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On the numerical solution of nonlinear fractional-integro differential equations

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Abstract: In the present study, a numerical method, perturbation-iteration algorithm (shortly PIA), has been employed to give approximate solutions of some nonlinear Fredholm and Volterra type fractional-integro differential equations (FIDEs). Comparing with the exact solution, the PIA produces reliable and accurate results for FIDEs.

Keywords: Fractional-integro differential equations, Caputo fractional derivative, Initial value problems, Perturbation-Iteration Algorithm.

1 Introduction

Scientists has been interested in fractional order calculus as long as it has been in classical integer order analysis. However, for many years it could not find practical applications in physical sciences. Recently, fractional calculus has been used in applied mathematics, viscoelasticity [41], control [34], electrochemistry [31], electromagnetic [14].

Developments in symbolic computation capabilities is one of the driving forces behind this rise. Different multidisciplinary problems can be handled with fractional derivatives and integrals.

[25] and [33] are studies that describe the fundamentals of fractional calculus give some applications. Existence and uniqueness of the solutions are also studied in [40,38].

Similar to the studies in physical sciences, fractional order integro differential equations (FIDEs) also gave scientists the opportunity of describing and modeling many important and useful physical problems.

In this manner, a remarkable effort has been expended to propose numerical methods for solving FIDEs, in recent years. Fractional variational iteration method [19,20], homotopy analysis method [23,7], finite element method [11,12], fractional differential transform method [29,3,10] and Adams-Bashforth-Moulton Method [4,5] are among these methods.

The aim of this study was to construct and test an algorithm using PIA to obtain approximate solutions of some nonlinear fractional order Fredholm and Volterra type integro-differential equation. In the present study we also give the convergence analysis of the method for the first time. This method can be applied to a wide range of problems without requiring any special assumptions and restrictions.

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A few fractional derivative definitions of an arbitrary order exists in the literature. Two most used of them are the Riemann-Liouville and Caputo fractional derivatives. The two definitions are quite similar but have different order of evaluation of derivation.

Due to the appropriateness of the initial conditions, fractional definition of Caputo is often used in recent years.

2 Basic definitions

Definition 1. *A real function $f(t)$, $t>0$ is said to be in the space C_μ, $(\mu>0)$ if there exists a real number $p(>\mu)$, such that $f(t)=t^p f_1(t)$ where $f_1\in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)}\in C_\mu, m\in \aleph$ [24].*

Definition 2. *The Riemann-Liouville fractional integral operator (J^α) of order $\alpha\geq 0$, of a function $f\in C_\mu$, $\mu\geq -1$ is defined as [25].*

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad \alpha,t>0, \tag{1}$$

and $J^0 f(t)=f(t)$, where Γ is the well-known gamma function. For $f\in C_\mu$, $\mu\geq -1$, $\alpha,\beta\geq 0$ and $\lambda>-1$, the following properties hold.

(i) $J^\alpha J^\beta f(t)=J^{\alpha+\beta}f(t)$,
(ii) $J^\alpha J^\beta f(t)=J^\beta J^\alpha f(t)$,
(iii) $J^\alpha t^\lambda=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)}t^{\lambda+\alpha}$.

Definition 3. *The Caputo fractional derivative of f of order α, $f\in C_{-1}^m$, $m\in\aleph\cup\{0\}$, is defined as [33].*

$$D^\alpha f(t)=J^{m-\alpha}f^{(m)}(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau, \quad \alpha,t>0, \tag{2}$$

where $m-1<\alpha<m$ with the following properties;

(i) $D^\alpha(af(t)+bg(t))=aD^\alpha f(t)+bD^\alpha g(t)$, $a,b\in\Re$,
(ii) $D^\alpha J^\alpha f(t)=f(t)$,
(iii) $J^\alpha D^\alpha f(t)=f(t)-\sum_{j=0}^{k-1}f^{(j)}(0)\frac{t^j}{j!}$, $t>0$.

After this introductory section, section 3 is reserved to a brief review of the Perturbation-Iteration Algorithm PIA, in section 4 convergence analysis of the present method is given, in section 5 some examples are illustrated to show the simplicity and effectiveness of the algorithm. Finally the paper ends with a conclusion in section 6.

