New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.203

Numerical solutions of the Fredholm integral equations of the second type

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Received: 12 May 2017, Accepted: 26 August 2017
Published online: 26 August 2017.

Abstract: We present in this paper, Bernstein Piecewise Polynomials Method(BPPM), Integral Mean Value Method(IMVM), Taylor Series Method(TSM),The Least Square Method(LSM) are used to solve the integral equations of the second kind numerically. We aim to compare the efficiency of BPPM, IMVM, TSM and LSM in solving the integral equations of the second kind. We solve some examples to illustrate the applicability and simplicity of the methods. The numerical results show that which method is more efficient and accurate. As all these 4 methods consider solutions in numerically it is important to know about their rapidity of convergence to the exact solution.

Keywords: Fredholm integral equations, mean value,Bernstein, least square taylor polynomials.

1 Introduction

In the survey of solutions of integral equations, a large number of analytical and approximate methods for solving numerically various classes of integral equations [1, 2] are available. Many different powerful methods have been proposed to obtain exact and approximate solutions of integral equations. This study is an effort to comparison of methods for solving linear Fredholm integral equations of the second kind.Since the piecewise polynomials are differentiable and integrable, the Bernstein polynomials [4, 5] are defined on an interval to form a complete basis over the finite interval. Moreover, these polynomials are positive and their sum is unity. For these advantages, Bernstein polynomials have been used to solve second order linear and nonlinear differential equations, which are available in the literature, e.g. Bhatti and Bracken [7]. Very recently, Mandal and Bhattacharya [6] have attempted to solve integral equations numerically using Bernstein polynomials, but they obtained the results in terms of finite series solutions. Mandal and Bhattacharya [6] has described a special approximate method of solution of Fredholm integral equations by using Bernstein polynomials which suits the integral equations associated with function spaces spanned by polynomials only. Available methods for solving such equations are various, such as spectral methods [8,9,10,11,12,13,14,15,16,17]. Taylor-series expansion method first presented in [18] for solving Fredholm integral equations of second kind. we give a short introduction of Bernstein Piecewise Polynomials Method(BPPM), Integral Mean Value Method(IMVM), Taylor Series Method(TSM),The Least Square Method(LSM) first. All the computations are performed using MATHEMATICA.

In this paper we consider the solutions the integral equations of the second kind by using BPPM, IMVM, TSM and LSM. The paper is organised as follows. In the next section we illustrate briefly the BPPM, IMVM, TSM and LSM. In section 3, we apply these four methods to three examples to solve the integral equations of the second kind. In section 4, we give a brief discussion and conclusion.

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2 The theory of methods

2.1 The mean value method

Consider the following Fredholm integral equation of second kind:

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt,\ x,t \in [a,b] \tag{1}$$

where λ is a real number, also f and K are given continuous functions, and u is unknown function to be determined. Now, we remind integral mean value theorem and apply it directly in this method.

If $s(x)$ is continuous on the closed interval $[a,b]$, then there is a number c with a $a \leq c \leq b$ such that

$$\int_a^b s(x)dx = (b-a)s(c) \tag{2}$$

Now, we illustrate the main idea of our method. By applying the above theorem to Eq. 1 we have

$$\varphi(x) = f(x) + \lambda(b-a)K(x,c)\varphi(c) \tag{3}$$

where $c \in [a,b]$. Now, we must just find c and $\varphi(c)$ as unknowns. Substitution of c into Eq. 3 gives the following equation

$$\varphi(c) = f(c) + \lambda(b-a)K(c,c)\varphi(c). \tag{4}$$

For constructing another equation concerning c and $\varphi(c)$ we substitute Eq. 3 into Eq. 1 and obtain

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)(f(t) + \lambda(b-a)K(t,c)\varphi(c))dt \tag{5}$$

and by substituting $x = c$ into Eq. 5 we obtain

$$\varphi(c) = f(c) + \lambda \int_a^b K(c,t)(f(t) + \lambda(b-a)K(t,c)\varphi(c))dt. \tag{6}$$

After consecutive substitutions, we obtain proper and enough equations. Now, we solve Eqs. 3 and 6 simultaneously. For solving the above system, we can use various methods.

