New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.188

Hermite-Hadamard type fractional integral inequalities for generalized (s,m,φ)-preinvex functions

Artion Kashuri and Rozana Liko

Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania

Received: 11 November 2016, Accepted: 17 April 2017
Published online: 25 August 2017.

Abstract: In the present paper, by using new identity for fractional integrals some new estimates on generalizations of Hermite-Hadamard type inequalities for the class of generalized (s,m,φ)-preinvex functions via Riemann-Liouville fractional integral are established. These results not only extend the results appeared in the literature (see [2]), but also provide new estimates on these types. At the end, some applications to special means are given.

Keywords: Hermite-Hadamard inequality, Hölder's inequality, power mean inequality, Riemann-Liouville fractional integral, *s*-convex in the second sense, *m*-invex, *P*-function.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a,b]$ is denoted by $L_1[a,b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a,b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

The following definition will be used in the sequel.

Definition 1. *The hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by*

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

for $c > b > 0$ and $|z| < 1$, where $\beta(x,y)$ is the Euler beta function for all $x,y > 0$.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained (see [20]-[22]). For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [18]) and the references cited therein, also (see [9]-[17]) and the references cited therein.

* Corresponding author e-mail: artionkashuri@gmail.com

Fractional calculus (see [18]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 2. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [18]-[25]) and the references cited therein.

Definition 3. *(see [4]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\circ}$ is said to be P-function or P-convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0,1].$$

Definition 4. (see [5]) *A function $f : \mathbb{R}_{\circ} \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{2}$$

for all $x, y \in \mathbb{R}_{\circ}$, $\lambda \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_{\circ}$ as usual. The s-convex functions in the second sense have been investigated in (see [5]).

Definition 5. (see [6]) *A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please (see [6], [7]) and the references therein.

Definition 6. (see [8]) *The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that*

$$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

Motivated by these results, the aim of this paper is to establish some generalizations of Hermite-Hadamard type inequalities using new identity given in Section 2 for generalized (s,m,φ)-preinvex functions via Riemann-Liouville fractional integral. In Section 3, some applications to special means are given. In Section 4, some conclusions and future research are given. These results not only extend the results appeared in the literature (see [2]), but also provide new estimates on these types.

2 Main results

Definition 7. (see [3]) *A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x,y \in K$ and any $t \in [0,1]$.*

Remark. In Definition 7, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

Definition 8. (see [1]) *Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ and $\varphi : I \longrightarrow K$ a continuous increasing function. For $f : K \longrightarrow \mathbb{R}$ and any fixed $s,m \in (0,1]$, if*

$$f(m\varphi(x) + t\eta(\varphi(y),\varphi(x),m)) \leq m(1-t)^s f(\varphi(x)) + t^s f(\varphi(y)), \tag{3}$$

is valid for all $x,y \in I$, $t \in [0,1]$, then we say that $f(x)$ is a generalized (s,m,φ)-preinvex function with respect to η.

Throughout this paper we denote the *generalized cumulative to the left α-gap with respect to η* by

$$\mathrm{L}_{\alpha}(x;\eta,\varphi,m,a) = \frac{1}{2}\left[f\left(m\varphi(a) + \frac{\eta(\varphi(x),\varphi(a),m)}{4}\right) + f\left(m\varphi(a) + \frac{3\eta(\varphi(x),\varphi(a),m)}{4}\right)\right]$$

$$-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(\varphi(x),\varphi(a),m)} \times \left[J^{\alpha}_{\left(m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{4}\right)^-} f(m\varphi(a)) + J^{\alpha}_{\left(m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{2}\right)^-} f\left(m\varphi(a) + \frac{\eta(\varphi(x),\varphi(a),m)}{4}\right)\right.$$

$$\left.+J^{\alpha}_{\left(m\varphi(a)+\frac{3\eta(\varphi(x),\varphi(a),m)}{4}\right)^-} f\left(m\varphi(a) + \frac{\eta(\varphi(x),\varphi(a),m)}{2}\right) + J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))^-} f\left(m\varphi(a) + \frac{3\eta(\varphi(x),\varphi(a),m)}{4}\right)\right].$$

Remark. If we choose $m = 1$, $\varphi(x) = x = b$ and $\eta(b,a,1) = b - a$, then we get the notion of *cumulative to the left α-gap* given in (see [2]).