3 Analysis of the PIA

Differential equations are naturally used to describe problems in engineering and other applied sciences. Searching approximate solutions for complicated equations has always attracted attention. Many different methods and frameworks exist for this purpose and perturbation techniques [30,22,37] are among them. These techniques can be used to find approximate solutions for both ordinary and partial differential equations.

Requirement of a small parameter in the equation that is sometimes artificially inserted is a major limitation of perturbation techniques that renders them valid only in a limited range. Therefore, to overcome the disadvantages come with the perturbation techniques, several methods have been proposed by authors [15,26,27,28,8,18,39,13,21,16].

Parallel to these attempts, a perturbation-iteration method has been proposed by Aksoy, Pakdemirli and their co-workers [1,32,2] previously. In the new technique, an iterative algorithm is constructed on the perturbation expansion. The present method has been tested on Bratu-type differential equations [1] and first order differential equations [32] with success. Then the algorithms were applied to nonlinear heat equations also [2]. The solutions of the Volterra and Fredholm type integral equations [9], first-order differential equations and systems [35] and solutions of ordinary frational differental equations [36] have been presented by the developed method, finally.

This new algorithm have not been used for any fractional integro differential equations yet. To obtain the approximate solutions of FIDEs, the most basic perturbation-iteration algorithm PIA(1,1) is employed by taking one correction term in the perturbation expansion and correction terms of only first derivatives in the Taylor series expansion. [1,32,2].

Take the fractional-integro differential equation.

$$F\left(u^{(\alpha)},u,\int_0^t g(t,s,u(s))\,ds,\varepsilon\right)=0, \tag{3}$$

where $u=u(t)$ and ε is a small parameter. The perturbation expansions with only one correction term is

$$\begin{aligned} u_{n+1}&=u_n+\varepsilon(u_c)_n,\\ u'_{n+1}&=u'_n+\varepsilon\left(u'_c\right)_n. \end{aligned} \tag{4}$$

Replacing Eq.(8) into Eq.(7) and writing in the Taylor series expansion for only first order derivatives gives

$$\begin{aligned} &F\left(u_n^{(\alpha)},u_n,\int_0^t g(t,s,u_n(s))\,ds,0\right)+F_u\left(u_n^{(\alpha)},u_n,\int_0^t g(t,s,u_n(s))\,ds,0\right)\varepsilon(u_c)_n\\ &+F_{u^{(\alpha)}}\left(u_n^{(\alpha)},u_n,\int_0^t g(t,s,u_n(s))\,ds,0\right)\varepsilon\left(u_c^{(\alpha)}\right)_n+F_{\int u}\left(u_n^{(\alpha)},u_n,\int_0^t g(t,s,u_n(s))\,ds,0\right)\varepsilon\int(u_c)_n\\ &+F_\varepsilon\left(u_n^{(\alpha)},u_n,\int_0^t g(t,s,u_n(s))\,ds,0\right)\varepsilon=0 \end{aligned} \tag{5}$$

or

$$\left(u_c^{(\alpha)}\right)_n\frac{\partial F}{\partial u^{(\alpha)}}+(u_c)_n\frac{\partial F}{\partial u}+\left(\int(u_c)_n\right)\frac{\partial F}{\partial(\int u)}+\frac{\partial F}{\partial\varepsilon}+\frac{F}{\varepsilon}=0. \tag{6}$$

Here $(.)'$ represents the derivative according to the independent variable and

$$F_\varepsilon=\frac{\partial F}{\partial\varepsilon},\ F_u=\frac{\partial F}{\partial u},\ F_{u'}=\frac{\partial F}{\partial u'},\ldots \tag{7}$$

The derivatives in the expansion are evaluated at $\varepsilon=0$. Beginning with an initial function $u_0(t)$, first $(u_c)_0(t)$ is calculated by the help of (21) and then substituted into Eq.(8) to calculate $u_1(t)$. Iteration procedure is continued using (21) and (8) until obtaining a reasonable solution.

4 Convergence analysis of the PIA

In this section we give a convergence analysis of the method.

Theorem 1. *PIA*$(1,1)$ *converges for Eq.*(3) *when* $\|u_{k+1}-u_k\|\le\varepsilon'$ *and* $\varepsilon'\to0$.