2.2 The Bernstein method

The general form of the Bernstein polynomials [4-7] of nth degree over the interval $[a,b]$ is defined by

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-\alpha)^i(\beta-x)^{n-i}}{(\beta-\alpha)^n}, i = 0,1,2,\ldots,n. \tag{7}$$

Note that each of these $n+1$ polynomials having degree n satisfies the following properties.

$$B_{i,n}(x) = 0, \text{if } i<0 \text{ or } i>n,$$
$$\textstyle\sum_{i=0}^{n} B_{i,n}(x) = 1;$$
$$B_{i,n}(a) = B_{i,n}(b) = 0,\ 1 \leq i \leq n-1$$

Using MATHEMATICA code, the first 11 Bernstein polynomials of degree ten over the interval $[a,b]$, are given below

$$
\begin{aligned}
&B_{0,10}(x) = (b-x)^{10}/(b-a)^{10}\\
&B_{1,10}(x) = 10(b-x)^{9}(x-a)/(b-a)^{10}\\
&B_{2,10}(x) = 45(b-x)^{8}(x-a)^{2}/(b-a)^{10}\\
&B_{3,10}(x) = 120(b-x)^{7}(x-a)^{3}/(b-a)^{10}\\
&B_{4,10}(x) = 210(b-x)^{6}(x-a)^{4}/(b-a)^{10}\\
&B_{5,10}(x) = 252(b-x)^{5}(x-a)^{5}/(b-a)^{10}\\
&B_{6,10}(x) = 210(b-x)^{4}(x-a)^{6}/(b-a)^{10}\\
&B_{7,10}(x) = 120(b-x)^{3}(x-a)^{7}/(b-a)^{10}\\
&B_{8,10}(x) = 45(b-x)^{2}(x-a)^{8}/(b-a)^{10}\\
&B_{9,10}(x) = 10(b-x)(x-a)^{9}/(b-a)^{10}\\
&B_{10,10}(x) = (b-x)^{10}/(b-a)^{10}
\end{aligned}
$$

Consider a general linear Fredholm integral equation of second kind is given by 7.

Now we use the technique of Galerkin method [Lewis, 3] to find an approximate solution $\varphi(x)$ of 1. For this, we assume that

$$\varphi(x) = \sum_{i=0}^{n} a_i B_{i,n}(x) \tag{8}$$

where$B_{i,n}(x)$ are Bernstein polynomials (basis) of degree i defined in eqn. 7, and a_i are unknown parameters, to be determined. Substituting 8 into 1, we obtain

$$\sum_{i=0}^{n} a_i B_{i,n}(x) - \lambda \int_a^b \left[K(x,t) \sum_{i=0}^{n} a_i B_{i,n}(t) \right] dt = f(x)$$

or

$$\sum_{i=0}^{n} a_i \left[B_{i,n}(x) - \lambda \int_a^b K(x,t) B_{i,n}(t) dt \right] = f(x). \tag{9}$$

Then the Galerkin equations [Lewis, 3] are obtained by multiplying both sides of 8 by multiplying both sides of 9 by $B_{j,n}(x)$ and then integrating with respect to x from a to b, we have

$$\sum_{i=0}^{n} a_i \left[\int_a^b \left[B_{i,n}(x) - \lambda \int_a^b K(x,t) B_{i,n}(t) dt \right] B_{j,n}(x) dx \right] = \int_a^b B_{j,n}(x) f(x) dx, \quad j = 0,1,\ldots..,n.$$

Since in each equation, there are three integrals, the inner integrand of the left side is a function of x and t and is integrated with respect to t from a to b. As a result the outer integrand becomes a function of x only and integration with respect to x yields a constant. Thus for each $j = 0,1,2,\ldots.,n$ we have a linear equation with $n+1$ unknowns a_i, $i = 0,1,\ldots..,n$. Finally 10 represents the system of $n+1$ linear equations in $n+1$ unknowns, are given by

$$\sum_{i=0}^{n} a_i C_{i,j} = F_j, \ j = 0,1,\ldots,n \tag{10}$$

where,

$$C_{i,j} = \int_a^b \left[B_{i,n}(x) - \lambda \int_a^b K(x,t) B_{i,n}(t) dt \right] B_{j,n}(x) dx. \tag{11}$$

$$F_j = \int_a^b B_{j,n}(x) f(x) dx. \tag{12}$$

Now the unknown parameters a_i are determined by solving the system of equations (10-12), and substituting these values of parameters in 8, we get the approximate solution $\varphi(x)$ of the integral equation of second type.

