New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.194

A Note on bilinear maps on vector lattices

Rusen Yilmaz

Department of Mathematics, Faculty of Arts and Science, Recep Tayyip Erdogan University, Rize, Turkey

Received: 18 January 2017, Accepted: 6 June 2017
Published online: 25 August 2017.

Abstract: In this paper we introduce a new concept of a d-bimorphism on a vector lattice and prove that, for vector lattices A and B, the Arens triadjoint $T^{***} : (A')'_n \times (A')'_n \to (B')'_n$ of a d-bimorphism $T : A \times A \to B$ is a d-bimorphism. This generalizes the concept of d-algebra and some results on the order bidual of d-algebras.

Keywords: Arens adjoint, vector lattice, order bidual, d-algebra, d-bimorphism.

1 Introduction and preliminaries

The Arens multiplications introduced in [2] on the bidual of various lattice ordered (or Riesz) algebras have been well documented (see, e.g., [3]). The more general question about Arens triadjoints of bilinear maps on products of vector lattices has recently aroused considerable interest (see, e.g., [7]). In Theorem 2.1 in [7] several properties of the Arens triadjoint maps are collected. For example, the adjoint of a bilinear map of order bounded variation is of order bounded variation and the triadjoint of such a map is separately order continuous. In this direction, as the extensions of the notions of classes of *f-algebras* [4] (a lattice ordered algebra A with the property that $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A^+$)and *almost f-algebras* [5] (a lattice ordered algebra A for which $a \wedge b = 0$ in A implies $ab = 0$), we studied the Arens triadjoints of some classes of bilinear maps on vector lattices (or Riesz spaces); mainly, bi-orthomorphisms and orthosymmetric bilinear maps (see [16]):

Definition 1. Let A and B be vector lattices. A bilinear map $T : A \times A \to B$ is said to be
(1) *orthosymmetric* if $x \wedge y = 0$ implies $T(x,y) = 0$ for all $x,y \in A$ (first appeared in a paper by G. Buskes and A. van Rooij in [10] in 2000).
(2) *a bi-orthomorphism* if it is a separately order bounded bilinear map such that $x \wedge y = 0$ in A implies $T(z,x) \wedge y = 0$ for all $z \in A^+$, when $A = B$ (first appears a paper by G. Buskes, R. Page Jr and R. Yilmaz in [11] in 2009).

The class of orthosymmetric bilinear maps was introduced in [10] by G. Buskes and A. van Rooij. Subsequent developments have been made as a result of contributions by the same authors [9], G. Buskes and A. G. Kusraev [8], and M. A. Toumi [14]. In [14] it is proved that if A,B are vector lattices, $(A')'_n, (B')'_n$ are their respective order continuous biduals and $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then the triadjoint $T^{***} : (A')'_n \times (A')'_n \to (B')'_n$ of T is a positive orthosymmetric bilinear map by the technique used in [3]. In [16] we extended this result to the whole $A'' \times A''$; that is, if A and B are Archimedean vector lattices, A'' and B'' are the order biduals of A and B respectively, then $T^{***} : A'' \times A'' \to B''$ is a positive orthosymmetric bilinear map whenever $T : A \times A \to B$ is so. Moreover we obtained similar results for the class of the Arens triadjoint of bi-orthomorphisms when $A = B$. So, we proved that all the results on the order biduals of f-algebras and almost f-algebras in the paper [3] could be reformulated and obtained the following results:

* Corresponding author e-mail: rusen.yilmaz@erdogan.edu.tr, ryilmaz00@yahoo.com

Theorem 1. *Let A and B Archimedean vector lattices. Then the following are satisfied.*
(i) *The Arens triadjoint $T^{***}: A'' \times A'' \to B''$ of a positive orthosymmetric bilinear map $T: A \times A \to B$ is positive orthosymmetric.*
(ii) *The Arens triadjoint $T^{***}: A'' \times A'' \to A''$ of a bi-orthomorphism $T: A \times A \to A$ is a bi-orthomorphism.*

R. Yilmaz and K. Rowlands in [15] in 2006 were the first to study bi-orthomorphisms what they called quasi-orthomorphisms. The notion of bi-orthomorphism, as given here, first appears in a paper by G. Buskes, R. Page Jr and R. Yilmaz in [11] in 2009, where it is proved that, under the certain conditions, the space of bi-orthomorphisms forms an f-algebra and we ask the question when exactly it is in general an f-algebra. Very recently K. Boulabiar and W. Brahmi in [6] have given a complete answer to this question, proving that the non-trivial space of bi-orthomorphisms is equipped with a structure of f-algebra.

