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A Note on genus problem and conjugation of the normalizer

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Abstract: In this paper, we find a certain part of the total order of ramification of $\Gamma_0(N)$ over its normalizer $Nor(N)$ in PSL(2,$\mathbb{R}$) and determine a suitable element of PSL(2,$\mathbb{R}$) which $\Gamma_0(N)$ is conjugate to a subgroup of $Nor(M)$ by this element for some square-free integer M.

Keywords: Normalizer, signature, ramification, genus

1 Introduction

Let PSL(2,$\mathbb{R}$) denote the group of all linear fractional transformations

$$T : z \longrightarrow \frac{az+b}{cz+d}, \text{ where } a,b,c,d \text{ are real and } ad-bc=1.$$

This is the automorphism group of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Γ, the modular group, is the subgroup of PSL(2,$\mathbb{R}$) such that a,b,c and d are integers. $\Gamma_0(N)$ is the subgroup of Γ with $N|c$. In terms of matrix representation, the elements of PSL(2,$\mathbb{R}$) correspond to the matrices

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1.$$

But here we omit the symbol $\pm$ for brevity. On the other hand, $Nor(N)$ will denote the normalizer of $\Gamma_0(N)$ in PSL(2,$\mathbb{R}$) consists exactly of the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \tag{1}$$

where all symbols represent integers, h is the largest divisor of 24 for which $h^2|N$, $e>0$ is an exact divisor of N/h^2 and the determinat of matrix is $e>0$. (We say that x is an exact divisor of y, denoted by $x \parallel y$, if $x \mid y$ and $(x, y/x) = 1$). One of the important subgroups of the normalizer $Nor(N)$ is the Atkin-Lehner Group

$$\Gamma_0^+(N) = \left\{ \begin{pmatrix} af & b \\ cN & df \end{pmatrix} \in Nor(N) : f \parallel N \right\} \tag{2}$$

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which is the subgroup generated by $\Gamma_0(N)$ with all Atkin-Lehner involutions. It is known that $Nor(N)$ is a finitely generated Fuchsian group whose fundamental domain has finite area, so it has a signature consisting of the geometric invariants

$$(g; m_1, \ldots, m_r, s) \tag{3}$$

where g is the genus of the compactified quotient space, $m_1, \ldots, m_r$ are the periods of the elliptic elements and s is the parabolic class number. From (3), the hyperbolic measure is

$$\mu\left(Nor(N)\right) = 2\pi \left\{ 2(g-1) + \sum_{i=1}^{r} (1 - \frac{1}{m_i}) + s \right\}. \tag{4}$$

In [4], Maclachlan found the signature of the normalizer when N is square-free. Akbaş and Singerman, in [1], when N is arbitrary; calculated all the invariant of the signature of the normalizer except for the genus g and the number of periods 2. If one of these two were known, anyone would find all signature by using (4). Lastly, for $\Gamma_0^+(kN^2)$ with some special conditions, Lang obtained the signature and computed classes of order 2 by using number theoretical results[3].

Since the genus g of the normalizer is not known yet when N is not square-free, we approach to the problem by different way. Actually, Lang also points out same idea in [3]. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two compact Riemann surfaces, and $f : \mathfrak{B} \to \mathfrak{B}'$ a holomorphic mapping. Then $(\mathfrak{B}', f)$ is called a *covering* of $\mathfrak{B}$. If g and g' are the genera of $\mathfrak{B}$ and $\mathfrak{B}'$, respectively, then these integers are connected by *Riemann-Hurwitz formula*

$$2g' - 2 = n(2g-2) + \sum_{z \in \mathfrak{B}'} (e_z - 1) \tag{5}$$

where n is the degree of the covering and e_z is the ramification index at z.

2 Ramification index

In this section we will find a certain part of the total order of ramification of $\Gamma_0(N)$ over $Nor(N)$ via Riemann-Hurwitz formula. We give some well-known facts as following lemmas without proof.

Lemma 1. *$Nor(N)$ is conjugate to $\Gamma_0^+(N/h^2)$ by* $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$.

So, if we find the genus g of $\Gamma_0^+(N)$ where N is not square-free, the genus of $Nor(N)$ is found.