2.3 The least square method

We observe that the Bernstein procedure of the determination of the coefficients $a_i, i = 0,1,2,\ldots,n$ gives rise to computational difficulties because of the fact that a large number of integrals need to be evaluated which involve the Bernstein polynomials, even by selecting n to be as small as $n = 4$. We have avoided these difficulties by recasting the expression 8 as

$$\varphi(x) = a_0 + a_1 x + a_1 x^2 + \cdots + a_n x^n \tag{13}$$

where, if $a = 0, b = 1$, we get

$$\begin{aligned} a_0 &= c_0 \\ a_1 &= -nc_0 + nc_1 \\ a_1 &= \tfrac{n(n-1)}{2})(c_0 + c_2) - n(n-1)c_1 \\ &\vdots \\ a_n &= (-1)^n c_0 + (-1)^{n-1} nc_1 + (-1)^{n-2} \tfrac{n(n-1)}{2} c_2 + \ldots + (-1) nc_{n-1} + c_n. \end{aligned} \tag{14}$$

We now make the following observations. If an approximate solution of the Eq. 1(for $\lambda = 1$) is expressed in the form of a polynomial, as given by

$$\varphi(x) = \sum_{i=0}^{n} a_i x^i \tag{15}$$

where $a_i(i = 0,1,2,\ldots,N)$ are unknown constants to be determined then it amounts to determining the values $\varphi(x)$ at $N+1$ points in its domain of definition. This forces us to approximate the integral term of the integral equation by a suitable quadrature formula requiring the knowledge of these $(N+1)$ value of φ.

But, if the integral in the above Eq. 1 is replaced by a quadrature formula (see Fox and Goodwin [19]), we get

$$\varphi(x) - \sum_{k=0}^{N} w_k \varphi(t_k) K(x,t_k) = f(x), \; a < x < b \tag{16}$$

where w_k are the weights and t_k's are appropriately chosen interpolation points.

The Eq. 16 represents an over-determined system of linear algebraic equations for the determination of $(N+1)$ unknowns $\varphi(t_k)(k = 0,\ldots,N)$.

So, if from theoretical considerations it is already known that the given integral Eq. 1 possesses a unique solution, then varieties of methods can be used to cast the over-determined system of Eq. 16 into a system of $(N+1)$ equations and the method of least-squares provides the most appropriate procedure to handle the situation completely.

Note that one can obtain exactly $(N+1)$ equations for the $(N+1)$ unknowns $\varphi_0, \varphi_1, \ldots, \varphi_N$ from the over-determined system of Eq. 16 by selecting $(N+1)$ interpolating points $x = t_k, k = 0,1,2,\ldots,N, (0 < x < 1)$.

Substituting the approximate solution 15 into the integral Eq. 1 we obtain the relation

$$\sum_{i=0}^{n} a_i \Psi_i(x) = f(x), a < x < b \tag{17}$$

giving rise to an over-determined system of linear algebraic equations for the determination of the unknown constants $a_i(i = 0,1,2,\ldots,N)$ where,

$$\Psi_i(x) = x^i + \int_{\alpha}^{\beta} K(x,t) t^i dt, i = 0,1,2,\ldots,N. \tag{18}$$

On using the least-squares method, we obtain the normal equations,

$$\sum_{i=0}^{N} a_i c_{i,j} = b_j,\ \ j = 1,2,\ldots,N+1 \tag{19}$$

$$c_{i,j} = \int_a^b \Psi_{i-1}(x)\Psi_{j-1}(x)dx, i = 1,2,\ldots,N+1, j = 1,2,\ldots,N+1 \tag{20}$$

and,

$$b_j = \int_a^b f(x)\Psi_{j-1}(x)dx,\ \ j = 1,2,\ldots,N+1. \tag{21}$$

The solution of the system of Eq. 19 along with the relation 15, finally determines an approximate solution $\varphi(x)$.