In this paper, for the sake of completion of our paper [16], we introduce a new concept of a d-bimorphism and prove that if A, B are vector lattices and a bilinear map $T: A \times A \to B$ is a d-bimorphism, then so is the bilinear map $T^{***}: (A')'_n \times (A')'_n \to (B')'_n$. This also extends the notion of a *d-algebra* [12] (a lattice ordered algebra A such that $a \wedge b = 0$ in A implies $ac \wedge bc = ca \wedge cb = 0$ for all $c \in A^+$) and generalizes results on the continuous order bidual of d-algebras given in [3].

From here on, let A, B, and C be Archimedean vector lattices and A', B', C' be their respective duals. A bilinear map $T: A \times B \to C$ can be extended in a natural way to the bilinear map $T^{***}: A'' \times B'' \to C''$ constructed in the following stages:

$$\begin{array}{ll} T^*: C' \times A \to B', & T^*(f,x)(y) = f(T(x,y)) \\ T^{**}: B'' \times C' \to A', & T^{**}(G,f)(x) = G(T^*(f,x)) \\ T^{***}: A'' \times B'' \to C'', & T^{***}(F,G)(f) = F(T^{**}(G,f)) \end{array}$$

for all $x \in A, y \in B, f \in C', F \in A'', G \in B''$ (so-called the *first Arens adjoint* of T). Another extension of a bilinear map $T: A \times B \to C$ is the map ${}^{***}T: A'' \times B'' \to C''$ constructed in the following stages:

$$\begin{array}{ll} {}^{*}T: B \times C' \to A', & {}^{*}T(y,f)(x) = f(T(x,y)) \\ {}^{**}T: C' \times A'' \to B', & {}^{**}T(f,F)(y) = F({}^{*}T(y,f)) \\ {}^{***}T: A'' \times B'' \to C'', & {}^{***}T(F,G)(f) = G({}^{**}T(f,F)) \end{array}$$

for all $x \in A, y \in B, f \in C', F \in A'', G \in B''$ (so-called the *second Arens adjoint* of T) [2].

In this work we shall concentrate on the first Arens adjoint; that is, we prove that the triadjoint $T^{***}: (A')'_n \times (A')'_n \to (B')'_n$ is a d-bimorphism whenever $T: A \times A \to B$ is so. Similar results hold for the second. For the elementary theory of vector lattices and terminology not explained here we refer to [1,13,17].

2 The Arens triadjoint of a d-bimorphism

In this section we define the notion of a d-bimorphism on a vector lattice and prove that the extension T^{***} of a d-bimorphism $T: A \times A \to B$ is again a d-bimorphism. We first recall some relevant notions. The *canonical mapping* $a \mapsto \widehat{a}$ of a vector latice A into its order bidual A'' is defined by $\widehat{a}(f) = f(a)$ for all $f \in A'$. For each $a \in A$, $\widehat{a}$ defines an order continuous algebraic lattice homomorphism on A' and the canonical image $\widehat{A}$ of A is a subalgebra of $(A')'_c$. Moreover the band

$$I_{\widehat{A}} = \{F \in (A')'_c : |F| \leq \widehat{x} \text{ for some } x \in A^+\}$$

generated by $\widehat{A}$ is order dense in $(A')'_c$; that is, for each $F \in (A')'_c$, there exists an upwards directed net $\{G_\lambda : \lambda \in \Lambda\}$ in $I_{\widehat{A}}$ such that $0 < G_\lambda \uparrow F$.

A bilinear operator $T : A \times B \to C$ is said to be *order bounded* if for all $(x,y) \in A^+ \times B^+$ we have

$$\{T(a,b) : 0 \le a \le y, 0 \le b \le y\}$$

is order bounded. T is *positive* if for all $x \in A^+$ and $y \in B^+$ we have $T(x,y) \in C^+$. Clearly every positive bilinear map is order bounded. Moreover if T is positive, then so is T^*.

