New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.184

On an application of Laplace transforms

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Received: 3 October 2016, Accepted: 9 November 2016
Published online: 25 August 2017

Abstract: In this study, complex differential equations are solved using laplace transform. Firstly we seperate real and imaginer parts of equation. Thus from one unknown equation is obtained two unknown equation system. Later we obtain laplace transforms of real and imaginer parts of solutions using laplace transform. In the latest we obtain real and imaginer parts of solution using inverse laplace transform.

Keywords: Laplace Transform.

1 Introduction

In real, general solutions of some equations, especially type of elliptic, are not found. Real partial differential equation systems when number of independent variables are even can be transformed to a complex partial differential equations. The solving a complex equation can more easier with complex methods. For example,

$$u_{xx}+u_{yy}=0 \tag{1}$$

Laplace equation hasn't got general solution in R^2, but it can be written

$$u_{z\overline{z}}=0$$

with the relation

$$\Delta=\frac{\partial^2}{\partial z\partial\overline{z}}$$

and the solution of this equation is

$$u=f(z)+g(\overline{z})$$

where f is analytic, g is anti analytic arbitrary functions. A partial differential equation system which has two real dependant and two real independant variables can be transformed to a complex equation. For example,

$$u_x-v_y=0$$
$$u_y+v_x=0.$$

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Cauchy Riemann system transforms to complex equation

$$w_{\overline{z}} = 0$$

where $w = u + iv, z = x + iy$. All solutions of this complex equation are analytic functions.

Moreover any order complex differential equation can be transformed to real partial differential equation system which has two unknowns, two independent variables by seperating the real and imaginer parts. The solution of complex equation can be put forward helping solutions of this real system.

In this study, we investigate solutions of first order constant coefficients complex equations with laplace transforms. Laplace transform using several areas of mathematics is a integral transform. We can solve ordinary differential equations, system of ordinary differential equation, integral equations, integro differential equations, difference equations, integro difference equations and also calculate some generalized integrals with laplace transform. Moreover we can use laplace transform in electrical circuits. Therefore we can solve fractional differential equations via laplace transforms[2,3]. Nonlinear differential equations can be solved laplace decomposition method[4].

2 Basic definitions and theorems

Definition 1. *Let $F(t)$ be a function for $t > 0$. Laplace transform of $F(t)$.*

$$L(F(t)) = f(s) = \int_0^\infty e^{-st}.f(t)\,dt \tag{2}$$

is defined. Since integral (1) is a function of s, $L(F(t)) = f(s)$ is written.

Theorem 1. *Laplace transforms of some functions are given in following.*

$$\begin{array}{cc} F(t) & L(F(t)) = f(s) \\ 1 & \frac{1}{s} \\ t & \frac{1}{s^2} \\ t^n & \frac{n!}{s^{n+1}}. \end{array}$$

Theorem 2. *If $F^{(n)}(t)$ is partial continuous then*

$$L(F^{(n)}(t)) = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - s^{n-2}F''(0) - \ldots - F^{(n-1)}(0)$$

where $L(F(t)) = f(s)$.

Theorem 3. *Laplace transforms of partial derivatives of $u(x,t)$ are following.*

(i) $L\left[\frac{\partial u}{\partial t}\right] = s.U(x,s) - u(x,0)$

(ii) $L\left[\frac{\partial u}{\partial x}\right] = \frac{\partial U(x,s)}{\partial x}$,

where $U(x,s) = L[u(x,t)]$.

2.1 Complex derivatives

Let $w = w(z,\overline{z})$ be a complex function. Here $z = x+iy$, $w(z,\overline{z}) = u(x,y)+i.v(x,y)$. First order derivatives according to z and $\overline{z}$ of $w(z,\overline{z})$ are defined as following:

$$\frac{\partial w}{\partial z} = \frac{1}{2}(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}). \tag{3}$$

$$\frac{\partial w}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}). \tag{4}$$

3 Solution of complex differential equations from first order which is constant coeffients

Theorem 4. *Let A,B,C are real constants ,$F(z,\overline{z})$ is a polynomial of $z,\overline{z}$ and $w = u+iv$ is a complex function. Then a solution of*

$$A.\frac{\partial w}{\partial z} + B.\frac{\partial w}{\partial \overline{z}} + C.w = F(z,\overline{z})$$

$$w(x,0) = f(x)$$

is

$$u = L^{-1}\left[\frac{(A+B)\frac{\partial}{\partial x}(2F_1^* + (A-B)v(x,0)) + 2C(2F_1^* + (A-B)v(x,0)) - s(A-B)(2F_2^* + (B-A)u(x,0))}{[(A+B)D+2C]^2 + s^2(A-B)^2}\right]$$

$$v = L^{-1}\left[\frac{(A+B)\frac{\partial}{\partial x}(2F_2^* + (B-A)u(x,0)) + 2C(2F_2^* + (B-A)u(x,0)) - s(B-A)(2F_1^* + (A-B)v(x,0))}{[(A+B)D+2C]^2 + s^2(A-B)^2}\right]$$

Proof.

$$A.\frac{\partial w}{\partial z} + B.\frac{\partial w}{\partial \overline{z}} + C.w = F(z,\overline{z}) \tag{5}$$

If it is used equalities (3) , (4) in equality (5), following equality is obtained .

$$A.\frac{1}{2}(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}) + B.\frac{1}{2}(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}) + Cw = F_1(x,y) + iF_2(x,y) \tag{6}$$

If in (6) $w = u+iv$ is written, following equality is obtained .

$$A(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}) + B.(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}) + 2C.(u+iv) = 2F_1(x,y) + 2iF_2(x,y) \tag{7}$$