2.4 The Taylor Method

Let us take the equation 1. We assume that $\varphi(x)$ a solution as

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\varphi^{(n)}(c)(x-c)^n \tag{22}$$

If we take n^{th} derivative of 1 we obtain that

$$\varphi^{(n)}(x) = f^{(n)}(x) + \lambda \int_a^b \frac{\partial^{(n)}K(x,t)}{\partial x^n}\varphi(t)dt. \tag{23}$$

Substituting $x = c$ in 23.

$$\varphi^{(n)}(c) = f^{(n)}(c) + \lambda \int_a^b \frac{\partial^{(n)}K(x,t)}{\partial x^n}\varphi(t)dt. \tag{24}$$

Eq.22 into Eq. 24.

$$\varphi^{(n)}(c) = f^{(n)}(c) + \lambda \int_a^b \frac{\partial^{(n)}K(x,t)}{\partial x^n}\sum_{m=0}^{\infty}\frac{1}{m!}\varphi^{(m)}(c)(x-c)^m dt. \tag{25}$$

If the equation is edited,

$$D_{nm} = \frac{1}{m!}\int_a^b \frac{\partial^{(n)}K(x,t)}{\partial x^n}(t-c)^m dt. \tag{26}$$

We substitute Eq. 26 in to Eq. 25

$$\varphi^{(n)}(c) = f^{(n)}(c) + \lambda \sum_{m=0}^{\infty} D_{nm}^{(m)}\varphi(c) \tag{27}$$

and

$$\lambda \sum_{m=0}^{\infty} D_{nm}^{(m)}\varphi(c) - \varphi^{(n)}(c) = -f^{(n)}(c) \tag{28}$$

the matrix equation is obtained, $D\varphi = F$ From here $D = \lambda[D_{nm}]$, $n,m = 0,1,2,\ldots\ldots,N.$,if $D \neq 0$ than 28 can be written in the form of equation
$\varphi = D^{-1}F$. If we solve this system than the Taylor solution 22 is obtained.

3 Numerical Applications

In this section we consider examples that show the efficiency of IMVM,BPPM,LSM and BPPM for solving fredholm integral equation of second type. We illustrate the above procedures through the following examples.

Example 1.

$$\varphi(x) = x + \int_0^1 (xt - x^2)\varphi(t)dt.$$

The exact solution of integral equation is obtained as follows

$$\varphi_{exact}(x) = \frac{96}{73}x - \frac{36}{73}x^2.$$

The approximate solutions of the integral equation are obtained using Integral Mean Value Method(IMVM), Bernstein Piecewise Polynomials Method(BPPM), The Least Square Method(LSM), Taylor Series Method(TSM) respectively.

$$\varphi_{IMVM}(x) = 1.397922046675241x - 0.6308106266346826x^2,$$

$$\varphi_{BPPM}(x) = 1.3150684931506849x - 0.4931506849315068x^2,$$

$$\varphi_{LSM}(x) = 0.6575342465753424x + 0.4931506849315068x^2,$$

$$\varphi_{TSM}(x) = 1.397922046675241x - 0.6308106266346826x^2.$$

The exact and the approximate solutions at various points of the domain are shown in Table 1.

Table 1

x	0	0.1	0.2	0.3	0.4
exact	0.126575	0.243288	0.350137	0.447123	0.534247
meanvalue	0.133484	0.2543524	0.362604	0.458239	0.541258
bernstein	0.126575	0.243288	0.350137	0.447123	0.534247
leastsquare	0.0706849	0.151233	0.241644	0.341918	0.452055
taylor	0.126575	0.243288	0.350137	0.447123	0.534247

x	0.5	0.6	0.7	0.8
exact	0.534247	0.611507	0.678904	0.736438
meanvalue	0.541258	0.611661	0.669448	0.714619
Bernstein	0.534247	0.611507	0.678904	0.736438
leastsquare	0.452055	0.572055	0.701918	0.841644
Taylor	0.534247	0.611507	0.678904	0.736438

x	0.9	1
exact	0.78411	0.821918
meanvalue	0.747173	0.767111
Bernstein	0.78411	0.821918
leastsquare	0.991233	1.15068
Taylor	0.78411	0.821918

Example 2.

$$\varphi(x) = e^{-x} + \int_0^1 e^{x+t}\varphi(t)dt.$$

The exact solution of integral equation is obtained as follows

$$\varphi_{exact}(x) = e^{-x} + \frac{2e^x}{3-e^2}.$$

The approximate solutions of the integral equation are obtained using Integral Mean Value Method(IMVM), Bernstein Piecewise Polynomials Method(BPPM), The Least Square Method(LSM), Taylor Series Method(TSM) respectively.

$$\begin{aligned}
\varphi_{IMVM}(x) &= e^{-x} - 0.455678e^x, \\
\varphi_{BPPM}(x) &= 0.5443631075322628 - 1.4560665935277939x + 0.27051618695259094x^2 - 0.22946402430534363x^3, \\
\varphi_{LSM}(x) &= 0.75298 - 1.13347x + 0.10864x^2, \\
\varphi_{TSM}(x) &= 0.4813656746174906 - 1.5186343253825094x + 0.2406828373087453x^2.
\end{aligned}$$

The exact and the approximate solutions at various points of the domain are shown in Table 2.

