New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.199

Boundary value problem for the nonlinear analogues of the Boussinesq equation: Numerical solution and its stability

Sherif Amirov[1] *and Mustafa Anutgan*[2]

[1]Department of Mathematics, Faculty of Science, Karabuk University, Karabuk, Turkey
[2]Department of Mechatronics Engineering, Faculty of Technology, Karabuk University, Karabuk, Turkey

Received: 6 April 2017, Accepted: 14 August 2017
Published online: 25 August 2017.

Abstract: The recent work on the solvability of the boundary value problem for the nonlinear analogue of the Boussinesq equation has been further extended to focus on the characteristics of the solution. Since this type of equation does not have a known analytical solution for arbitrary boundary conditions, the problem has been solved numerically. The stability of the solution and the effect of the input function on the stability have been investigated from the physics point of view. For the special case of a discontinuous function at the right hand side of the equation, the solution has been analyzed around the discontinuity points.

Keywords: Boussinesq equation, nonlinear nonhomogeneous partial differential equation, solvability, stability of solution.

1 Introduction

Boussinesq equation is a type of partial differential equation that appears in several problems of fluid dynamics and related branches. Some interesting examples for such problems include

(i) Longitudinal waves on elastic rods where the elastic medium is nonlinear and makes a transverse motion [1],
(ii) Plasma waves where the behavior of ions and electrons are described by the equations of hydrodynamics [2],
(iii) Rise of a nuclear explosion cloud in the atmosphere [3],
(iv) Heat transfer through a porous medium between two walls of different temperatures with insulated horizontal boundaries [4].

In addition, there is a great number of works dealing with the boundary value problems for the Boussinesq equation to find the solitary or travelling wave solutions of water waves [5,6,7,8,16,17,18,19,20]. For instance, Wang found the specific solitary wave solutions for two types of variant Boussinesq equations [6]. In addition, Zufiria formed a weakly nonlinear Hamiltonian model for two dimensional irrotational laterally unbounded waves of finite depth and long wavelength [7]. Applying several procedures on this Hamiltonian including change of variables and Fourier transformation, he reached a Boussinesq-type differential equation. Seadawy et al. found the solitary wave solutions for Zufiria's high-order Boussinesq equation using a series expansion method [8]. They showed that this approach can be used to find the analytical solitary wave solutions for several types of partial differential equations [9,10,11,12,13,14,15]. Another interesting paper was published by Moutsopoulos who presented an analytical solution for the case of a sudden change in the depth of the water by matching the Adomian decomposition method and Tolikas's polynomial expression for upstream and downstream regions, respectively [16,17]. Also, Basha et al. [18] and Rupp et al. [19] improved the work of Brutseard et al. on homogeneous and horizontal aquifers [20], and found the solutions for aquifers where the slope has an important effect on the flow.

* Corresponding author e-mail: mustafaanutgan@karabuk.edu.tr

As for the particular case of strong nonlinearity in the Boussinesq equation, an important example can be given as the electrical signal propagation on a dispersive transmission line which has components exhibiting nonlinear outputs in response to a linear excitation (e.g. capacitance characteristics of a semiconductor diode as a function of applied voltage) [21].

In the previous work, solvability of initial boundary value problem for a strongly nonlinear Boussinesq equation was proved [22]. The present work aims to find the corresponding numerical solution for a specific case of nonlinearity and to test the stability of that solution.

2 Formulation of the problem

Let Ω be the interval (0,1) on the x axis and let Q be a rectangle defined by $Q = \Omega \times (0,T)$ where $0<T<\infty$. $f(t,x)$, $u_0(x)$, $u_1(x)$ and $q(\xi)$ are the given functions where $x \in \Omega$, $t \in [0,T]$ and $\forall \xi \in R$. The initial boundary value problem for the Boussinesq equation:

$$\frac{\partial^2 q(u)}{\partial t^2} - u_{xx} - u_{xxtt} = f(t,x) \tag{1}$$

is to find the solution which satisfies the following conditions:

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega \tag{2}$$

$$u(t,0) = u(t,1) = 0, \quad t \in (0,T). \tag{3}$$

Let the solution space for this problem be defined by

$$\begin{aligned} V = \{\, &\upsilon(t,x), \quad \upsilon(t,x) \in L_\infty(0,T;W_2^2(\Omega)), \\ &\upsilon_t(t,x) \in L_\infty(0,T;W_2^2(\Omega)), \quad \upsilon_{tt} \in L_2(0,T;W_2^2(\Omega))\} \end{aligned} \tag{4}$$

where the norm in the space V is given as:

$$\|\upsilon\|_V = [\|\upsilon\|^2_{L_\infty(0,T;W_2^2(\Omega))} + \|\upsilon_t\|^2_{L_\infty(0,T;W_2^2(\Omega))} + \|\upsilon_{tt}\|^2_{L_2(0,T;W_2^2(\Omega))}]^{1/2}.$$

It is obvious that the space V is a Banach space with respect to the given norm. Now let us summarize the two theorems to be used in this work whose proofs were provided in [22].

Theorem 1. *If the function $q(\xi)$ satisfies the following conditions*

$$\begin{aligned} &q(\xi) = q_0(\xi) + q_1(\xi),\ q_0(\xi) \in C^2(R),\ q_1(\xi) \in C^2(R); \\ &q_0'(\xi) \geq 0,\ |q_1'(\xi)| \leq q_1 < 1,\ \forall \xi \in R; \end{aligned} \tag{5}$$

then, the initial boundary value problem (1), (2), (3) has a unique solution in the V space [22].

Theorem 2. *If the function $q(\xi)$ satisfies both the conditions given in (5) and the conditions $q(\xi) \in C^3(R)$, $q_0(0) = q_1(0) = 0$; then for any function $f(t,x) \in L_2(Q)$ and $u_0(x)$, $u_1(x) \in W_2^2(\Omega) \bigcap \overset{o}{W}{}_2^1(\Omega)$, there exists at least one solution, in the space V, for the initial boundary value problem (1), (2), (3) [22].*

$q(\xi)$ and the functions $u_0(x)$, $u_1(x)$, $f(t,x)$ must be chosen in accordance with Theorem 1 and Theorem 2. For example, if $q(\xi) = \xi^3 - e^{-\xi^2}$, Eq. (1) becomes

$$\frac{\partial^2(u^3 - e^{-u^2})}{\partial t^2} - u_{xx} - u_{xxtt} = f(t,x) \tag{6}$$

The open form of Eq. (6) can be written as

$$-\frac{\partial^2 u}{\partial x^2} + 2\exp^{-u^2}(\frac{\partial u}{\partial t})^2 + 6u(\frac{\partial u}{\partial t})^2 - 4\exp^{-u^2}u^2(\frac{\partial u}{\partial t})^2 + 2\exp^{-u^2}u\frac{\partial^2 u}{\partial t^2} + 3u^2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^2 \partial t^2} = f(t,x). \tag{7}$$

Now, let us consider the initial boundary conditions compatible with (2) and (3)

$$u_0(x) = u_1(x) = 0, \quad u(t,0) = u(t,1) = 0 \tag{8}$$

Then, Eq. (7) is a strongly nonlinear nonhomogeneous partial differential equation that does not have an analytical solution satisfying the conditions (8). Nonetheless, the solution can be obtained numerically using a suitable computer software. In this work, NDSolve command of Mathematica is used with MaxStepSize of 0.2 and PrecisionGoal of 3 for all of the solved equations in order that the comparison of the results be reliable.

3 Results and discussion

3.1 Physical interpretation of the problem

Provided that sufficient information about a dynamical system is known at an arbitrary time, the unique future of that system can be determined using its characteristic differential equation [23]. In other words, the initial boundary value problem for a differential equation describing a physical system must have a unique solution.

A Boussinesq-type equation is a kind of partial differential equation which can be obtained from the Hamiltonian of the system as provided in Zufiria's work [7]. From the mathematical point of view, it is possible to derive the Hamiltonian of that system starting from the Boussinesq equation by a reverse processing. Also, from the physical point of view, that Hamiltonian can be reached from the solution of the problem. Therefore, Boussinesq equation and its corresponding Hamiltonian system are closely related and can be transformed from one to another. The crucial restrictions for such a Hamiltonian to be the Hamiltonian of a physical system are that the problem must have a solution and this solution must be unique.

