New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.213

A generalized fixed point theorem in non-Newtonian calculus

Mehmet Kir

Department of Engineering, Faculty of Engineering, Sirnak University, Sirnak, Turkey

Received: 6 April 2016, Accepted: 26 September 2016
Published online: 20 October 2017.

Abstract: In this paper, a generalized fixed point theorem and its results are established in the concept of multiplicative distance which was introduced by Agamirza et.al [3] to improve the non-Newtonian calculus. Our results include some existing results in the concept of multiplicative metric space.

Keywords: Fixed point, multiplicative distance, multiplicative metric,non-Newtonian calculus.

1 Introduction

The differentiation and integration are basic tools in the set of Newton-Leibnitz calculus. Grossman and Kantz [1] introduced definitions of a new type of derivate and integral, by changing the roles of addition and subtraction to multiplication and division respectively. Thus, they established a new type calculus that called multiplicative calculus.

Agamirza et. al. [3] established multiplicative distance d^*on $\mathbb{R}_{++} = (0,\infty)$ as an alternative to the usual distance on $\mathbb{R}_{++}$ and introduced some related properties and definitions on multiplicative calculus. Authors introduced multiplicative derivative, multiplicative integral etc. as well as showed that some facts of Newtonian calculus and some problems of semigroups of linear operators were proved easily through multiplicative calculus.

A natural question is that whether one can introduce a multiplicative metric as alternative to the usual metric.

In 2012, Özavşar and Çevikel [5] introduced the definition of multiplicative metric space and they investigated some topological properties. They also introduced an analogue of Banach contraction principle in the context of multiplicative metric space.

In this short note, we are aim to give a generalized fixed point theorem and related results in the framework multiplicative metric space.

2 Mathematical preliminaries

Let $\mathbb{R}^n_{++}$ be the collection of $n-$tubles of positive real numbers. Let $d^* : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}$ be defined as follows

$$d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*$$

* Corresponding author e-mail: mehmetkir04@gmail.com

where $x=(x_1,x_2,\cdots,x_n)$, $y=(y_1,y_2,\cdots,y_n)\in\mathbb{R}^n_{++}$ and $||^*:\mathbb{R}_{++}\to\mathbb{R}_{++}$ such that

$$|a|^*=\begin{cases} a \ if \ a\geq 1\\ \frac{1}{a} \ if \ a<1\end{cases}$$

Note that d^* has the following properties

(m1) $d^*(x,y)>1$ for all $x,y\in\mathbb{R}_{++}$ and $d^*(x,y)=1\Longleftrightarrow x=y$,
(m2) $d^*(x,y)=d^*(y,x)$ for all $x,y\in\mathbb{R}_{++}$,
(m3) $d^*(x,z)\leq d^*(x,y)\,d^*(y,z)$ for all $x,y,z\in\mathbb{R}_{++}$.

The mapping d^* is called multiplicative distance which was introduced by Agamirza et.al [3].

Özavşar and Çelikel [5], by moving the roles of multiplicative distance to any nonempty set X, introduced multiplicative metric space and its topological concept.

Definition 1. *[5] Let X be a nonempty set. Define the mapping $M:X\times X\to\mathbb{R}$ such that*

(M1) *$M(x,y)>1$ for all $x,y\in X$ and $M(x,y)=1\Longleftrightarrow x=y$,*
(M2) *$M(x,y)=M(y,x)$ for all $x,y\in X$,*
(M3) *$M(x,z)\leq M(x,y)\,M(y,z)$ for all $x,y,z\in X$ [multiplicative triangle inequality]*

then, M is called a multiplicative metric and the pair (X,M) is called a multiplicative metric space.

Note that, the definition of metric and definition of the multiplicative metric are independent. The following examples show that neither a metric is a multiplicative metric nor a multiplicative metric is a usual metric.

Example 1. Let $X=\mathbb{R}$ and define the usual metric d as $d(x,y)=|x-y|$ for all $x,y\in\mathbb{R}$.We have

$$d(1,5)=4>3=d(1,2)\,d(2,5)$$

that is, d is not a multiplicative metric on $\mathbb{R}$.