Proof. The general iteration formula of $PIA(m,n)$ is converted to $PIA(1,1)$ in recursive relation (21) by substituting $m=1$ and $n=1$ that can be stated as follows:

$$u_k^{'}(t)+\frac{F_{u_k}}{F_{u_k^{'}}}u_k(t)=-\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_k^{'}}}. \tag{8}$$

By changing k to $k+1$ in Eq.(22) to obtain a relation with respect to $u_{k+1}^{'}(t)$ and $u_k(t)$ and imposing norm 2 on both sides of equations, we have:

$$\left\|u_k^{'}\right\|\leq\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_k^{'}}}\right\|+\left\|\frac{F_{u_k}}{F_{u_k^{'}}}\right\|.\left\|u_k\right\|, \tag{9}$$

$$\left\|u_{k+1}^{'}\right\|\leq\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_{k+1}^{'}}}\right\|+\left\|\frac{F_{u_{k+1}}}{F_{u_{k+1}^{'}}}\right\|.\left\|u_{k+1}\right\|. \tag{10}$$

Now, we need to obtain $\|u_{k+1}-u_k\|$ from $\left\|u_{k+1}^{'}-u_k^{'}\right\|$. By rewriting inequalities with respect to$\|u_k\|$ and $\|u_{k+1}\|$ and by using the magnitude rules in calculus, we get:

$$\|u_{k+1}-u_k\|\geq\|u_{k+1}\|-\|u_k\|=\left\|\frac{F_{u_{k+1}^{'}}}{F_{u_{k+1}}}\right\|.\left(\left\|u_{k+1}^{'}\right\|-\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_{k+1}^{'}}}\right\|\right)-\left\|\frac{F_{u_k^{'}}}{F_{u_k}}\right\|.\left(\left\|u_k^{'}\right\|-\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_k^{'}}}\right\|\right). \tag{11}$$

So we need to obtain bound for $\|u_{k+1}-u_k\|$. As a result, we can find out that $\{u_n\}$ is a Cauchy sequence in Banach space, then it is convergent. It holds due to this fact that $f\in C^m[a,b]$ and $f^m\in C_\mu^m$ according to main definition of Banach space in real and functional analysis. All elements of $\|u_{k+1}-u_k\|$ in right hand side of inequality are known except $\left\|u_k^{'}\right\|$ and $\left\|u_{k+1}^{'}\right\|$.

$$\|u_{k+1}-u_k\|\geq\left\|\frac{F_{u_{k+1}^{'}}}{F_{u_{k+1}}}\right\|.\left\|u_{k+1}^{'}\right\|-\left\|\frac{F_{u_{k+1}^{'}}}{F_{u_{k+1}}}\right\|.\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_{k+1}^{'}}}\right\|-\left\|\frac{F_{u_k^{'}}}{F_{u_k}}\right\|.\left\|u_k^{'}\right\|+\left\|\frac{F_{u_k^{'}}}{F_{u_k}}\right\|.\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_k^{'}}}\right\|. \tag{12}$$

Since $F_{u_{k+1}^{'}}$, $F_{u_{k+1}}$, F_{u_k} and $F_{u_k^{'}}\in C_\mu^m$, $\mu\geq-1$, then they are bounded and we have:

$$\left\|\frac{F_{u_{k+1}^{'}}}{F_{u_{k+1}}}\right\|\leq M_1\ ,\ \left\|\frac{F_{u_k^{'}}}{F_{u_k}}\right\|\leq M_2, \tag{13}$$

$$\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_{k+1}^{'}}}\right\|\leq M_1^{'}\ ,\ \left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_k^{'}}}\right\|\leq M_2^{'}. \tag{14}$$

Therefore, we obtain

$$\|u_{k+1}-u_k\|\geq M_1\left\|u_{k+1}^{'}\right\|-M_1.M_1^{'}-M_2.\left\|u_k^{'}\right\|+M_2.M_2^{'}. \tag{15}$$

Now we consider.

$$\begin{aligned} u_k^{'}&=L\left[u_k\right] \\ u_{k+1}^{'}&=L\left[u_{k+1}\right], \end{aligned} \tag{16}$$

where L is a linear operator that is defined as $L=\frac{d}{d()}\left[\right]$. Since any linear operator is bounded in theory of operators from pure mathematics, then, we can define

$$\left\|u_k^{'}\right\|=\|L\left[u_k\right]\|\leq N_1\left\|u_{k+1}^{'}\right\|=\|L\left[u_{k+1}\right]\|\leq N_2. \tag{17}$$