We assume that the Hamiltonian of an arbitrary dynamical system leads to the nonlinear analogue of the Boussinesq equation given in Eq. (7). Although the specification of the Hamiltonian is beyond the scope of this paper, this assumption is physically reasonable since the initial boundary value problem for Eq. (7) has a unique solution [22]. Then, the solution $u(t,x)$ of Eq. (7) becomes the behavior of that arbitrary system as a function of time t and horizontal distance x (Fig.s (1-a,-d) and (2-a,-d)). In this respect, the nonhomogeneity term $f(t,x)$ will be regarded as an excitation source disturbing the equilibrium of the system (e.g. external force in a harmonic oscillator, applied voltage in an electrical circuit, earthquake creating a tsunami, etc.).

3.2 Stability of the solution

While the solution of an initial boundary value problem lets us predict the behavior of the constituent(s) of a physical system at a later time, it does not alone show whether the system is sustainable. In order to see the sustainability of the system, the solution must be tested in terms of stability, a general description of which was given by Lyapunov [24] as the tendency of the system to reestablish its equilibrium state.

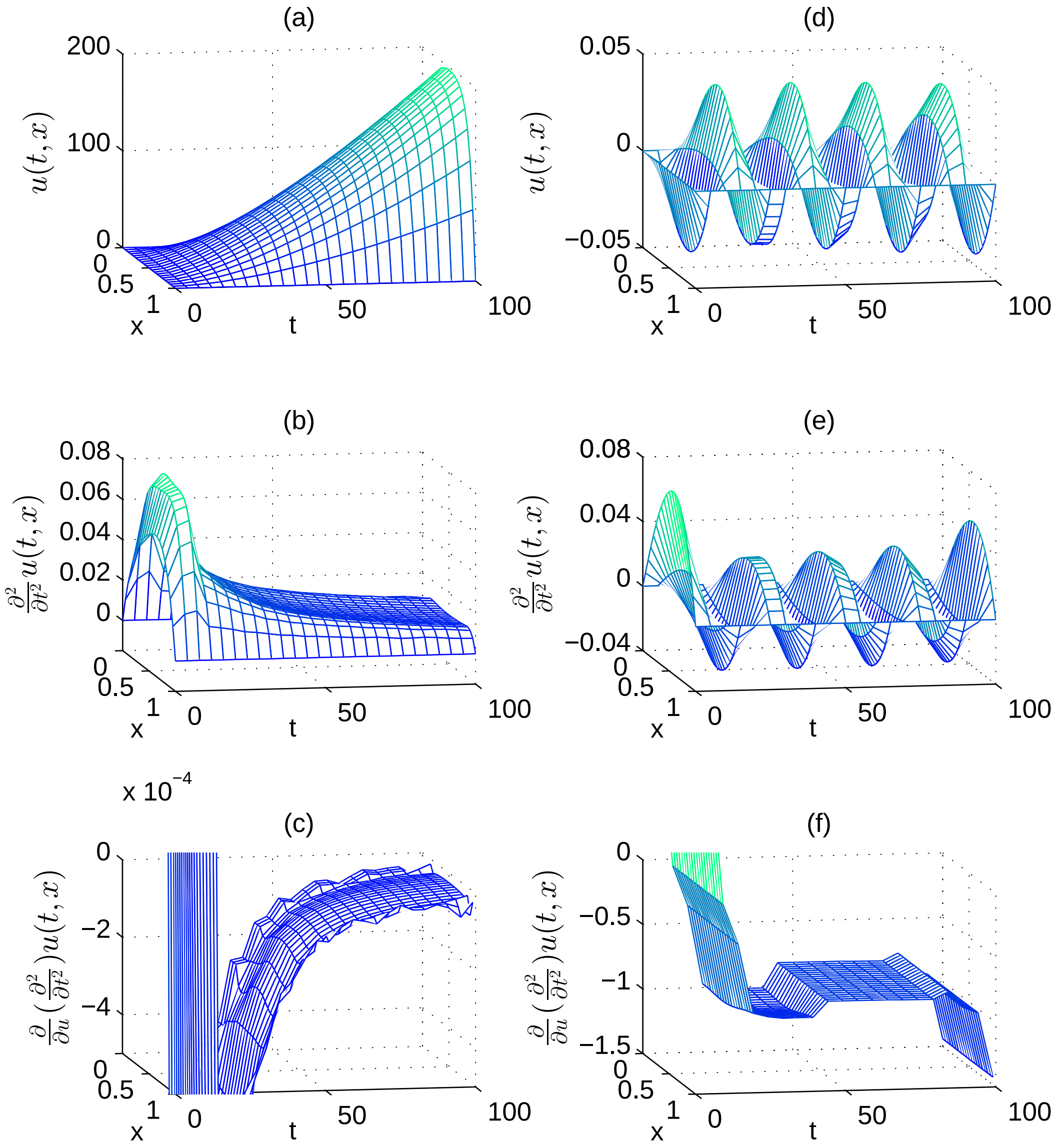

Fig. 1: (Color online) Numerical solution of the nonlinear Boussinesq equation with the nonhomogeneous term (a) $f(t,x) = x^2 + t^2$ and (d) $f(t,x) = x \cdot \exp(-t)$. (b) and (e) are the second derivatives of $u(t,x)$ with respect to time. (c) and (f) are the first derivatives of $\partial^2 u(t,x)/\partial t^2$ with respect to $u(t,x)$ whose negative values correspond to the stability condition.

More specifically, for the case of solitary and travelling wave solutions of the initial boundary value problems, stability condition was usually analyzed via energy considerations [25,26,27,28,9]. According to Benjamin [25], solitary waves of constant momentum must be local energy minimizers for stability. Similarly, Bona et al. [26] summed the invariants of

