overline{m_{zt}}$. That is, $\overline{m}\overline{m''} \in \mathrm{P}(x,z)$ for all $\overline{m} \in \mathrm{P}(x,y)$, $\overline{m''} \in \mathrm{P}(y,z)$ and so the family $\mathrm{P} = \{\mathrm{P}(x,y) : x,y \in \mathrm{A}_0\}$ is a pre-R-algebroid.

Thus, since $\overline{\mathrm{V}} = \left\{\overline{\mathrm{V}(x,y)} : x,y \in \mathrm{A}_0\right\}$ is a generator family of the pre-R-algebroid $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$ it must be the case that $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y) \subseteq \mathrm{P}(x,y)$ for all $x,y \in \mathrm{A}_0$ which means $\mathrm{P}(x,y) = \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$ and $\mathrm{P} = \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$.

The result we get from II is that for all $\overline{v}, \overline{v'} \in \overline{\mathrm{V}}$ and $\overline{m} \in \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$ with $tv = sm$ and $tm = sv'$ the equalities ${}^{\overline{\mu}\,\overline{v}}\overline{m} = \overline{v}\,\overline{m}$ and $\overline{m}^{\overline{\mu}\,\overline{v}} = \overline{m}\,\overline{v}$ hold.

(III) For all $x,y \in \mathrm{A}_0$ let $\mathrm{Q}(x,y)$ be the set of all $\overline{m} \in \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$ satisfying the equalities ${}^{\overline{\mu}\,\overline{m}}\overline{m''} = \overline{m}\overline{m''}$ and $\overline{m'}^{\overline{\mu}\,\overline{m}} = \overline{m'}\overline{m}$ for all $w,z \in \mathrm{A}_0$, $\overline{m'} \in \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(w,x)$ and $\overline{m''} \in \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(y,z)$. From II the inclusion $\overline{\mathrm{V}(x,y)} \subseteq \mathrm{Q}(x,y) \subseteq \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$ is clear. In addition, a similar procedure to that of II shows that $\mathrm{Q}(x,y)$ is an R-submodule of $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$ for all $x,y \in \mathrm{A}_0$ and the family $\mathrm{Q} = \{\mathrm{Q}(x,y) : x,y \in \mathrm{A}_0\}$ is a pre-R-algebroid. Thus, since $\overline{\mathrm{V}} = \left\{\overline{\mathrm{V}(x,y)} : x,y \in \mathrm{A}_0\right\}$ is a generator family of the pre-R-algebroid $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$ we get $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y) \subseteq \mathrm{Q}(x,y)$ for all $x,y \in \mathrm{A}_0$ which means $\mathrm{Q}(x,y) = \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}(x,y)$ and $\mathrm{Q} = \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$.

So the equalities ${}^{\overline{\mu}\,\overline{m}}\,\overline{m''} = \overline{m}\,\overline{m''}$ and $\overline{m'}^{\,\overline{\mu}\,\overline{m}} = \overline{m'}\,\overline{m}$ hold for all $\overline{m'}, \overline{m}, \overline{m''} \in \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$ having compatible sources and targets. That is, $\overline{\mu} : \frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]} \longrightarrow \mathrm{A}$ is a crossed module.

As a result, for all $x,y,z \in \mathrm{A}_0$, $m \in \mathrm{M}(x,z)$ and $m' \in \mathrm{M}(z,y)$ the equality

$$\overline{m^{\mu m'}} = \overline{m}^{\mu m'} = \overline{m}^{\overline{\mu m'}} = \overline{m}\overline{m'} = \overline{mm'}$$