Lemma 2. *The index of $\Gamma_0(N)$ in $\Gamma_0^+(N)$ is 2^r where r is the number of distinct prime factors of N, the elements of $\Gamma_0^+(N) - \Gamma_0(N)$ are of the form $\omega\delta$ where ω is an Atkin-Lehner involution and $\delta \in \Gamma_0(N)$.*

We suppose that N is not square-free. Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and the set of cusps of $\Gamma_0(N)$. We can endow the quotient space $\mathbb{H}^*/\Gamma_0(N)$ with a topological structure so that $\mathbb{H}^*/\Gamma_0(N)$ is a compact Riemann surface. By the idea of *commensurability* of groups, we conclude that $\mathbb{H}^*/\Gamma_0^+(N)$ is a compact Riemann surface. Applying the Riemann-Hurwitz formula to the map from $\mathbb{H}^*/\Gamma_0(N)$ to $\mathbb{H}^*/\Gamma_0^+(N)$ we have

$$g_0 - 1 = \frac{1}{2}\eta(N) + 2^r(g-1)$$

where g_0 is the genera of $\Gamma_0(N)$, as in [5].

Lemma 3. *Let $\hat{\mathbb{Q}}$ be $\mathbb{Q} \cup \infty$ the set of cusps of Γ. The number of orbits of $\hat{\mathbb{Q}}$ under $\Gamma_0(N)$ is $\sum_{0<d|N} (\varphi((d, N/d))$ when $N \geq 2$, where φ is Euler's function. Moreover the representatives of $\hat{\mathbb{Q}}$ under $\Gamma_0(N)$ can be chosen as a/d with $d|N$ with $0 \leq a \leq N$ and can be identifications between such elections a/d.*

It can be easily seen that the square of every element of $\Gamma_0^+(N)$ is in $\Gamma_0(N)$. The order of an elliptic fixed point of $\Gamma_0^+(N)$ is once or twice its order in $\Gamma_0(N)$, hence one of the numbers 2, 3, 4, and 6. As for parabolic elements of $\Gamma_0(N)$, if $4 \parallel N$ then there are parabolic elements of the form

$$q = \begin{pmatrix} 4a & b \\ 4N_1c & 4(1-a) \end{pmatrix} \in \Gamma_w(N)(\text{ of det } 4\text{ }),\ \text{where } 4N_1 = N,$$

which are not in $\Gamma_0(N)$. If $4 \nparallel N$, then every parabolic element is in $\Gamma_0(N)$. As we see above all ramification indexes are 2. Therefore $\eta(N)$ is simply the number $m+n$ where m (respectively n) is the number of points of $\hat{\mathbb{Q}}$ *mod* $\Gamma_0(N)$(respectively $\mathbb{H}$ *mod* $\Gamma_0(N)$) whose stabilizers in $\Gamma_0(N)$ have index 2 in their stabilizer in $\Gamma_0^+(N)$ by Riemann-Roch formula. Let us define

$$\eta'(N) := \sum_{\substack{d|N \\ 2\|d}} \varphi\left((d, \frac{N}{d})\right), \text{where } 4 \parallel N,$$

and φ is Euler's function.

Theorem 1. *Suppose* $4 \parallel N$. *Then the number of orbits of* $\hat{\mathbb{Q}}$ *under* $\Gamma_0(N)$ *whose stabilizers in* $\Gamma_0(N)$ *have index* 2 *in their stabilizer in* $\Gamma_0^+(N)$ *are exactly* $\eta'(N)$.

Proof. We see that there is only one type of parabolic element in $\Gamma_0^+(N)\backslash\Gamma_0(N)$, which is

$$q = \begin{pmatrix} 4a & b \\ 4N_1c & 4(1-a) \end{pmatrix}, \text{ of determinant } 4 \parallel N.$$

Furthermore, q fixes $\frac{2a-1}{2N_1c}$. This means that fixed point of any parabolic elements is of the form $\frac{k}{2d}$, $d|N_1$ (Because $(2N_1c, N) = 2d$, $d|N_1$, d is not even).