Therefore, we get

$$\|u_{k+1}-u_k\| \geq M_1.N_2 - M_1.M_1^{'} - M_2.N_1 + M_2.M_2^{'} = M_1\left(N_2 - M_1^{'}\right) + M_2\left(M_2^{'} - N_1\right). \tag{18}$$

If $\|u_{k+1}-u_k\| \to 0$ then, we have

$$\lim_{\varepsilon' \to 0}\left(M_1\left(N_2 - M_1^{'}\right) + M_2\left(M_2^{'} - N_1\right)\right) = 0. \tag{19}$$

If $M_1\left(N_2 - M_1^{'}\right) = 0$ then $M_1 = 0$ or and if $M_2\left(M_2^{'} - N_1\right) = 0$ then $M_2 = 0$ or $M_2^{'} = N_1$. The proof is complete. The proof can be done in similar manner for $PIA(1,2)$, $PIA(2,2)$ and so on. Therefore, we can find such these conditions for $PIA(1,2)$, $PIA(2,2)$ states and so on. In fact, we have found the condition of stop process for PIA method by all involved expressions in the iteration algorithm. Therefore, all the governing conditions needs to be imposed on PIA method to become convergent just in a few of computational iterations.

5 Applications

Example 1. Consider the following nonlinear fractional Fredholm integro-differential equation [17].

$$\frac{d^{\alpha}u(t)}{dt^{\alpha}} - \int_0^1 tsu^2(s)ds = 1 - \frac{t}{4}, \quad t > 0, \quad 0 \leq t < 1, \quad 0 < \alpha \leq 1, \tag{20}$$

with the initial condition $u(0) = 0$ and the known exact solution for $\alpha = 1$ is

$$u(t) = t. \tag{21}$$

Before iteration process rewriting Eq.(23) with adding and subtracting $u'(t)$ to the equation gives

$$\varepsilon\frac{d^{\alpha}u(t)}{dt^{\alpha}} - u^{'}(t) + \varepsilon u^{'}(t) - \varepsilon\int_0^1 ts(u(s))^2ds - 1 + \frac{t}{4} = 0. \tag{22}$$

In this case for

$$F\left(u^{'},u,\varepsilon\right) = \frac{1}{\Gamma(1-\alpha)}\varepsilon\int_0^t \frac{u'(s)}{(t-s)^{\alpha}}ds - u_n^{'}(t) + \varepsilon u_n^{'}(t) - \varepsilon\int_0^1 ts(u_n(s))^2ds - 1 + \frac{t}{4}, \tag{23}$$

and the iteration formula

$$u^{'}(t) + \frac{F_u}{F_{u'}}u(t) = -\frac{F_{\varepsilon} + \frac{F}{\varepsilon}}{F_{u'}} \tag{24}$$

the terms that will be replaced in, are

$$\begin{aligned}
F &= u_n^{'}(t) - 1 + \frac{t}{4},\\
F_u &= 0,\\
F_{u'} &= 1,\\
F_{\varepsilon} &= -u_n^{'}(t) + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(s)}{(t-s)^{\alpha}}ds - \int_0^1 ts(u(s))^2ds.
\end{aligned} \tag{25}$$

After substitution the differential equation for this problem, Eq.(21) becomes

$$\frac{\int_0^t (-s+t)^{-\alpha} u_n{}'(s)ds}{\Gamma(1-\alpha)} + \left(u_c'(t)\right)_n = \int_0^1 st(u_n(s))^2 ds + \frac{4-t+4(-1+\varepsilon)u_n'(t)}{4\varepsilon}. \tag{26}$$

Appropriate to the initial conditions, chosen $u_0(t) = 0$ and, solving Eq.(29) for $n = 0$ gives

$$(u_c(t))_0 = t - \frac{t^2}{8} + C_1. \tag{27}$$

This expression written in

$$u_1 = u_0 + \varepsilon(u_c(t))_0. \tag{28}$$

gives

$$u_1(x,t) = u_0(x,t) + \varepsilon(t - \frac{t^2}{8} + C_1) \tag{29}$$

or

$$u_1(x,t) = \varepsilon(t - \frac{t^2}{8} + C_1). \tag{30}$$

Solving this equation for

$$u_1(0) = 0, \tag{31}$$

we obtain

$$C_1 = 0. \tag{32}$$

For this value and $\varepsilon = 1$ reorganizing $u_1(t)$

$$u_1(t) = t - \frac{t^2}{8}, \tag{33}$$

gives the first iteration result. If the iteration procedure is continued in a similar way, we obtain the following iterations.