Example 2. Let $M:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, be defined by $M(x,y)=e^{|x-y|}$ for all $x,y\in\mathbb{R}$, then M is a multiplicative metric on $\mathbb{R}$. It is clear that M satisfies all conditions $M1-M3$. But it is not usual metric since M does not satisfy triangular inequality of the usual metric. Indeed, we have

$$M(1,5)=e^4>2e^2=M(1,3)+M(3,5).$$

Some basic notions given by Özavşar and Çevikel [5] are following.

Definition 2. *Let (X,M) be a multiplicative metric space, $x\in X$ and $\varepsilon>1$. Multiplicative open ball of radious r with center x is defined by*

$$B_\varepsilon(x)=\{y\in X: M(x,y)<\varepsilon\}$$

Similarly, the multiplicative closed ball is defined by

$$\overline{B_\varepsilon(x)}=\{y\in X: M(x,y)\leq\varepsilon\}$$

Corollary 1. *Every multiplicative metric space is a topological space based on the set of all multiplicative open sets.*

Definition 3. *A sequence* $\{x_n\}$ *in a multiplicative metric space* (X,M) *is said to be multiplicative convergent to a point* $x \in X$ *if for an abritrary* $\varepsilon > 1$ *there exists a positive integer* n_0 *such that*

$$M(x_n, x) < \varepsilon$$

for all $n \geq n_0$.

Definition 4. *A sequence* $\{x_n\}$ *in a multiplicative metric space* (X,M) *is said to be multiplicative Cauchy sequence if for all* $\varepsilon > 1$ *there exists a positive integer* n_0 *such that*

$$M(x_n, x_m) < \varepsilon$$

for all $m, n \geq n_0$.

Definition 5. *A multiplicative metric space* (X,M) *is complete if every multiplicative Cauchy sequence is multiplicative convergent.*

Definition 6. *[Multiplicative Continuity] Let* (X, M_X) *and* (Y, M_Y) *be two multiplicative metric spaces and* $T : X \to Y$ *be a function. If for every* $\varepsilon > 1$, *there exists* $\delta > 1$*such that*

$$T(B_\delta(x)) \subset B_\varepsilon(T(x))$$

then, T *called multiplicative continuous at* $x \in X$.

Definition 7. *Let* (X,M) *be a multiplicative metric space. A mapping* T *is called multiplicative contraction if there exists* $\lambda \in [0,1)$ *such that*

$$M(Tx, Ty) \leq [M(x,y)]^{\lambda}$$

for all $x, y \in X$.

Theorem 1. *[Multiplicative Banach Contraction Principle] Let* (X,M) *be a multiplicative complete metric space. If the mapping* $T : X \to X$ *is a multiplicative contraction, then* T *has a unique fixed point in* X.

Some other fixed point results for multiplicative metric spaces can be find Refs. [5]-[9].

3 Main results

Definition 8. *Let* (X,M) *be a multiplicative metric space. A self mapping* T *on* X *is called multiplicative Ciric type contraction mapping, if for all* $x, y \in X$ *there exists* $\alpha, \beta, \gamma, \delta \geq 0$ *such that*

$$M(Tx, Ty) \leq [M(x,y)]^{\alpha} [M(x, Tx)]^{\beta} [M(y, Ty)]^{\gamma} [M(x, Ty) M(y, Tx)]^{\delta} \tag{1}$$

and $\alpha + \beta + \gamma + 2\delta < 1$.

Theorem 2. *Let* (X,M) *be a complete multiplicative metric space. If a self mapping* T *on* X *is multiplicative Ciric type contraction, then* T *has a unique fixed point in* X.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for $n \geq 1$. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then, we get the proof. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then we have

$$\begin{aligned} M(x_{n+1}, x_n) &= M(Tx_n, Tx_{n-1}) \\ &\leq [M(x_n, x_{n-1})]^{\alpha} . [M(x_n, x_{n+1})]^{\beta} . [M(x_{n-1}, x_n)]^{\gamma} [M(x_n, x_n) . M(x_{n-1}, x_{n+1})]^{\delta} \\ &\leq [M(x_n, x_{n-1})]^{\alpha} . [M(x_n, x_{n+1})]^{\beta} . [M(x_{n-1}, x_n)]^{\gamma} [M(x_{n-1}, x_n)]^{\delta} . [M(x_n, x_{n+1})]^{\delta} \\ &= [M(x_n, x_{n-1})]^{\alpha+\gamma+\delta} [M(x_n, x_{n+1})]^{\beta+\delta} . \end{aligned} \tag{2}$$