and similarly the equality $\overline{{}^{\mu m}m'} = \overline{mm'}$ hold in $\frac{\mathrm{M}}{[\![\mathrm{V},\mathrm{V}]\!]}$ which means $m^{\mu m'} - mm', {}^{\mu m}m' - mm' \in [\![\mathrm{V},\mathrm{V}]\!](x,y)$, i.e. $[\![m,m']\!]_1, [\![m,m']\!]_2 \in [\![\mathrm{V},\mathrm{V}]\!](x,y)$. That is, $[\![\mathrm{M},\mathrm{M}]\!](x,y) \subseteq [\![\mathrm{V},\mathrm{V}]\!](x,y)$ for all $x,y \in \mathrm{A}_0$. But we know that $[\![\mathrm{V},\mathrm{V}]\!](x,y) \subseteq [\![\mathrm{M},\mathrm{M}]\!](x,y)$. So $[\![\mathrm{V},\mathrm{V}]\!](x,y) = [\![\mathrm{M},\mathrm{M}]\!](x,y)$ for all $x,y \in \mathrm{A}_0$ and $[\![\mathrm{V},\mathrm{V}]\!] = [\![\mathrm{M},\mathrm{M}]\!]$.

4 Applications to coproduct of crossed modules of R-algebroids

In [6], we've constructed in $\mathrm{XAlg}(R)/\mathrm{A}$ the coproduct of given two crossed A-modules $\mathscr{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ and $\mathscr{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ of R-algebroids by two different methods.

In the first method to get the coproduct $\mathscr{M} \circ_* \mathscr{N} = (\mu \circ_* \eta : \mathrm{M} \circ_* \mathrm{N} \longrightarrow \mathrm{A})$ firstly we've constructed the free product $\mathrm{M} * \mathrm{N}$ of pre-R-algebroids M and N, using reduced words, and secondly the precrossed module $\mu * \eta : \mathrm{M} * \mathrm{N} \longrightarrow \mathrm{A}$ which is defined as identity on A_0 and as $(\mu * \eta)\left(\sum\limits_i \widehat{w}_i\right) = \sum\limits_i (\mu * \eta)(\widehat{w}_i)$ on morphisms such that

$$(\mu * \eta)(\widehat{w}_i) = (\mu * \eta)\left(\widehat{k_{i_1}...k_{i_n}}\right) = (\mu * \eta)\left(\widehat{k_{i_1}}...\widehat{k_{i_n}}\right) = (\mu * \eta)\left(\widehat{k_{i_1}}\right)...(\mu * \eta)\left(\widehat{k_{i_n}}\right)$$

where $(\mu * \eta)\left(\widehat{k_{i_\alpha}}\right) = \mu k_{i_\alpha}$ if $k_{i_\alpha} \in \mathrm{M}$ and $(\mu * \eta)\left(\widehat{k_{i_\alpha}}\right) = \eta k_{i_\alpha}$ if $k_{i_\alpha} \in \mathrm{N}$ for each $\alpha \in \{1,2,...,n\}$. After that we've divided $\mathrm{M} * \mathrm{N}$ by its two-sided ideal $\mathbf{I}_{\mathrm{M}*\mathrm{N}}$ generated by all elements of the forms

$$\widehat{m^n - mn},\ \widehat{{}^m n - mn},\ \widehat{n'^m - n'm},\ \widehat{{}^{n'}m - n'm} \tag{1}$$

to get the quotient pre-R-algebroid $\mathrm{M} \circ_* \mathrm{N} = \frac{\mathrm{M}*\mathrm{N}}{\mathbf{I}_{\mathrm{M}*\mathrm{N}}}$, where the elements m^n, ${}^m n$, n'^m and ${}^{n'}m$ stands respectively for $m^{\eta n}$, ${}^{\mu m}n$, $n'^{\mu m}$ and ${}^{\eta n'}m$. Then we've defined $\mu \circ_* \eta$ as induced from $\mu * \eta$.