Conversely we show that the rational $\frac{k}{2d}$, $d|N_1$, is fixed by parabolic elements. Since det $q = 4^2a(1-a) - 4bN_1c = 4$, then $(2a-1)^2 = -bN_1c$. So, if we are given a rational number $\frac{k}{2d}$ with $d|N_1$, we can find a parabolic element q of the above form, as follows.

Suppose $N_1 = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ and $d = p_1^{\beta_1}\cdots p_\ell^{\beta_\ell}$, where $0 \le \beta_i \le \alpha_i, i = 1,2,\ldots,\ell$ and $\ell \le r$.
And now suppose $\beta_i \le \left[\frac{\alpha_i}{2}\right]$ for all i. Since $(2a-1)^2 = -bcp_1^{\alpha_1}\cdots p_r^{\alpha_r}$, if we

choose $c = 1$ and
$b = -k^2 p_1^{\alpha_1-2\beta_1} p_2^{\alpha_2-2\beta_2}\cdots p_\ell^{\alpha_\ell-2\beta_\ell} p_{\ell+1}^{\alpha_{\ell+1}}\cdots p_r^{\alpha_r}$, we get

$(2a-1)^2 = k^2 p_1^{2\alpha_1-2\beta_1} p_2^{2\alpha_2-2\beta_2}\cdots p_\ell^{2\alpha_\ell-2\beta_\ell} p_{\ell+1}^{2\alpha_{\ell+1}}\cdots p_r^{2\alpha_r}$.

$$\frac{2a-1}{2p_1^{\alpha_1}\cdots p_r^{\alpha_r}} = \frac{k}{2p_1^{\beta_1}\cdots p_\ell^{\beta_\ell}} = \frac{k}{2d}.$$

If we are given $\frac{k}{2p_1^{\beta_1}\cdots p_\ell^{\beta_\ell}}$ so that some $\beta_i > \left[\frac{\alpha_i}{2}\right]$, then the fixed point of q is

$$\frac{2a-1}{2p_1^{\alpha_1}\cdots p_r^{\alpha_r}c} = \frac{2a-1}{2p_1^{\alpha_1}\cdots p_\ell^{\alpha_\ell} p_{\ell+1}^{\alpha_{\ell+1}}\cdots p_r^{\alpha_r}c}.$$

Suppose $p_1^{\beta_1}\cdots p_m^{\beta_m}$, $m \le \ell$ so that $\beta_i > \left[\frac{\alpha_i}{2}\right]$, $i = 1,2,\ldots,m$, then take c to be $p_1^{s_1}\cdots p_m^{s_m}$ so that $\beta_i \le \left[\frac{\alpha_i+s_i}{2}\right]$, $i = 1,2,\ldots,m$. And then take b to be as above.

3 Conjugation

The normalizer $Nor(N)$ is commensurable with the modular group Γ. In [2], Helling has shown that any subgroup of PSL(2,$\mathbb{R}$) commensurable with Γ is conjugate to a subgroup of $Nor(M)$, for some square-free M.

We will now find an element in PSL(2,$\mathbb{R}$) by which $Nor(N)$ is conjugate to a subgroup A of $Nor(M)$ for some square-free integer M.

Theorem 2. *Let $N = 2^{\alpha}3^{\beta}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$ be the prime power decomposition of N. Then $Nor(N)$ is conjugate to a subgroup A of $Nor(2^{\varepsilon_1}3^{\varepsilon_2}p_3^{\varepsilon_3}\cdots p_r^{\varepsilon_r})$, where $\varepsilon_1 = \alpha - 2\left[\frac{\alpha}{2}\right]$, $\varepsilon_2 = \beta - 2\left[\frac{\beta}{2}\right]$, $\varepsilon_i = \alpha_i - 2\left[\frac{\alpha_i}{2}\right]$ for $i = 3,\cdots,r$, where [x] is the largest integer $\leq x$.*

Proof. Now we find an element $T \in$ PSL(2,$\mathbb{R}$) which makes $Nor(N)$ conjugate to A. Let us take T of the form $\begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$ and let $M := 2^{\varepsilon_1}3^{\varepsilon_2}p_3^{\varepsilon_3}\cdots p_r^{\varepsilon_r}$, where a is an integer. Then

$$\begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} N & N-1 \\ N & N \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & a \end{pmatrix} \in Nor(M).$$

So the matrix

$$S := \begin{pmatrix} (a+1)N/a & [(a^2-1)N/a]-a \\ N/a & (a-1)N/a \end{pmatrix}$$

has determinant N.