This implies that

$$M(x_{n+1}, x_n) \leq [M(x_n, x_{n-1})]^k$$

where

$$k = \frac{\alpha+\gamma+\delta}{1-(\beta+\delta)} < 1.$$

Continuing this way, we have

$$M(x_{n+1}, x_n) \leq M(x_n, x_{n-1})^k \leq M(x_{n-1}, x_{n-2})^{k^2} \vdots \leq M(x_1, x_0)^{k^n} . \tag{3}$$

Letting $n \to \infty$ in (3), we get that

$$\lim_{n\to\infty} M(x_{n+1}, x_n) = 1.$$

Let $m, n \in \mathbb{N}$ with $m > n$, then from multiplicative triangle inequality, we have

$$M(x_m, x_n) \leq M(x_m, x_{m-1}) . M(x_{m-1}, x_{m-2}) \cdots M(x_{n-1}, x_n) . \tag{4}$$

By using (3), we have

$$M(x_m, x_n) \leq \quad M(x_1, x_0)^{\left[k^{m-1}+k^m+\cdots+k^n\right]} \leq M(x_1, x_0)^{\frac{k^n}{1-k}} . \tag{5}$$

Letting $m, n \to \infty$, we have

$$\lim_{n,m\to\infty} M(x_m, x_n) = 1$$

this implies that the sequence $\{x_n\}$ is a multiplicative Cauchy sequence. Since (X, M) is complete, there exists $u \in X$ such that $x_n \to u$, that is

$$\lim_{n\to\infty} M(x_n, u) = 1$$

Now, we claim that $u \in X$ is a fixed point of T. In fact, from multiplicative triangle inequality, we have

$$M(Tu, u) \leq M(Tu, Tx_n) M(Tx_n, u) \leq [M(u, x_n)]^{\alpha} . [M(u, Tu)]^{\beta} [M(x_n, x_{n+1})]^{\gamma} [M(u, Tx_n) . M(x_n, Tu)]^{\delta} M(Tx_n, u) . \tag{6}$$

Letting $n \to \infty$ in (6), we have

$$M(Tu, u) \leq [M(Tu, u)]^{\beta+2\delta+1}$$

this is a contradiction unless $M(Tu, u) = 1$. This implies that $u = Tu$. To show the uniqueness of the fixed point, if possible assume that $u' \in X$ is an other fixed point of T, that is $Tu' = u'$. Thus, we have

$$\begin{aligned} M(u, u') &= M(Tu, Tu') \leq [M(u, u')]^{\alpha} [M(u, Tu)]^{\beta} [M(u', Tu')]^{\gamma} [M(u, Tu') M(u', Tu)]^{\delta} \\ &= [M(u, u')]^{\alpha+2\delta} . \end{aligned} \tag{7}$$

The equation (7) is a contradiction unless $M(u,u')=1$, that is $u=u'$. Thus, the fixed point is unique.

Example 3. Let $X=[0,2]$ endowed with multiplicative metric $M(x,y)=e^{|x-y|}$ and define the mapping $T:X\to X$ as following

$$Tx=\begin{cases}\frac{x}{9} & if\ x\in[0,1]\\ \frac{x}{10} & if\ x\in(1,2]\end{cases}$$

Note that if $x=\frac{999}{1000}, y=\frac{1001}{1000}$, then $|Tx-Ty|>5|x-y|$ which implies that

$$e^{|Tx-Ty|}>\left[e^{|x-y|}\right]^5 \Longleftrightarrow M(Tx,Ty)>[M(x,y)]^5$$

that is T is not multiplicative contraction on whole X. In the other hand, T is multiplicative Ciric type contraction with $\alpha=\frac{1}{10}, \beta=\gamma=\frac{1}{4}, \delta=\frac{1}{6}$. As a result T is satisfies all conditions of Theorem 2, and hence T has a unique fixed point in X. Indeed,

$$M(T0,0)=1=M(0,T0)$$

which implies that $u=0\in X$ is the unique fixed point of T.