Definition 2. Let A and B be vector lattices. A bilinear map $T : A \times A \to B$ is said to be a *d-bimorphism* if $x \wedge y = 0$ in A implies $T(z,x) \wedge T(z,y) = 0$ for all $z \in A^+$.

The following result is obvious from the definitions.

Theorem 2. *Every bi-orthomorphism is boht orthosymmetric and a d-bimorphism.*

We are in a position to prove the main result of this paper.

Theorem 3. *Let A, B be vector lattices and $T : A \times A \to B$ be a d-bimorphism. Then the bilinear map $T^{***} : (A')'_n \times (A')'_n \to (B')'_n$ is a d-bimorphism.*

Proof. Let A, B be vector lattices and $T : A \times A \to B$ be a d-bimorphism. We have to show that T^{***} is a d-bimorphism. The proof of this is in two steps, as follows.

Step 1. We first show that if $x \in A^+$ and $0 \le F, G, H \in (A')'_n$ satisfy $F, G, H \le \widehat{x}$ and $G \wedge H = 0$, then

$$T^{***}(F,H) \wedge T^{***}(F,G) = 0,$$

which is the main step of the proof.

Let $0 \le f \in B'$ and $x \in A^+$. Then $0 \le T^*(f,x) \in A'$, and so, by Corollary 1.2 of [3], there exist $g, h \in A'$ with $g \wedge h = 0$, and $G(g) = 0 = H(h)$ such that

$$T^*(f,x) = g + h.$$

By the Riesz-Kontorovič Theorem ([1, Theorem 1.13]),

$$\inf\{g(y) + h(z) : x = y + z, y, z \in A^+\} = (g \wedge h)(x) = 0,$$

which implies that, for $\varepsilon > 0$, there exist $y, z \in A^+$ such that

$$x = y + z, \quad g(y) < \frac{\varepsilon}{2} \quad \text{and} \quad h(z) < \frac{\varepsilon}{2}.$$

We now define the linear functionals G_1 and H_1 on A' by

$$G_1 = G \wedge (\widehat{y - y \wedge z}) \quad \text{and} \quad H_1 = H \wedge (\widehat{z - y \wedge z}).$$

Clearly, $0 \leq G_1, H_1 \in (A')'_c$ and the following inequalities hold.

$$\begin{aligned} 0 \leq H - H_1 &= (H - (z - y \wedge z))^+ \leq (\widehat{x} - (\widehat{z - y \wedge z}))^+ \\ &= (y + z - \widehat{(z - y \wedge z)})^+ = (\widehat{y + y \wedge z})^+ \leq 2\widehat{y}, \end{aligned} \tag{1}$$

and similarly

$$0 \leq G - G_1 \leq 2\widehat{z}. \tag{2}$$

Since T^{***} is a d-bimorphism (and so positive) and $T^{***}(\widehat{a}, \widehat{b}) = \widehat{T(a,b)}$ for all $a, b \in A$, it follows that

$$\begin{aligned} 0 &\leq T^{***}(F, G_1) \wedge T^{***}(F, H_1) \\ &\leq T^{***}(\widehat{x}, \widehat{y - y \wedge z}) \wedge T^{***}(\widehat{x}, \widehat{z - y \wedge z}) = 0; \\ &\text{i.e., } T^{***}(F, G_1) \wedge T^{***}(F, H_1) = 0. \end{aligned} \tag{3}$$

We next consider the elements

$$0 \leq T^{***}(F, G - G_1), T^{***}(F, H - H_1)$$

of $(A')'_n$. Then, by the positivity of T^{***} and (1),

$$\begin{aligned} T^{***}(F, H - H_1)(f) &\leq T^{***}(\widehat{x}, H - H_1)(f) = \widehat{x}(T^{**}(H - H_1, f)) \\ &= T^{**}(H - H_1, f)(x) = (H - H_1)(T^{*}(f, x)) \\ &= (H - H_1)(g + h) = (H - H_1)(g) + (H - H_1)(h) \\ &\leq (H - H_1)(g) + (H)(h) \leq 2\widehat{y}(g) + 0 = 2g(y). \end{aligned} \tag{4}$$