In order to prove our main results regarding some Hermite-Hadamard type inequalities for generalized (s,m,φ)-preinvex function via fractional integrals, we need the following new Lemma:

Lemma 1. *Let $\varphi : I \longrightarrow K$ be a continuous increasing function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m \in (0,1]$ and let $\eta(\varphi(b),\varphi(a),m) \neq 0$. Assume that $f : K \longrightarrow \mathbb{R}$ is a differentiable function on K° and $f' \in L_1[m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)]$. Then for $\alpha > 0$, the following identity holds:*

$$L_{\alpha}(x;\eta,\varphi,m,a) = \frac{\eta(\varphi(x),\varphi(a),m)}{16}$$

$$\times\left[\int_0^1 t^{\alpha} f'\left(m\varphi(a) + \frac{t}{4}\eta(\varphi(x),\varphi(a),m)\right)dt + \int_0^1 (t^{\alpha}-1) f'\left(m\varphi(a) + \left(\frac{t+1}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)dt\right.$$

$$\left.+\int_0^1 t^{\alpha} f'\left(m\varphi(a) + \left(\frac{t+2}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)dt + \int_0^1 (t^{\alpha}-1) f'\left(m\varphi(a) + \left(\frac{t+3}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)dt\right]. \tag{4}$$

Proof. A simple proof of the equality (4) can be done by performing an integration by parts in the integrals from the right side and changing the variables. The details are left to the interested reader.

Remark. If we choose $m = 1$, $\varphi(x) = x = b$ and $\eta(b,a,1) = b - a$, then we get (see [2], Lemma 1).

Now we turn our attention to establish new integral inequalities of Hermite-Hadamard type for generalized (s,m,φ)-preinvex functions via fractional integrals. Using Lemma 1, the following results can be obtained for the corresponding version for power of the absolute value of the first derivative.

Theorem 2. *Let $\varphi : I \longrightarrow A$ be a continuous increasing function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A\times A\times(0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m \in (0,1]$ and let $\eta(\varphi(b),\varphi(a),m) \neq 0$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s,m,φ)-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q>1$, $p^{-1}+q^{-1}=1$, then for any $\alpha>0$, the following inequality for fractional integrals holds:*

$$
\begin{aligned}
|L_\alpha(x;\eta,\varphi,m,a)| \le \frac{|\eta(\varphi(x),\varphi(a),m)|}{16(4^s(s+1))^{1/q}} \times \Bigg[\frac{1}{(1+p\alpha)^{\frac{1}{p}}}\Bigg\{\Big[m(4^{s+1}-3^{s+1})|f'(\varphi(a))|^q + |f'(\varphi(x))|^q\Big]^{\frac{1}{q}} \\
+\Big[m(2^{s+1}-1)|f'(\varphi(a))|^q + (3^{s+1}-2^{s+1})|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}\Bigg\} + \left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}} \\
\times\Bigg\{\Big[m(3^{s+1}-2^{s+1})|f'(\varphi(a))|^q + (2^{s+1}-1)|f'(\varphi(x))|^q\Big]^{\frac{1}{q}} + \Big[m|f'(\varphi(a))|^q + (4^{s+1}-3^{s+1})|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}\Bigg\}\Bigg].
\end{aligned} \tag{5}
$$

Proof. Suppose that $q>1$. Using Lemma 1, the fact that $|f'|^q$ is a generalized (s,m,φ)-preinvex function, property of the modulus and Hölder's inequality, we have