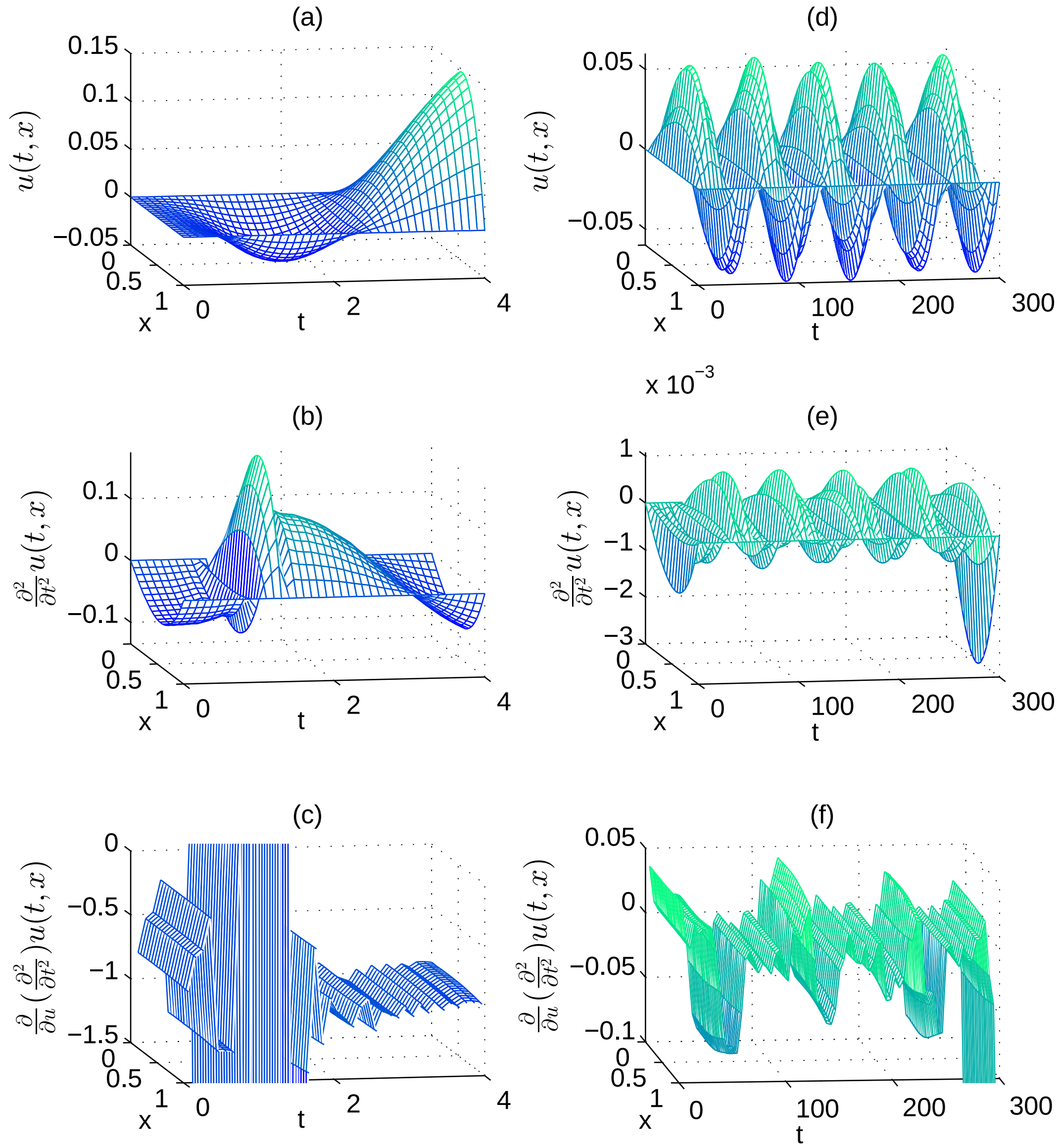

Fig. 2: (Color online) Numerical solution of the nonlinear Boussinesq equation with the nonhomogeneous term (a) $f(t,x) = x \cdot \mathrm{Sign}(t-1)$ and (d) $f(t,x) = x \cdot \mathrm{Sin}(0.1t)$. (b) and (e) are the second derivatives of $u(t,x)$ with respect to time. (c) and (f) are the first derivatives of $\partial^2 u(t,x)/\partial t^2$ with respect to $u(t,x)$ whose negative values correspond to the stability condition.

motion as a function of wave speed c (i.e. $d(c)$) and concluded that $d(c)$ must be a strictly convex function of c around the equilibrium for a stable solution. Bona stated that his conclusion was equivalent to that of Shatah [29] where the energy must have a local minimum for the stability condition. Using this argument as a starting point, Liu proved that the solitary wave solution was unstable if $d(c)$ is a concave function of c [27]. In addition, Grillakis et al. [28] studied the stability of solitary waves for symmetric systems and used the condition $E''(\phi)\geq 0$, where E is the energy and ϕ is the solitary wave solution. Seadawy et al. [9] found the travelling wave solutions for small amplitude waves in shallow waters supposing $\zeta=\chi - kt$ where χ is the position in the direction of propagation and k is the propagation speed of the wave. Defining the Hamiltonian system for the momentum as v, they expressed the sufficient condition for stability as $\partial v/\partial k>0$ [9,10,11,12,13].

The main idea behind the stability analyses including but not limited to the works mentioned above may be summarized as follows: the potential energy of the system must be originated from a restoring force or visa versa, such that the resultant force acting on the constituent(s) of the system must tend to retain the equilibrium state in case of any deviation from it. More simply, the force $\mathbf{F}$ and the displacement ϕ vectors are in opposite directions in a stable system. This statement can be written as [30]:

$$\frac{\partial F}{\partial \phi} < 0 \tag{9}$$

Considering the general relation between force and acceleration $\mathbf{F}\propto\partial^2\phi/\partial t^2$, inequality (9) can be adapted to our problem and the stability condition can be given as the following