Similarly, by (2),

$$T^{***}(F, G - G_1)(f) \leq 2h(z). \tag{5}$$

Hence, using the fact that $(a + b) \wedge c \leq a \wedge c + b \wedge c$ in ℓ-spaces and (3), we find

$$\begin{aligned} T^{***}(F, G) \wedge T^{***}(F, H) &= T^{***}(F, G - G_1 + G_1) \wedge T^{***}(F, H - H_1 + H_1) \\ &= (T^{***}(F, G - G_1) + T^{***}(F, G_1)) \wedge (T^{***}(F, H - H_1) + T^{***}(F, H_1)) \\ &\leq (T^{***}(F, G - G_1) \wedge (T^{***}(F, H - H_1) + T^{***}(F, H_1))) \\ &+ (T^{***}(F, G_1) \wedge (T^{***}(F, H - H_1) + T^{***}(F, H_1))) \\ &\leq T^{***}(F, G - G_1) + (T^{***}(F, G_1) \wedge (T^{***}(F, H - H_1) + T^{***}(F, H_1))) \\ &\leq T^{***}(F, G - G_1) + (T^{***}(F, G_1) \wedge T^{***}(F, H - H_1)) + (T^{***}(F, G_1) \wedge T^{***}(F, H_1)) \\ &\leq T^{***}(F, G - G_1) + T^{***}(F, H - H_1)) + 0. \end{aligned}$$

Therefore, by (4) and (5),

$$\begin{aligned} 0 \leq (T^{***}(F, G) \wedge T^{***}(F, H))(f) &\leq (T^{***}(F, G - G_1) + T^{***}(F, H - H_1))(f) \\ &\leq 2g(y) + 2h(z) \\ &< 2\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since this holds for an arbitrary $\varepsilon > 0$, we have

$$(T^{***}(F,G)\wedge T^{***}(F,H))(f)=0$$

for all $0\leq f\in B'$. It now follows that for all $f\in B'$

$$\begin{aligned}(T^{***}(F,G)\wedge T^{***}(F,H))(f) &= (T^{***}(F,G)\wedge T^{***}(F,H))(f^{+})-(T^{***}(F,G)\wedge T^{***}(F,H))(f^{-})\\ &= 0-0=0,\end{aligned}$$

and so $T^{***}(F,G)\wedge T^{***}(F,H)=0$.

Step 2. So far, we have proved that the restriction map $T^{***}|_{I_{\widehat{A}}\times I_{\widehat{A}}}$ is a d-bimorphism whenever $T:A\times A\to B$ is so. We now extend the result to the whole $(A')'_n\times(A')'_n$. To do this, let $0\leq F,G,H\in(A')'_n$ such that $G\wedge H=0$. We have to show that $(T^{***}(F,G)\wedge T^{***}(F,H))(f)=0$ for all $0\leq f\in B'$.

Since the band $I_{\widehat{A}}$ is order dense in $(A')'_n$, there exist $G_\alpha,H_\beta\in I_{\widehat{A}}$ such that $0\leq G_\alpha\uparrow G$ and $0\leq H_\beta\uparrow H$ with $0\leq G_\alpha\leq\widehat{x}_\alpha$ and $0\leq H_\beta\leq\widehat{y}_\beta$ for some $x_\alpha,y_\beta\in A^{+}$. It follows from $G\wedge H=0$ that $G_\alpha\wedge H_\beta=0$ for all α,β. Furthermore, $0\leq G_\alpha,H_\beta\leq\widehat{x_\alpha+y_\beta}$. Hence, by above, we see that

$$T^{***}(F,G_\alpha)\wedge T^{***}(F,H_\beta)=0 \tag{6}$$

for all α and β. Now let $0\leq f\in B'$. Since $0\leq G_\alpha\uparrow G$, we have