$$u_2(t) = 2t - \frac{571t^2}{3840} + \frac{t^{2-\alpha}(t+4(-3+\alpha))}{4\Gamma(4-\alpha)}, \tag{34}$$

$$\begin{aligned} u_3(t) = {} & 3t + \frac{29844889t^2}{176947200} - \frac{t^{3-2\alpha}(t+8(-2+\alpha))}{4\Gamma(5-2\alpha)} \\ & + \frac{t^2(3379230 + 8t^{-\alpha}(1051t + 5760(-3+\alpha))(-7+\alpha)(-6+\alpha)(-5+\alpha))}{15360(-7+\alpha)(-6+\alpha)(-5+\alpha)\Gamma(4-\alpha)} \\ & - \frac{2240277\alpha + (450151 - 28436\alpha)\alpha^2}{15360(-7+\alpha)(-6+\alpha)(-5+\alpha)\Gamma(4-\alpha)} - \frac{t^2(-4+\alpha)(-1159 + 2\alpha(529 + 16(-10+\alpha)\alpha))}{64(-7+2\alpha)\Gamma(5-\alpha)^2}. \end{aligned} \tag{35}$$

The other iterations contain large inputs and are not given. A computational software program could help to calculate the other iterations up to any order. In Table 1. some of the PIA iteration results are compared with the exact solutions. The results express that the present method gives highly approximate solutions. Also in Figure 1. the obtained results are illustrated graphically.

Example 2. Consider the following nonlinear Volterra type fractional integro-differential equation [6].

$$\frac{d^\alpha u(t)}{dt^\alpha} - \int_0^t e^{-s} u^2(s)ds = 1, \quad t > 0, \quad 0 \le t < 1, \quad 0 < \alpha \le 1, \tag{36}$$

Table 1: Numerical results of Example 1. for u_5 values for different values of α.

	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$		$\alpha=1.0$	
t	*PIA*	*PIA*	*PIA*	*PIA*	Exact Solution	Absolute Error
0.0	0.00000	0.00000	0.00000	0.000000	0.000000	0.000000
0.1	0.40400	0.31001	0.19212	0.099981	0.100000	1.872712E-6
0.2	0.70027	0.51165	0.32983	0.199992	0.200000	7.490848E-6
0.3	0.92857	0.66680	0.45097	0.299983	0.300000	1.685440E-5
0.4	1.10862	0.79495	0.56312	0.399970	0.400000	2.996339E-5
0.5	1.25343	0.90616	0.66959	0.499953	0.500000	4.681780E-5
0.6	1.37247	1.00645	0.77217	0.599932	0.600000	6.741763E-5
0.7	1.47293	1.09971	0.87196	0.699908	0.700000	9.176289E-5
0.8	1.56044	1.18858	0.969682	0.799880	0.800000	1.198535E-4
0.9	1.63945	1.27490	1.06583	0.899848	0.900000	1.516896E-4
1.0	1.71351	1.35996	1.16078	0.999812	1.000000	1.872712E-4

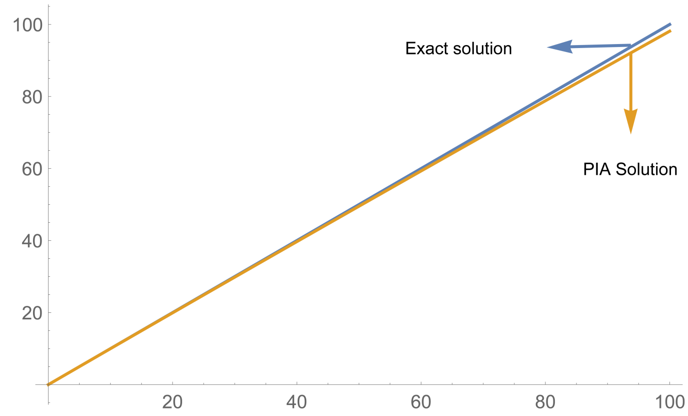

Fig. 1: Comparison of the PIA solution $u_5(t)$ and exact solution for Example 1. when $\alpha=1$

with the initial condition $u(0)=1$ and the known exact solution for $\alpha=1$ is

$$u(t)=e^t. \tag{37}$$

By applying similar procedures as in the first example, we obtain the following iteration results.