If S is an element of $Nor(M)$ then det S must be $2^{\varepsilon_1}3^{\varepsilon_2}p_3^{\varepsilon_3}\cdots p_r^{\varepsilon_r}$. So we define $K := 2^{t_1}3^{t_2}p_3^{t_3}\cdots p_r^{t_r}$, where $t_1 = \left[\frac{\alpha}{2}\right], t_2 = \left[\frac{\beta}{2}\right], t_i = \left[\frac{\alpha_i}{2}\right], i = 3,\cdots,r$. Then dividing all letters of S by K, so that we get the matrix

$$\begin{pmatrix} (a+1)N/aK & * \\ N/aK & (a-1)N/aK \end{pmatrix}.$$

But

$$\frac{N}{aK} = \frac{2^{\varepsilon_1}3^{\varepsilon_2}p_3^{\varepsilon_3}\cdots p_r^{\varepsilon_r}}{a\cdot 2^{[\alpha/2]}3^{[\beta/2]}p_3^{[\alpha_3/2]}\cdots p_r^{[\alpha_r/2]}}$$

must divide $2^{\varepsilon_1}3^{\varepsilon_2}p_3^{\varepsilon_3}\cdots p_r^{\varepsilon_r}$. So, we obtain $a = 2^{[\alpha/2]}3^{[\beta/2]}p_3^{[\alpha_3/2]}\cdots p_r^{[\alpha_r/2]}$, that is K.

Now, take K instead of a and examine the general case.

$$\begin{pmatrix} K & 1 \\ 0 & 1 \end{pmatrix} Nor(N) \begin{pmatrix} 1 & -1 \\ 0 & K \end{pmatrix} \leq Nor(2^{\varepsilon_1}3^{\varepsilon_2}p_3^{\varepsilon_3}\cdots p_r^{\varepsilon_r}).$$

In this case,

$$S^* = \begin{pmatrix} K & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & K \end{pmatrix}$$

$$= \begin{pmatrix} ae+(cN/hK) & -ae-(cN/hK)+Kb/h+de \\ cN/hK & -cN/hK+de \end{pmatrix}, \ \det S^* = e.$$

Suppose that $N=2^{\alpha}3^{\beta}L$, where L is $p_3^{\alpha_3}\cdots p_r^{\alpha_r}$. Then $h=2^{\min\{3,[\alpha/2]\}}3^{\min\{1,[\beta/2]\}}$ so that $\det=e:=2^{\alpha_1}3^{\beta_1}\parallel\frac{N}{h^2}=2^{\alpha-2\min\{3,[\alpha/2]\}}3^{\beta-2\min\{1,[\beta/2]\}}L$. In this case, e could be

$$1,\ 2^{\alpha-2\min\{3,[\alpha/2]\}},\ 3^{\beta-2\min\{1,[\beta/2]\}}\text{ or }2^{\alpha-2\min\{3,[\alpha/2]\}}3^{\beta-2\min\{1,[\beta/2]\}}.$$

Now we make e square-free. For this, we divide all letters of S^* by

$$\ell:=2^{[\alpha_1/2]}\ 3^{[\beta_1/2]}$$

Let $e_0:=e/\ell=2^{\alpha-[\alpha/2]}\ \ 3^{\beta-[\beta/2]}$ and $\frac{N}{hK}=2^{\alpha-[\alpha/2]-\min\{3,[\alpha/2]\}}\ \ 3^{\beta-[\beta/2]-\min\{1,[\beta/2]\}}L_0$ where $L_0=p_3^{\alpha_3-[\alpha_3/2]}\cdots p_r^{\alpha_r-[\alpha_r/2]}$. If we divide $\frac{N}{hK}$ by ℓ, we get the number

$$\xi:=2^{\alpha-[\alpha/2]-[\alpha_1/2]-\min\{3,[\alpha/2]\}}\ 3^{\beta-[\beta/2]-[\beta_1/2]-\min\{1,[\beta/2]\}}L_0.$$