$$0\leq G_\alpha(T^{**}(f,x))\uparrow G(T^{**}(f,x));$$

$$\text{i.e., } 0\leq T^{***}(G_\alpha,f)(x)\uparrow T^{***}(G,f)(x),$$

for all $0\leq x\in A$. Hence

$$0\leq T^{**}(G_\alpha,f)\uparrow T^{**}(G,f),$$

and so, by the order continuity of F,

$$0\leq F(T^{**}(G_\alpha,f))\uparrow F(T^{**}(G,f));$$

$$\text{i.e., } 0\leq T^{***}(F,G_\alpha)(f)\uparrow T^{***}(F,G)(f)$$

for all $0\leq f\in B'$. Thus

$$0\leq T^{***}(F,G_\alpha)\uparrow T^{***}(F,G). \tag{7}$$

Similarly, it follows from $0\leq H_\beta\uparrow H$ that

$$0\leq H_\beta(T^{**}(f,x))\uparrow H(T^{**}(f,x));$$

$$\text{i.e., } 0\leq T^{**}(H_\beta,f)(x)\uparrow T^{**}(H,f)(x)$$

for all $0\leq x\in A$. This shows that

$$0\leq T^{**}(H_\beta,f)\uparrow T^{**}(H,f).$$

Hence, by the order continuity of F again,

$$0 \leq F(T^{**}(H_\beta, f)) \uparrow F(T^{**}(H, f));$$

$$\text{i.e., } 0 \leq T^{***}(F, H_\beta)(f) \uparrow T^{***}(F, H)(f)$$

for all $0 \leq f \in B'$. Therefore

$$0 \leq T^{***}(F, H_\beta) \uparrow T^{***}(F, H). \tag{8}$$

Now it follows from (7) and (8) that

$$0 \leq T^{***}(F, G_\alpha) \wedge T^{***}(F, H_\beta) \uparrow T^{***}(F, G) \wedge T^{***}(F, H),$$

for all α and β. This leads that, by (6),

$$T^{***}(F, G) \wedge T^{***}(F, H) = 0,$$

as required.

As every Arens multiplication on the bidual of lattice ordered algebras defines the Arens triadjoint map and every commutative d-algebra is an almost f-algebra, we immediately obtain the following corollary [3, Theorems 4.1 and 4.2].

Corollary 1. (i) *The order continuous bidual of a d-algebra is a Dedekind complete (and hence Archimedean) d-algebra.*
(ii) *The order bidual of a commutative d-algebra is a Dedekind complete d-algebra.*

Finally we point out that the triadjoints on the whole biduals is still an open problem. One has to obtain a way to handle the singular parts of biduals, as the cases of orthosymmetric bilinear maps and bi-orthomorphisms [16], in order to prove that the triadjoint $T^{***} : A'' \times A'' \to B''$ of a d-bimorphism $T : A \times A \to B$ is a d-bimorphism.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, 1985.

[2] R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. 2 (1951), 839-848.

[3] S. J. Bernau and C. B. Huijsmans, *The order bidual of almost f-algebras and d-algebras*, Trans. Amer. Math. Soc. 347 (1995), 4259-4275.

[4] G. Birkhoff and R. S. Pierce, *Lattice-ordered rings*, An. Acad. Brasil. Ciénc. 28 (1956), 41-49.

[5] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. No. 25 (1967).

[6] K. Boulabiar and W. Brahmi, *Multiplicative structure of biorthomorphisms and embedding of orthomorphisms*, Indagationes Mathematicae 27 (2016), 786-798.

[7] K. Boulabiar, G. Buskes and R. Pace, *Some properties of bilinear maps of order bounded variation*, Positivity 9 (2005), 401-414.

[8] A. G. Kusraev, *Representation and extension of orthoregular bilinear operators*, Vladikavkaz Mat. Zh. 9 (2007), 16-29.

[9] G. Buskes and A. van Rooij, *Squares of Riesz spaces*, Rocky Mountain J. Math.31 (2001), 45-56.

[10] G. Buskes and A. van Rooij, *Almost f-algebras: commutativity and Cauchy-Schwarz inequality*, Positivity 4 (2000), 227-231.

[11] G. Buskes, R. Page Jr and R. Yilmaz, *A note on bi-orthomorphisms*, Vector Measures, Integration and Related Topics, Operator Theory: Advances and Applications, Vol. 201 (2009), 99-107.

[12] V. Kudláček, *On some types of ℓ-rings*, Sborni Vysokého Učeni Techn v Brně 1-2 (1962), 179-181.

[13] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland, 1971.

[14] M. A. Toumi, *The triadjoint of an orthosymmetric bimorphism*, Czechoslovak Mathematical Journal, 60 (135) (2010), 85-94.

[15] R. Yilmaz and K. Rowlands, *On orthomorphisms, quasi-orthomorphisms and quasi-multipliers*, J. Math. Anal. Appl. 313 (2006), 120-131.

[16] R. Yılmaz, *The Arens triadjoints of some bilinear maps*, Filomat 28:5 (2014), 963-979.

[17] A. C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer, 1997.