4 Some Results

Corollary 2. *[Multiplicative Kannan Type Contraction] Let (X,M) be a complete multiplicative metric space and T be a self mapping on X. If for all $x,y\in X$ we have*

$$M(Tx,Ty)\le[M(x,Tx)]^{\beta}[M(y,Ty)]^{\gamma}$$

where $\beta+\gamma<1$, then T has a unique fixed point in X.

Corollary 3. *[Multiplicative Chatterjea Type Contraction] Let (X,M) be a complete multiplicative metric space and T be a self mapping on X. If for all $x,y\in X$ we have*

$$M(Tx,Ty)\le[M(x,Ty)M(y,Tx)]^{\delta}$$

where $0\le\delta<1$, then T has a unique fixed point in X.

Corollary 4. *[Multiplicative Reich Type Contraction] Let (X,M) be a complete multiplicative metric space and T be a self mapping on X. If for all $x,y\in X$ we have*

$$M(Tx,Ty)\le[M(x,y)]^{\alpha}[M(x,Tx)]^{\beta}[M(y,Ty)]^{\gamma}$$

where $\alpha+\beta+\gamma<1$, then T has a unique fixed point in X.

5 Conclusions

We established a generalized fixed point theorem that involve some previous results in the framework multiplicative metric space.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] M. Grossman, R. Kantz, Non-Newtonian Calculus, Lee Press, Pigeon Cove, MA, 1972.

[2] D. Stanley, A multiplicative calculus, Primus IX (4)1999,310-326.

[3] Agamirza E. Bashirov, Emine. M. Kurpınar, A. Özyapıcı, Multiplicative calculus and its applications, J. Math. Anal. Appl. 337(2008)36-48.

[4] L. Florack, H. van Assen, Multiplicative calculus in Biomedical image analysis, J. Math. Imaging Vis. (2012) 42: 64-75.

[5] M., Özsavar and A. C. Çevikel, Fixed point of multipcative contraction mappings on multiplicative metric space. arXiv:1205.5131v1[matn.GN] (2012).

[6] M. Sarwar and Badshah-e-Rome, Some unique fixed point theorems in multiplicative metric space, arXiv:1410.3384v2[matn.GN] (2014).

[7] K. Abodayeh, A. Pitea, W. Shatanawi and T. Abdeljawad, Remarks on multiplicative metric spaces and related fixed point results, arXiv:1512.03771v1[matn.GN] (2015)

[8] O. Yamaod and W. Sintunavarat, Some fixed point results for generalized contraction mappings with cyclic $(\alpha,\beta)-$admissible mapping in multiplicative metric space, Journal of inequalities and applications, 2014:488 (2014).

[9] X. He, M. Song and D. Chen, Common fixed points for weak commutative mappings on a multiplicative metric space, Fixed point theory and appl., 2014/1/48 (2014).

[10] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integerales", Fundamenta Mathematicae, vol.3 (1922) pp. 133-181.

[11] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., 60, 1968,71-76.

[12] Chatterjea, S. K.: Fixed point theorems, C.R. Acad. Bulgare Sci. 25,1972, 727-730.

[13] T. Zamfirescu, Fixed point theorems in metric space. Arch. Math. (Basel), 23,292-298,1972.

[14] S. Reich, Some remarks concerning contraction mappings, Canadian Mathematical Bulletin 14 (1971) 121–124.

[15] Lj. B. Ciric, Generalized contractions and fixed point theorems, Publ. Inst.Math. 12 (26)(1971),19-26.

[16] Lj. B. Ciric, Ageneralization of Banach contraction princible, Proc. Amer. Math. Soc. 45.267-273,1974.

[17] Hardy, G.E., Rogers, T.D., A generalization of a fixed point theorem of Reich, Canad. Math. Bull., No.2, 201-206,1973.