There are two possibilities ; $\alpha_1=0$ or $\alpha_1=\alpha-2\min\{3,[\alpha/2]\}$.
(I) Suppose that $\alpha_1=0$, then $\xi=2^{\alpha-[\alpha/2]-\min\{3,[\alpha/2]\}}\ 3^{\beta-[\beta/2]-[\beta_1/2]-\min\{1,[\beta/2]\}}L_0$.
As for β_1, again there are two possibilities; $\beta_1=0$ or $\beta_1=\beta-2\min\{1,[\beta/2]\}$. If $\beta_1=0$, then $e_0=1$. If $\beta_1=\beta-2\min\{1,[\beta/2]\}$, then

$$\lambda:=\frac{\xi}{e_0}=2^{\alpha-[\alpha/2]-\min\{3,[\alpha/2]\}}\ 3^{[\beta_1/2]-[\beta/2]+\min\{1,[\beta/2]\}}L_0$$

is an integer.
(II) Suppose that $\alpha_1=\alpha-2\min\{3,[\alpha/2]\}$. In this case,

$$\lambda=2^{[\alpha_1/2]-[\alpha/2]+\min\{3,[\alpha/2]\}}\ 3^{\beta-\beta_1+[\beta_1/2]-[\beta/2]-\min\{1,[\beta/2]\}}L_0.$$

If $\beta_1=0$, then $\lambda_1=2^{[\alpha_1/2]-[\alpha/2]+\min\{3,[\alpha/2]\}}\ 3^{\beta-[\beta/2]-\min\{1,[\beta/2]\}}L_0$.

If $\beta_1=\beta-2\min\{1,[\beta/2]\}$, then

$$\lambda_2=2^{[\alpha_1/2]-[\alpha/2]+\min\{3,[\alpha/2]\}}\ 3^{[\beta_1/2]-[\beta/2]+\min\{1,[\beta/2]\}}L_0.$$

As above both λ_1 and λ_2 will be integers.

Consequently we get one of the elements T to be $\begin{pmatrix}2^{t_1}3^{t_2}p_3^{t_3}\cdots p_r^{t_r} & 1\\ 0 & 1\end{pmatrix}$.

Theorem 3. *Let $N=p_1^{\alpha_1}\cdots p_i^{\alpha_i}\cdots p_r^{\alpha_r}$ be the prime power decomposition of N such that the α_t are even, where $t=1,2,\ldots,i-1$. And let A be as in Theorem 1. Then*

$$|Nor(p_i\cdots p_r):A|=\frac{2^{r+1}p_1^{\alpha_1}\cdots p_{i-1}^{\alpha_{i-1}}p_i^{\alpha_i-1}\cdots p_r^{\alpha_r-1}}{2^{\rho+i}h^2}\prod_{p|N/\left(h\left(p_1^{\alpha_1}\cdots p_{i-1}^{\alpha_{i-1}}\right)\right)^2}\left(1+\frac{1}{p}\right)$$

where ρ is the number of prime factors of $\frac{N}{h^2}$.

Proof. $\mu(A) = \mu(Nor(N))$, due to conjugacy, therefore

$$|Nor(p_i \cdots p_r) : A| = \frac{\mu(Nor(N))}{\mu(Nor(p_i \cdots p_r))} = \frac{2\pi \frac{N}{3 \cdot 2^{\rho+1} h^2} \prod_{p|\frac{N}{h^2}} \left(1+\frac{1}{p}\right)}{2\pi \frac{p_i \cdots p_r}{3 \cdot 2^{r-i+2}} \prod_{p|p_i \cdots p_r} \left(1+\frac{1}{p}\right)}$$
$$= \frac{2^{r+1} p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} \cdots p_r^{\alpha_r - 1}}{2^{\rho+i} h^2} \prod_{p|N/\left(h\left(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}\right)\right)^2} \left(1+\frac{1}{p}\right),$$

where ρ is the number of prime factors of $\frac{N}{h^2}$ and $h(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}})$ is the biggest integer dividing 24 and its square divides N; i-1 is the number of primes whose exponents are even in N.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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