$$u_1(t)=1+t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-\frac{t^5}{120}, \tag{38}$$

$$u_2(t)=1+2t+t^2+\frac{t^4}{12}-\frac{t^5}{30}-\frac{t^7}{420}+\frac{t^8}{1344}-\frac{t^9}{576}+\frac{67t^{10}}{86400}-\frac{217t^{11}}{950400}+\frac{581t^{12}}{11404800}-\frac{101t^{13}}{13478400}+\frac{t^{14}}{1209600}-\frac{t^{15}}{18144000}$$
$$+\frac{t^{2-\alpha}\left(t^4+t^3(\alpha-6)+t^2(\alpha-6)(\alpha-5)+(t-1)((\alpha-6)(\alpha-5)(\alpha-4)-(\alpha-3))\right)}{\Gamma(7-\alpha)}, \tag{39}$$

and so on. The third iteration result (u_3) is calculated in this manner. In Table 2. some of the PIA iteration results are compared with the results from Laplace variational iteration method, and exact solutions. The results express that the present method gives highly approximate solutions. Also , in Figure 2. the obtained results are illustrated graphically.

6 Conclusion

In this study, Perturbation-Iteration Algorithm was introduced for some Fredholm and Volterra type fractional-integro differential equations and the convergence analysis of the method is given for the first time. The application and results

Table 2: Numerical results of Example 2. for u_3 values for different values of α.

	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$			$\alpha=1.0$		
t	*PIA*	*PIA*	*PIA*	*LVIM*	*PIA*	*Exact*	Absolute Error LVIM	Absolute Error PIA
0.0	1.00000	1.00000	1.00000	1.000000	1.00000	1.00000	0.000000	0.000000
0.1	1.28212	1.24644	1.18464	1.11051	1.10517	1.10517	7.51413E-6	1.73951E-9
0.2	1.55024	1.46729	1.35031	1.22151	1.22140	1.22140	1.07525E-4	1.35655E-7
0.3	2.81414	1.68260	1.51800	1.35034	1.34986	1.34986	4.81654E-4	1.84719E-6
0.4	2.07933	1.90032	1.69403	1.49315	1.49181	1.49182	1.32812E-3	1.22371E-5
0.5	2.35017	2.12576	1.88236	1.65149	1.64867	1.64872	2.77549E-3	5.44797E-5
0.6	2.63047	2.36307	2.08602	1.82691	1.82193	1.82212	4.79581E-3.	1.88391E-4
0.7	2.92353	2.61564	2.30752	2.02086	2.01321	2.01375	7.11310E-3	5.46883E-4
0.8	3.23214	2.88627	2.54898	2.23464	2.22414	2.22554	9.10286E-3	1.39627E-3
0.9	3.55846	3.17710	2.81216	2.46928	2.45637	2.45960	9.67744E-3	3.23130E-3
1.0	3.90380	3.48951	3.09828	2.72543	2.71136	2.71828	7.15178E-3	6.00691E-3

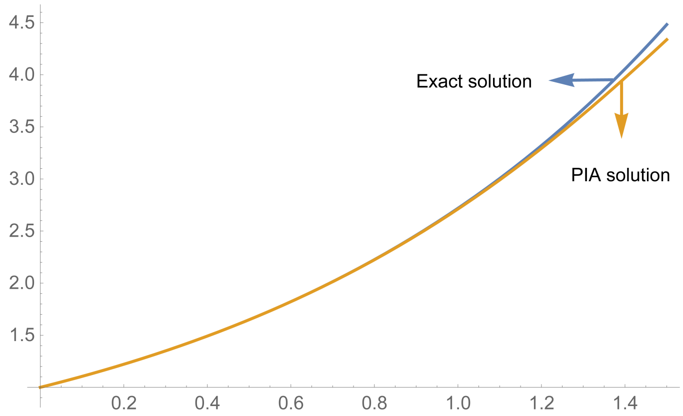

Fig. 2: Comparison of the PIA solution $u_3(t)$ and exact solution for Example 2. when $\alpha = 1$

show that the method is very simple and reliable perturbation-iteration technique and producing highly approximate results. We expect that the present method can be used to calculate the approximate solutions of other types of fractional differential equations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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