$$
\begin{aligned}
|\mathrm{L}_\alpha(x;\eta,\varphi,m,a)| &\le \frac{|\eta(\varphi(x),\varphi(a),m)|}{16} \\
&\times\Bigg[\int_0^1 t^\alpha\left|f'\left(m\varphi(a)+\frac{t}{4}\eta(\varphi(x),\varphi(a),m)\right)\right|dt + \int_0^1(1-t^\alpha)\left|f'\left(m\varphi(a)+\left(\frac{t+1}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt \\
&+\int_0^1 t^\alpha\left|f'\left(m\varphi(a)+\left(\frac{t+2}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt + \int_0^1(1-t^\alpha)\left|f'\left(m\varphi(a)+\left(\frac{t+3}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt\Bigg] \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|}{16}\times\Bigg[\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(m\varphi(a)+\frac{t}{4}\eta(\varphi(x),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1(1-t^\alpha)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(m\varphi(a)+\left(\frac{t+1}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(m\varphi(a)+\left(\frac{t+2}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1(1-t^\alpha)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(m\varphi(a)+\left(\frac{t+3}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg] \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|}{16}\times\Bigg[\left(\frac{1}{1+p\alpha}\right)^{1/p}\left(\int_0^1\left(m\left(1-\frac{t}{4}\right)^s|f'(\varphi(a))|^q+\left(\frac{t}{4}\right)^s|f'(\varphi(x))|^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left(\int_0^1\left(m\left(1-\frac{t+1}{4}\right)^s|f'(\varphi(a))|^q+\left(\frac{t+1}{4}\right)^s|f'(\varphi(x))|^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{1+p\alpha}\right)^{1/p}\left(\int_0^1\left(m\left(1-\frac{t+2}{4}\right)^s|f'(\varphi(a))|^q+\left(\frac{t+2}{4}\right)^s|f'(\varphi(x))|^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left(\int_0^1\left(m\left(1-\frac{t+3}{4}\right)^s|f'(\varphi(a))|^q+\left(\frac{t+3}{4}\right)^s|f'(\varphi(x))|^q\right)dt\right)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

$$= \frac{|\eta(\varphi(x),\varphi(a),m)|}{16(4^s(s+1))^{1/q}} \times \left[\frac{1}{(1+p\alpha)^{\frac{1}{p}}} \left\{ \left[m(4^{s+1}-3^{s+1})|f'(\varphi(a))|^q + |f'(\varphi(x))|^q \right]^{\frac{1}{q}} \right.\right.$$

$$\left. + \left[m(2^{s+1}-1)|f'(\varphi(a))|^q + (3^{s+1}-2^{s+1})|f'(\varphi(x))|^q \right]^{\frac{1}{q}} \right\} + \left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)} \right)^{\frac{1}{p}}$$

$$\left.\times \left\{ \left[m(3^{s+1}-2^{s+1})|f'(\varphi(a))|^q + (2^{s+1}-1)|f'(\varphi(x))|^q \right]^{\frac{1}{q}} + \left[m|f'(\varphi(a))|^q + (4^{s+1}-3^{s+1})|f'(\varphi(x))|^q \right]^{\frac{1}{q}} \right\} \right].$$

The proof of Theorem 2 is completed.

Corollary 1. *Under the conditions of Theorem 2, if we choose $m = s = 1$, $\varphi(x) = x = b$ and $\eta(b,a,1) = b-a$, then we get (see [2], Theorem 2).*

Theorem 3. *Let $\varphi : I \longrightarrow A$ be a continuous increasing function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m \in (0,1]$ and let $\eta(\varphi(b),\varphi(a),m) \neq 0$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s,m,φ)-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q \geq 1$, then for any $\alpha > 0$, the following inequality for fractional integrals holds:*

$$|L_\alpha(x;\eta,\varphi,m,a)| \leq \frac{|\eta(\varphi(x),\varphi(a),m)|}{16}$$

$$\times \left[\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left\{ \left[\frac{m \cdot {}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{4}\right)|f'(\varphi(a))|^q}{\alpha+1} + \frac{|f'(\varphi(x))|^q}{4^s(\alpha+s+1)} \right]^{\frac{1}{q}} \right.\right.$$

$$\left. + \left[\frac{m \cdot {}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{2}\right)|f'(\varphi(a))|^q}{2^s(\alpha+1)} + \frac{{}_2F_1\left(-s,\alpha+1;\alpha+2;-\frac{1}{2}\right)|f'(\varphi(x))|^q}{2^s(\alpha+1)} \right]^{\frac{1}{q}} \right\}$$

$$+ \left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}} \cdot \frac{1}{4^{s/q}} \left\{ \left[m\left(\frac{3^{s+1}-2^{s+1}}{s+1}\right)|f'(\varphi(a))|^q + \left(\frac{2^{s+1}-1}{s+1}\right)|f'(\varphi(x))|^q \right.\right.$$

$$\left. - \frac{m3^s \cdot {}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{3}\right)|f'(\varphi(a))|^q}{\alpha+1} - B(\alpha,s)|f'(\varphi(x))|^q \right]^{\frac{1}{q}}$$

$$+ \left[\frac{m|f'(\varphi(a))|^q}{s+1} + \left(\frac{4^{s+1}-3^{s+1}}{s+1}\right)|f'(\varphi(x))|^q - m\frac{\Gamma(s+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)}|f'(\varphi(a))|^q \right.$$