In the second method to get the coproduct $\mathscr{M} \circ_\ltimes \mathscr{N} = (\mu \circ_\ltimes \eta : \mathrm{M} \circ_\ltimes \mathrm{N} \longrightarrow \mathrm{A})$ firstly we've constructed the semidirect product $\mathrm{M} \ltimes \mathrm{N}$ of M and N via μ, and secondly the precrossed module $\mu \ltimes \eta : \mathrm{M} \ltimes \mathrm{N} \longrightarrow \mathrm{A}$ which is defined on A_0 as identity and on $\mathrm{M} \ltimes \mathrm{N}$ as $(\mu \ltimes \eta)(m,n) = \mu m + \eta n$. Then we've divided $\mathrm{M} \ltimes \mathrm{N}$ by its two-sided ideal $\mathbf{I}_{\mathrm{M}\ltimes\mathrm{N}}$ generated by all elements of the forms

$$(m^n, -{}^m n)\ ,\ \left({}^{n'}m, -n'^m\right) \tag{2}$$

to get the quotient pre-R-algebroid $\mathrm{M} \circ_\ltimes \mathrm{N} = \frac{\mathrm{M}\ltimes\mathrm{N}}{\mathbf{I}_{\mathrm{M}\ltimes\mathrm{N}}}$ and defined $\mu \circ_\ltimes \eta$ as induced from $\mu \ltimes \eta$.

Now we claim that the ideals $\mathbf{I}_{\mathrm{M}*\mathrm{N}}$ and $\mathbf{I}_{\mathrm{M}\ltimes\mathrm{N}}$ are both Peiffer ideals:

Proposition 2. $\mathbf{I}_{\mathrm{M}*\mathrm{N}}$ *is the Peiffer ideal of the pre-R-algebroid* $\mathrm{M} * \mathrm{N}$ *of the precrossed module* $\mu * \eta : \mathrm{M} * \mathrm{N} \longrightarrow \mathrm{A}$.

Proof. Clearly $\mathrm{M} * \mathrm{N}$ is generated as a pre-R-algebroid by the family $\widehat{\mathrm{M} \cup \mathrm{N}} = \left\{\widehat{\mathrm{M} \cup \mathrm{N}}(x,y) = \widehat{\mathrm{M}}(x,y) \cup \widehat{\mathrm{N}}(x,y) : x,y \in \mathrm{A}_0\right\}$ where $\widehat{\mathrm{M}}(x,y) = \{\widehat{m} | m \in \mathrm{M}(x,y)\}$, $\widehat{\mathrm{N}}(x,y) = \{\widehat{n} | n \in \mathrm{N}(x,y)\}$ and $\widehat{\mathrm{M} \cup \mathrm{N}}$ is closed under the actions of R and A. Then from Theorem 1 the Peiffer ideal $[\![\mathrm{M} * \mathrm{N}, \mathrm{M} * \mathrm{N}]\!]$ of $\mathrm{M} * \mathrm{N}$ is generated by the family $[\![\widehat{\mathrm{M} \cup \mathrm{N}}, \widehat{\mathrm{M} \cup \mathrm{N}}]\!]_{\mathrm{g}} = \left\{ [\![\widehat{\mathrm{M} \cup \mathrm{N}}, \widehat{\mathrm{M} \cup \mathrm{N}}]\!]_{\mathrm{g}}(x,y) \middle| x,y \in \mathrm{A}_0\right\}$ of Peiffer commutators where

$$[\![\widehat{\mathrm{M} \cup \mathrm{N}}, \widehat{\mathrm{M} \cup \mathrm{N}}]\!]_{\mathrm{g}}(x,y) = \left\{[\![\widehat{k}, \widehat{k'}]\!]_1, [\![\widehat{k}, \widehat{k'}]\!]_2 : \widehat{k}, \widehat{k'} \in \mathrm{M} \cup \mathrm{N},\ x = sk,\ tk' = y\right\}$$

for all $x,y\in \mathrm{A}_0$. Replacing each k and k' with m or n according to their belongings we see that the generators of $[\![\mathrm{M}*\mathrm{N},\mathrm{M}*\mathrm{N}]\!]$ are in fact of the forms