Table 2

x	0	0.1	0.2	0.3	0.4
exact	0.544321	0.401234	0.262163	0.125716	-0.0094729
meanvalue	0.544332	0.401235	0.262164	0.125717	-0.0094716
Bernstein	0.544363	0.401232	0.262135	0.125694	-0.0094666
leastsquare	0.75298	0.640719	0.530632	0.422717	0.3169743
Taylor	0.481366	0.331909	0.187266	0.047436	-0.0875788

x	0.5	0.6	0.7	0.8
exact	-0.144757	-0.281489	-0.421039	-0.564803
meanvalue	-0.144755	-0.281488	-0.421038	-0.564801
Bernstein	-0.144724	-0.281455	-0.421037	-0.564845
leastsquare	0.213405	0.112008	0.012784	-0.084266
Taylor	-0.217781	-0.343169	-0.463744	-0.579505

x	0.9	1
exact	-0.714219	-0.870784
meanvalue	-0.714217	-0.870782
Bernstein	-0.714258	-0.870651
leastsquare	-0.179145	-0.27185
Taylor	-0.690452	-0.796586

.

Example 3.

$$\varphi(x) = \cos x + \int_0^{\pi} \sin(x-t) g(t)dt.$$

The exact solution of integral equation is obtained as follows

$$g_{exact}(x) = \frac{2(2\cos x + \pi \sin x)}{4+\pi^2}.$$

The approximate solutions of the integral equation are obtained using Integral Mean Value Method(IMVM), Bernstein Piecewise Polynomials Method(BPPM), The Least Square Method(LSM), Taylor Series Method(TSM) respectively.

$$\varphi_{IMVM}(x) = \cos x - 0.9528905138210743\sin(1.87896539 + x),$$
$$\varphi_{BPPM}(x) = 0.2630471333126811 + 0.6192370186520357x - 0.3865370100549862x^2 + 0.04185410139410817x^3,$$
$$\varphi_{LSM}(x) = 0.37717147291814945 - 0.82073705756559x + 0.18930504421154654x^2,$$
$$\varphi_{TSM}(x) = 0.2980567356721341 + 0.3121242837135104x - 0.14902836783606704x^2.$$

The exact and the approximate solutions at various points of the domain are shown in Table 3.

Table 3

x	0	0.1	0.2	0.3	0.4
exact	0.2884	0.332186	0.372652	0.409396	0.442048
meanvalue	0.0919997	0.120394	0.147586	0.173304	0.197289
Bernstein	0.263047	0.321147	0.371768	0.41516	0.451575
leastsquare	0.377171	0.297037	0.220689	0.148127	0.0793508
Taylor	0.298067	0.327779	0.35452	0.378281	0.39062

x	0.5	0.6	0.7	0.8
exact	0.470284	0.493821	0.512906	0.525906
meanvalue	0.219303	0.239127	0.256561	0.271431
Bernstein	0.481263	0.504477	0.521466	0.532482
leastsquare	0.0143609	-0.0468429	-0.104261	-0.157892
Taylor	0.416862	0.431681	0.44352	0.452378

x	0.9	1
exact	0.534134	0.537025
meanvalue	0.283589	0.292914
Bernstein	0.537777	0.537601
leastsquare	-0.207738	-0.253797
Taylor	0.458296	0.461153

4 Conclusion

The integral equations are solved numerically. We have obtained the approximate solution of the unknown function by Bernstein Piecewise Polynomials Method(BPPM), Integral Mean Value Method(IMVM), Taylor Series Method (TSM)and Least Square Method(LSM). We have verified the derived formulas with the appropriate numerical examples. Several illustrative examples are examined in detail.

Least squares method approximates to the exact solution only at a specific point on the defined interval. The more we go farther the higher the error becomes. Bernstein solution gives us a good approximation for kernels formed by elementary functions. Taylor solution is better for elementary and trigonometric functions but deviates from the exact solution for exponential and logarithmic kernels. Mean value method approximates the exact solution for only the exponential functions.

The success of numerical methods used for solving Fredholm integral equation depends on the type of kernel function.

Acknowledgment

This work was supported by Research Fund of the Marmara University. ProjectNumber: 130116-0014.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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