$$\left.\left.\left. - \frac{3^s \cdot {}_2F_1\left(-s,\alpha+1;\alpha+2;-\frac{1}{3}\right)|f'(\varphi(x))|^q}{\alpha+1} \right]^{\frac{1}{q}} \right\} \right], \tag{6}$$

where

$$B(\alpha,s) = \int_0^1 t^\alpha (1+t)^s dt.$$

Proof. Suppose that $q \geq 1$. Using Lemma 1, the fact that $|f'|^q$ is a generalized (s,m,φ)-preinvex function, property of the modulus and the well-known power mean inequality, we have

$$|\mathrm{L}_\alpha(x;\eta,\varphi,m,a)| \leq \frac{|\eta(\varphi(x),\varphi(a),m)|}{16}$$

$$\times \left[\int_0^1 t^\alpha \left| f'\left(m\varphi(a) + \frac{t}{4}\eta(\varphi(x),\varphi(a),m)\right) \right| dt + \int_0^1 (1-t^\alpha) \left| f'\left(m\varphi(a) + \left(\frac{t+1}{4}\right)\eta(\varphi(x),\varphi(a),m)\right) \right| dt \right.$$

$$+\int_0^1 t^{\alpha}\left|f'\left(m\varphi(a)+\left(\frac{t+2}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt+\int_0^1(1-t^{\alpha})\left|f'\left(m\varphi(a)+\left(\frac{t+3}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt\Bigg]$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|}{16}\times\left[\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'\left(m\varphi(a)+\frac{t}{4}\eta(\varphi(x),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})\left|f'\left(m\varphi(a)+\left(\frac{t+1}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'\left(m\varphi(a)+\left(\frac{t+2}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})\left|f'\left(m\varphi(a)+\left(\frac{t+3}{4}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|}{16}\times\left[\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left(m\left(1-\frac{t}{4}\right)^s|f'(\varphi(a))|^q+\left(\frac{t}{4}\right)^s|f'(\varphi(x))|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})\left(m\left(1-\frac{t+1}{4}\right)^s|f'(\varphi(a))|^q+\left(\frac{t+1}{4}\right)^s|f'(\varphi(x))|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left(m\left(1-\frac{t+2}{4}\right)^s|f'(\varphi(a))|^q+\left(\frac{t+2}{4}\right)^s|f'(\varphi(x))|^q\right)dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})\left(m\left(1-\frac{t+3}{4}\right)^s|f'(\varphi(a))|^q+\left(\frac{t+3}{4}\right)^s|f'(\varphi(x))|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{|\eta(\varphi(x),\varphi(a),m)|}{16}\times\left[\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\left[\frac{m\cdot{}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{4}\right)|f'(\varphi(a))|^q}{\alpha+1}+\frac{|f'(\varphi(x))|^q}{4^s(\alpha+s+1)}\right]^{\frac{1}{q}}\right.\right.$$

$$\left.+\left[\frac{m\cdot{}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{2}\right)|f'(\varphi(a))|^q}{2^s(\alpha+1)}+\frac{{}_2F_1\left(-s,\alpha+1;\alpha+2;-\frac{1}{2}\right)|f'(\varphi(x))|^q}{2^s(\alpha+1)}\right]^{\frac{1}{q}}\right\}$$

$$+\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\cdot\frac{1}{4^{s/q}}\left\{\left[m\left(\frac{3^{s+1}-2^{s+1}}{s+1}\right)|f'(\varphi(a))|^q+\left(\frac{2^{s+1}-1}{s+1}\right)|f'(\varphi(x))|^q\right.\right.$$

$$\left.-\frac{m3^s\cdot{}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{3}\right)|f'(\varphi(a))|^q}{\alpha+1}-B(\alpha,s)|f'(\varphi(x))|^q\right]^{\frac{1}{q}}$$

$$+\left[\frac{m|f'(\varphi(a))|^q}{s+1}+\left(\frac{4^{s+1}-3^{s+1}}{s+1}\right)|f'(\varphi(x))|^q-m\frac{\Gamma(s+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)}|f'(\varphi(a))|^q\right.$$

$$\left.\left.\left.-\frac{3^s\cdot{}_2F_1\left(-s,\alpha+1;\alpha+2;-\frac{1}{3}\right)|f'(\varphi(x))|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\}\right].$$

The proof of Theorem 3 is completed.