$$[\![\widehat{m},\widehat{n}]\!]_1,\ [\![\widehat{m},\widehat{n}]\!]_2,\ [\![\widehat{n'},\widehat{m}]\!]_1,\ [\![\widehat{n'},\widehat{m}]\!]_2,\ [\![\widehat{m},\widehat{m'}]\!]_1,\ [\![\widehat{m},\widehat{m'}]\!]_2,\ [\![\widehat{n''},\widehat{n}]\!]_1,\ [\![\widehat{n''},\widehat{n}]\!]_2$$

for all $m,m'\in\mathrm{M}$, $n,n',n''\in\mathrm{N}$ with $tm=tn''=sn=sm'$, $tn'=sm$. But

$$\begin{aligned}
[\![\widehat{m},\widehat{n}]\!]_1 &= \widehat{m}^{(\mu*\eta)(\widehat{n})}-\widehat{m}\widehat{n}=\widehat{m}^{\eta n}-\widehat{mn}=\widehat{m^{\eta n}}-\widehat{mn}=\widehat{m^{n}-mn},\\
[\![\widehat{m},\widehat{n}]\!]_2 &= {}^{(\mu*\eta)(\widehat{m})}\widehat{n}-\widehat{m}\widehat{n}={}^{\mu m}\widehat{n}-\widehat{mn}=\widehat{{}^{\mu m}n}-\widehat{mn}=\widehat{{}^{m}n-mn},\\
[\![\widehat{n'},\widehat{m}]\!]_1 &= \widehat{n'}^{(\mu*\eta)(\widehat{m})}-\widehat{n'}\widehat{m}=\widehat{n'}^{\mu m}-\widehat{n'm}=\widehat{n'^{\mu m}}-\widehat{n'm}=\widehat{n'^{m}-n'm},\\
[\![\widehat{n'},\widehat{m}]\!]_2 &= {}^{(\mu*\eta)\left(\widehat{n'}\right)}\widehat{m}-\widehat{n'}\widehat{m}={}^{\eta n'}\widehat{m}-\widehat{n'm}=\widehat{{}^{\eta n'}m}-\widehat{n'm}=\widehat{{}^{n'}m-n'm}
\end{aligned}$$

and

$$\begin{aligned}
[\![\widehat{m},\widehat{m'}]\!]_1 &= \widehat{m}^{(\mu*\eta)\left(\widehat{m'}\right)}-\widehat{m}\widehat{m'}=\widehat{m}^{\mu m'}-\widehat{mm'}=\widehat{m^{\mu m'}}-\widehat{mm'}=\widehat{mm'}-\widehat{mm'}=0,\\
[\![\widehat{m},\widehat{m'}]\!]_2 &= {}^{(\mu*\eta)(\widehat{m})}\widehat{m'}-\widehat{m}\widehat{m'}={}^{\mu m}\widehat{m'}-\widehat{mm'}=\widehat{{}^{\mu m}m'}-\widehat{mm'}=\widehat{mm'}-\widehat{mm'}=0.
\end{aligned}$$

Similarly it can be shown that $[\![\widehat{n''},\widehat{n}]\!]_1=[\![\widehat{n''},\widehat{n}]\!]_2=0$. As a result $[\![\mathrm{M}*\mathrm{N},\mathrm{M}*\mathrm{N}]\!]$ is generated by the elements of the forms

$$\widehat{m^{n}-mn},\ \widehat{{}^{m}n-mn},\ \widehat{n'^{m}-n'm},\ \widehat{{}^{n'}m-n'm}$$

which are the same as those in (1). That is, $[\![\mathrm{M}*\mathrm{N},\mathrm{M}*\mathrm{N}]\!]$ and $\mathbf{I}_{\mathrm{M}*\mathrm{N}}$ have exactly the same generators meaning that $\mathbf{I}_{\mathrm{M}*\mathrm{N}}=[\![\mathrm{M}*\mathrm{N},\mathrm{M}*\mathrm{N}]\!]$ as required.

Corollary 1. $\mathscr{M}\circ_*\mathscr{N}=(\mathscr{M}*\mathscr{N})^{\mathrm{cr}}$.