$$\frac{\partial}{\partial u}\frac{\partial^2 u}{\partial t^2} < 0. \tag{10}$$

3.3 Numerical analysis

The initial boundary value problem (7), (8) has been solved numerically by Mathematica (Fig.s (1-a, -d) and (2-a, -d)). After the calculation of $\partial^2 u/\partial t^2$ (Fig.s (1-b, -e) and (2-b, -e)), these data have been exported as XLS files to be used in Matlab for further processing. Then, all data have been rearranged to get the proper matrix forms of u and $\partial^2 u/\partial t^2$ for the t and x intervals of interest. Both u and $\partial^2 u/\partial t^2$ have been numerically differentiated for each t and x, whose ratio gives out the derivative seen in inequality (10). Negative values of this function indicates a stable solution implying the existence of a *restoring force* when the system is deviated from equilibrium.

In order to see the response of the system to different inputs, four types of $f(t,x)\in L_2(Q)$ have been chosen (Table 1). Given the boundary conditions (8), the solution is almost symmetric in reference to the midline $x = 0.5$ as seen in Fig.s (1-a,-d) and (2-a,-d). Hence, x dependence of $f(t,x)$ seems to have a minor effect on $u(t,x)$. On the other hand, t dependence of $f(t,x)$ determines the eventual behavior of the solution. This has a particular importance in our case since the problem at hand does not have an analytical solution.

Table 1: List of input functions $f(t,x)$ used in this work and the corresponding stability conditions of the solutions.

$f(t,x)$	Stability
x^2+t^2	stable
$x\cdot\exp(-t)$	stable
$x\cdot\text{Sign}(t-1)$	stable except for a local instability
$x\cdot\text{Sin}(0.1t)$	alternating stability condition

Looking at the (c) and (f) of both Fig.s (1) and (2), it is mostly common (except for Fig. (2-c)) that $u(t,x)$ has a strong instability at the beginning of the *motion*. This could be understood considering the initial conditions (8). At t=0, the *motion* starts from equilibrium and rest. Therefore at the very first moment, the initiation of motion from a stationary state results in a large derivative of $u(t,x)$ with respect to time. Instead, one should focus on the behavior of the system away from origin for the evaluation of stability, as is done in the following analysis.

When $f(t,x) \propto t^2$ and $f(t,x) \propto \exp(-t)$ (Fig.s (1-c) and (1-f), respectively), the solution is stable. As for $f(t,x) \propto \text{Sign}(t-1)$, the input function is discontinuous at t=1 where its direction is reversed. Although this discontinuity does not have a discernable effect on the solution (Fig. (2-a)), it is clearly more effective when the second derivative of the solution with respect to time (Fig. (2-b)) and especially the change in this second derivative with respect to $u(t,x)$ (Fig. (2-b)) are considered. It is obvious that the solution experiences a serious deterioration of stability around the point of discontinuity at t=1 (Fig. (2-c)). Nevertheless, this deterioration fades out with time and the solution eventually becomes stable. For the case of a periodic input function $f(t,x) \propto \mathrm{Sin}(0.1\mathrm{t})$, a statement on whether the solution is stable or not cannot be reached. The solution is sometimes stable and sometimes unstable indicated by the vertical axis of Fig. (2-f) being sometimes negative and sometimes positive, respectively. This behavior resembles that of a harmonic oscillator driven by a force with frequency different from the resonance frequency of the system. In that case, the force and the displacement vectors are sometimes in the opposite direction and sometimes in the same direction, corresponding to stable and unstable solutions, respectively. The results of the stability analysis for each input function are summarized in Table 1.

4 Conclusion

An initial boundary value problem for the nonlinear analogue of the Boussinesq equation, whose solvability was shown in the previous work, has been solved numerically for different input functions. The Boussinesq equation at hand can be possibly derived from the Hamiltonian of a dynamical system. Keeping that Hamiltonian undefined in the present work, the solution of the problem has been assigned to the behavior of an arbitrary physical system. The stability of the solution has been analyzed for four different input functions by testing whether there existed a *restoring force* in the system. When the input function is discontinuous, the solution is unstable around the point of discontinuity. Also, an alternating stability condition has been observed for a sinusoidal input function. Except for these two cases, the solution can be concluded to be stable within the frame of our analysis.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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