Corollary 2. *Under the conditions of Theorem 3, if we choose $m=s=1$, $\varphi(x)=x=b$ and $\eta(b,a,1)=b-a$, then we get (see [2], Theorem 3).*

Remark. It is worthwhile to note that for different choices of values x and functions φ, we can get some interesting Hermite-Hadamard type fractional integral inequalities by our theorems mentioned in this paper.

Remark. By considering the *generalized cumulative to the right α-gap with respect to η* defined as

$$\mathrm{R}_{\alpha}(x;\eta,\varphi,m,a)=-\frac{1}{2}\left[\frac{f(m\varphi(a))+f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{2}+f\left(m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{2}\right)\right]$$

$$+\frac{4^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(\varphi(x),\varphi(a),m)}\times\left[J^{\alpha}_{(m\varphi(a))+}f\left(m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{4}\right)+J^{\alpha}_{\left(m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{4}\right)+}f\left(m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{2}\right)\right.$$

$$\left.+J^{\alpha}_{\left(m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{2}\right)+}f\left(m\varphi(a)+\frac{3\eta(\varphi(x),\varphi(a),m)}{4}\right)+J^{\alpha}_{\left(m\varphi(a)+\frac{3\eta(\varphi(x),\varphi(a),m)}{4}\right)+}f\left(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)\right)\right],$$

one can obtain similar results. The details are left to the interested reader.

3 Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 9. (see [26]) *A function $M:\mathbb{R}^2_+\longrightarrow\mathbb{R}_+$, is called a Mean function if it has the following properties:*

1. *Homogeneity:* $M(ax,ay)=aM(x,y)$, *for all* $a>0$,
2. *Symmetry:* $M(x,y)=M(y,x)$,
3. *Reflexivity:* $M(x,x)=x$,
4. *Monotonicity: If* $x\le x'$ *and* $y\le y'$, *then* $M(x,y)\le M(x',y')$,
5. *Internality:* $\min\{x,y\}\le M(x,y)\le\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α,β $(\alpha\neq\beta)$.

1. The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}$$
2. The geometric mean:
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta}$$
3. The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
4. The power mean:
$$P_r:=P_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\ \ r\ge 1.$$
5. The identric mean:
$$I:=I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right), & \alpha\neq\beta;\\ \alpha, & \alpha=\beta.\end{cases}$$
6. The logarithmic mean:
$$L:=L(\alpha,\beta)=\frac{\beta-\alpha}{\ln(\beta)-\ln(\alpha)}.$$

7.The generalized log-mean:

$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}};\ p\in\mathbb{R}\setminus\{-1,0\}.$$

8.The weighted p-power mean:

$$M_p\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & ,\alpha_n\\ u_1, & u_2, & \cdots & ,u_n\end{pmatrix} = \left(\sum_{i=1}^{n}\alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0\le\alpha_i\le 1$, $u_i>0\,(i=1,2,\ldots,n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p\in\mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H\le G\le L\le I\le A$. Now, let a and b be positive real numbers such that $a<b$. Consider the function $M := M(\varphi(a),\varphi(b)) : [\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))]\times[\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I\longrightarrow A$ be a continuous increasing function, therefore one can obtain various inequalities using the results of Section 2 for these means as follows.

Replace $\eta(\varphi(y),\varphi(x),m)$ with $\eta(\varphi(y),\varphi(x))$ and setting $\eta(\varphi(a),\varphi(x)) = M(\varphi(a),\varphi(x))$, $\forall x\in I$, for value $m=1$ in (**??**) and (6), one can obtain the following interesting inequalities involving means:

$$\Bigg|\frac{1}{2}\left[f\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{4}\right)+f\left(\varphi(a)+\frac{3M(\varphi(a),\varphi(x))}{4}\right)\right]-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{M^{\alpha}(\varphi(a),\varphi(x))}$$
$$\times\Bigg[J^{\alpha}_{\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{4}\right)-}f(\varphi(a))+J^{\alpha}_{\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{2}\right)-}f\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{4}\right)$$
$$+J^{\alpha}_{\left(\varphi(a)+\frac{3M(\varphi(a),\varphi(x))}{4}\right)-}f\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{2}\right)+J^{\alpha}_{(\varphi(a)+M(\varphi(a),\varphi(x)))-}f\left(\varphi(a)+\frac{3M(\varphi(a),\varphi(x))}{4}\right)\Bigg]\Bigg|$$
$$\le\frac{M(\varphi(a),\varphi(x))}{16(4^s(s+1))^{1/q}}\times\Bigg[\frac{1}{(1+p\alpha)^{\frac{1}{p}}}\Bigg\{\left[(4^{s+1}-3^{s+1})|f'(\varphi(a))|^q+|f'(\varphi(x))|^q\right]^{\frac{1}{q}}$$
$$+\left[(2^{s+1}-1)|f'(\varphi(a))|^q+(3^{s+1}-2^{s+1})|f'(\varphi(x))|^q\right]^{\frac{1}{q}}\Bigg\}+\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}$$
$$\times\left\{\left[(3^{s+1}-2^{s+1})|f'(\varphi(a))|^q+(2^{s+1}-1)|f'(\varphi(x))|^q\right]^{\frac{1}{q}}+\left[|f'(\varphi(a))|^q+(4^{s+1}-3^{s+1})|f'(\varphi(x))|^q\right]^{\frac{1}{q}}\right\}\Bigg],\qquad(7)$$

$$\Bigg|\frac{1}{2}\left[f\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{4}\right)+f\left(\varphi(a)+\frac{3M(\varphi(a),\varphi(x))}{4}\right)\right]-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{M^{\alpha}(\varphi(a),\varphi(x))}$$
$$\times\Bigg[J^{\alpha}_{\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{4}\right)-}f(\varphi(a))+J^{\alpha}_{\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{2}\right)-}f\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{4}\right)$$
$$+J^{\alpha}_{\left(\varphi(a)+\frac{3M(\varphi(a),\varphi(x))}{4}\right)-}f\left(\varphi(a)+\frac{M(\varphi(a),\varphi(x))}{2}\right)+J^{\alpha}_{(\varphi(a)+M(\varphi(a),\varphi(x)))-}f\left(\varphi(a)+\frac{3M(\varphi(a),\varphi(x))}{4}\right)\Bigg]\Bigg|$$
$$\le\frac{M(\varphi(a),\varphi(x))}{16}\times\Bigg[\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg\{\left[\frac{{}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{4}\right)|f'(\varphi(a))|^q}{\alpha+1}+\frac{|f'(\varphi(x))|^q}{4^s(\alpha+s+1)}\right]^{\frac{1}{q}}$$

$$+\left[\frac{{}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{2}\right)|f'(\varphi(a))|^q}{2^s(\alpha+1)}+\frac{{}_2F_1\left(-s,\alpha+1;\alpha+2;-\frac{1}{2}\right)|f'(\varphi(x))|^q}{2^s(\alpha+1)}\right]^{\frac{1}{q}}\Bigg\}$$

$$+\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\cdot\frac{1}{4^{s/q}}\Bigg\{\Bigg[\left(\frac{3^{s+1}-2^{s+1}}{s+1}\right)|f'(\varphi(a))|^q+\left(\frac{2^{s+1}-1}{s+1}\right)|f'(\varphi(x))|^q$$

$$-\frac{3^s\cdot{}_2F_1\left(-s,\alpha+1;\alpha+2;\frac{1}{3}\right)|f'(\varphi(a))|^q}{\alpha+1}-B(\alpha,s)|f'(\varphi(x))|^q\Bigg]^{\frac{1}{q}}$$

$$+\Bigg[\frac{|f'(\varphi(a))|^q}{s+1}+\left(\frac{4^{s+1}-3^{s+1}}{s+1}\right)|f'(\varphi(x))|^q-\frac{\Gamma(s+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)}|f'(\varphi(a))|^q$$

$$-\frac{3^s\cdot{}_2F_1\left(-s,\alpha+1;\alpha+2;-\frac{1}{3}\right)|f'(\varphi(x))|^q}{\alpha+1}\Bigg]^{\frac{1}{q}}\Bigg\}\Bigg]. \quad (8)$$

Letting $M(\varphi(a),\varphi(x)) = A,G,H,P_r,I,L,L_p,M_p$, $\forall x\in I$ in (7) and (8), we get the inequalities involving means for a particular choices of a differentiable generalized $(s,1,\varphi)$-preinvex function f. The details are left to the interested reader. They also can obtain similar results considering the *generalized cumulative to the right α-gap with respect to η*.