Proof. $\mathrm{M}\circ_*\mathrm{N}=\frac{\mathrm{M}*\mathrm{N}}{\mathbf{I}_{\mathrm{M}*\mathrm{N}}}=\frac{\mathrm{M}*\mathrm{N}}{[\![\mathrm{M}*\mathrm{N},\mathrm{M}*\mathrm{N}]\!]}=(\mathrm{M}*\mathrm{N})^{\mathrm{cr}}$ from proposition 2 and $\mu\circ_*\eta=(\mu*\eta)^{\mathrm{cr}}$ since they are both defined as induced from $\mu*\eta$.

Proposition 3. $\mathbf{I}_{\mathrm{M}\ltimes\mathrm{N}}$ *is the Peiffer ideal of* $\mathrm{M}\ltimes\mathrm{N}$.

Proof. The Peiffer ideal $[\![\mathrm{M}\ltimes\mathrm{N},\mathrm{M}\ltimes\mathrm{N}]\!]$ of $\mathrm{M}\ltimes\mathrm{N}$ is generated by the elements of the forms $[\![(m,n),(m',n')]\!]_1$ and $[\![(m,n),(m',n')]\!]_2$. But

$$\begin{aligned}
[\![(m,n),(m',n')]\!]_1 &= (m,n)^{(\mu\ltimes\eta)\left(m',n'\right)}-(m,n)\left(m',n'\right)=\left(m^{\mu m'+\eta n'},n^{\mu m'+\eta n'}\right)-\left(mm',n^{m'}+{}^{m}n'+nn'\right)\\
&=\left(mm'+m^{n'}-mm',n^{m'}+nn'-n^{m'}-{}^{m}n'-nn'\right)=\left(m^{n'},-{}^{m}n'\right),\\
[\![(m,n),(m',n')]\!]_2 &= {}^{(\mu\ltimes\eta)(m,n)}\left(m',n'\right)-(m,n)\left(m',n'\right)=\left({}^{\mu m+\eta n}m',{}^{\mu m+\eta n}n'\right)-\left(mm',n^{m'}+{}^{m}n'+nn'\right)\\
&=\left(mm'+{}^{n}m'-mm',{}^{m}n'+nn'-n^{m'}-{}^{m}n'-nn'\right)=\left({}^{n}m',-n^{m'}\right),
\end{aligned}$$

i.e. $[\![\mathrm{M}\ltimes\mathrm{N},\mathrm{M}\ltimes\mathrm{N}]\!]$ is generated by elements of the forms $\left(m^{n'},-{}^{m}n'\right)$ and $\left({}^{n}m',-n^{m'}\right)$ which are the same as those in (2). That is $[\![\mathrm{M}\ltimes\mathrm{N},\mathrm{M}\ltimes\mathrm{N}]\!]$ and $\mathbf{I}_{\mathrm{M}\ltimes\mathrm{N}}$ have exactly the same generators which means $\mathbf{I}_{\mathrm{M}\ltimes\mathrm{N}}=[\![\mathrm{M}\ltimes\mathrm{N},\mathrm{M}\ltimes\mathrm{N}]\!]$ as required.

Corollary 2. $\mathscr{M}\circ_\ltimes\mathscr{N}=(\mathscr{M}\ltimes\mathscr{N})^{\mathrm{cr}}$.

Proof. $\mathrm{M}\circ_\ltimes\mathrm{N}=\frac{\mathrm{M}\ltimes\mathrm{N}}{\mathbf{I}_{\mathrm{M}\ltimes\mathrm{N}}}=\frac{\mathrm{M}\ltimes\mathrm{N}}{[\![\mathrm{M}\ltimes\mathrm{N},\mathrm{M}\ltimes\mathrm{N}]\!]}=(\mathrm{M}\ltimes\mathrm{N})^{\mathrm{cr}}$ from proposition 3 and $\mu\circ_\ltimes\eta=(\mu\ltimes\eta)^{\mathrm{cr}}$ since they are both defined as induced from $\mu\ltimes\eta$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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