4 Conclusions

We have considered and investigated the class of generalized (s,m,φ)-preinvex functions. Moreover, using new integral identity, some Hermite-Hadamard type inequalities for generalized (s,m,φ)-preinvex functions that are differentiable via Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, conformable fractional integrals and k-fractional integrals.

5 Competing Interests

The authors declare that they have no competing interests.

6 Authors' Contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (s,m,φ)-preinvex functions, *Aust. J. Math. Anal. Appl.*, **13**, (1) (2016), Article 16, 1-11.

[2] V. M. Mihai, Some Hermite-Hadamard type inequalities via Riemann-Liouville fractional calculus, *Tamkang J. Math.*, **44**, (4) (2013), 411-416.

[3] T. S. Du, J. G. Liao and Y. J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized (s,m)-preinvex functions, *J. Nonlinear Sci. Appl.*, **9**, (2016), 3112-3126.

[4] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.*, **21**, (1995), 335-341.

[5] H. Hudzik and L. Maligranda, Some remarks on s-convex functions, *Aequationes Math.*, **48**, (1994), 100-111.

[6] T. Antczak, Mean value in invexity analysis, *Nonlinear Anal.*, **60**, (2005), 1473-1484.

[7] X. M. Yang, X. Q. Yang and K. L. Teo, Generalized invexity and generalized invariant monotonicity, *J. Optim. Theory Appl.*, **117**, (2003), 607-625.

[8] R. Pini, Invexity and generalized convexity, *Optimization.*, **22**, (1991), 513-525.

[9] H. Kavurmaci, M. Avci and M. E. Özdemir, New inequalities of Hermite-Hadamard type for convex functions with applications, *arXiv:1006.1593v1 [math. CA]*, (2010), 1-10.

[10] Y. M. Chu, G. D. Wang and X. H. Zhang, Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.*, **13**, (4) (2010), 725-731.

[11] X. M. Zhang, Y. M. Chu and X. H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, *J. Inequal. Appl.*, (2010), Article ID 507560, 11 pages.

[12] Y. M. Chu, M. A. Khan, T. U. Khan and T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.*, **9**, (5) (2016), 4305-4316.

[13] M. Adil Khan, Y. Khurshid, T. Ali and N. Rehman, Inequalities for three times differentiable functions, *J. Math., Punjab Univ.*, **48**, (2) (2016), 35-48.

[14] M. Adil Khan, Y. Khurshid and T. Ali, Hermite-Hadamard inequality for fractional integrals via α-convex functions, *Acta Math. Univ. Comenianae*, **79**, (1) (2017), 153-164.

[15] Y. M. Chu, M. Adil Khan, T. Ullah Khan and T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.*, **9**, (2016), 4305-4316.

[16] H. N. Shi, Two Schur-convex functions related to Hadamard-type integral inequalities, *Publ. Math. Debrecen*, **78**, (2) (2011), 393-403.

[17] F. X. Chen and S. H. Wu, Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions, *J. Nonlinear Sci. Appl.*, **9**, (2) (2016), 705-716.

[18] W. Liu, W. Wen and J. Park, Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals, *J. Nonlinear Sci. Appl.*, **9**, (2016), 766-777.

[19] W. Liu, W. Wen and J. Park, Ostrowski type fractional integral inequalities for MT-convex functions, *Miskolc Mathematical Notes.*, **16**, (1) (2015), 249-256.

[20] Y. M. Chu, G. D. Wang and X. H. Zhang, Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.*, **13**, (4) (2010), 725-731.

[21] X. M. Zhang, Y. M. Chu and X. H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, *J. Inequal. Appl.*, (2010), Article ID 507560, 11 pages.

[22] Y. M. Chu, M. A. Khan, T. U. Khan and T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.*, **9**, (5) (2016), 4305-4316.

[23] M. Tunç, Ostrowski type inequalities for functions whose derivatives are MT-convex, *J. Comput. Anal. Appl.*, **17**, (4) (2014), 691-696.

[24] W. J. Liu, Some Simpson type inequalities for h-convex and (α,m)-convex functions, *J. Comput. Anal. Appl.*, **16**, (5) (2014), 1005-1012.

[25] F. Qi and B. Y. Xi, Some integral inequalities of Simpson type for $GA-\varepsilon$-convex functions, *Georgian Math. J.*, **20**, (5) (2013), 775-788.

[26] P. S. Bullen, Handbook of Means and Their Inequalities, *Kluwer Academic Publishers*, Dordrecht, (2003).