If (7) is seperated to real and imaginer parts, then following equation system is obtained

$$(A+B)\frac{\partial u}{\partial x} + (A-B)\frac{\partial v}{\partial y} + 2Cu = 2F_1(x,y) \tag{8}$$

$$(A+B)\frac{\partial v}{\partial x} + (B-A)\frac{\partial u}{\partial y} + 2Cv = 2F_2(x,y) \tag{9}$$

If we use laplace transform for above (8) , (9) equalities, then we get following equalities

$$(A+B)\frac{\partial U}{\partial x} + (A-B)(sV - v(x,0)) + 2CU = 2F_1^* \tag{10}$$

$$(A+B)\frac{\partial V}{\partial x}+(B-A)(sU-u(x,0))+2CV=2F_2^* \tag{11}$$

Where U,V,F_1^*,F_2^* are laplace transforms of u,v,F_1,F_2 respectively. If (10), (11) is rerugulate and is used cramer rule, then (12), (13) equalities are obtained.

$$(A+B)\frac{\partial U}{\partial x}+2CU+s(A-B)V=2F_1^*+(A-B)v(x,0)$$

$$s(B-A)U+(A+B)\frac{\partial V}{\partial x}+2CV=2F_2^*+(B-A)u(x,0)$$

$$\begin{vmatrix} (A+B)D+2C & s(A-B) \\ s(B-A) & (A+B)D+2C \end{vmatrix}=[(A+B)D+2C]^2+s^2(A-B)^2$$

$$U=\frac{\begin{vmatrix} 2F_1^*+(A-B)v(x,0) & s(A-B) \\ 2F_2^*+(B-A)u(x,0) & (A+B)D+2C \end{vmatrix}}{[(A+B)D+2C]^2+s^2(A-B)^2}$$

$$U=\frac{(A+B)\frac{\partial}{\partial x}(2F_1^*+(A-B)v(x,0))+2C\,(2F_1^*+(A-B)v(x,0))-s(A-B)(2F_2^*+(B-A)u(x,0))}{[(A+B)D+2C]^2+s^2(A-B)^2} \tag{12}$$

$$V=\frac{\begin{vmatrix} (A+B)D+2C & 2F_1^*+(A-B)v(x,0) \\ s(B-A) & 2F_2^*+(B-A)u(x,0) \end{vmatrix}}{[(A+B)D+2C]^2+s^2(A-B)^2}$$

$$V=\frac{(A+B)\frac{\partial}{\partial x}(2F_2^*+(B-A)u(x,0))+2C\,(2F_2^*+(B-A)u(x,0))-s(B-A)(2F_1^*+(A-B)v(x,0))}{[(A+B)D+2C]^2+s^2(A-B)^2} \tag{13}$$

Followings are obtained from inverse laplace transform of (12), (13).

$$u(x,y)=L^{-1}\left[\frac{(A+B)\frac{\partial}{\partial x}(2F_1^*+(A-B)v(x,0))+2C\,(2F_1^*+(A-B)v(x,0))-s(A-B)(2F_2^*+(B-A)u(x,0))}{[(A+B)D+2C]^2+s^2(A-B)^2}\right] \tag{14}$$

$$v(x,y)=L^{-1}\left[\frac{(A+B)\frac{\partial}{\partial x}(2F_2^*+(B-A)u(x,0))+2C\,(2F_2^*+(B-A)u(x,0))-s(B-A)(2F_1^*+(A-B)v(x,0))}{[(A+B)D+2C]^2+s^2(A-B)^2}\right] \tag{15}$$

Example 1. Solve the following problem

$$\frac{\partial w}{\partial z}+2\frac{\partial w}{\partial z}=3z^2+2 \tag{16}$$

with the condition

$$w(x,0)=x^3+x \tag{17}$$

Coefficients of equation are $A=1, B=2, C=0$ and $F(z,\bar{z})=3z^2+2$. Real and imaginar parts of $F(z,\bar{z})$ are $F_1(x,y)=3x^2-3y^2+2, F_1(x,y)=6xy$. Laplace transforms of $F_1(x,y)$ and $F_2(x,y)$ are following.

$$F_1^*(x,s)=L[F_1(x,y)]=(3x^2+2)/s-6/s^3.$$

$$F_2^*(x,s)=L[F_2(x,y)]=6x/s^2.$$

From theorem

$$\begin{aligned}
u(x,y) &= L^{-1}[(3\frac{\partial}{\partial x}((6x^2+4)/s-12/s^3)+s(12x/s^2+x^3+x))/(9D^2+s^2)] \\
&= L^{-1}[(48x/s+s(x^3+x))/(9D^2+s^2)] \\
&= L^{-1}[\frac{1}{s^2(1+\frac{9D^2}{s^2})}(\frac{48x}{s}+s(x^3+x))] \\
&= L^{-1}[\frac{1}{s^2}(\frac{48x}{s}+s(x^3+x)-\frac{54x}{s})] \\
&= L^{-1}[\frac{-6x}{s^3}+\frac{x^3+x}{s}]=x^3+x-3xy^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
v(x,y) &= L^{-1}[\frac{3\frac{\partial}{\partial x}(\frac{12x}{s^2}+x^3+x)-s((\frac{6x^2+4)}{s}-\frac{12}{s^3}))}{9D^2+s^2}] \\
&= L^{-1}[\frac{(\frac{48}{s^2}+3x^2-1)}{9D^2+s^2}] \\
&= L^{-1}[\frac{1}{s^2(1+\frac{9D^2}{s^2})}(\frac{48}{s^2}+3x^2-1)] \\
&= L^{-1}[\frac{1}{s^2}(-\frac{6}{s^2}+3x^2-1)] \\
&= L^{-1}[\frac{-6}{s^4}+\frac{3x^2-1}{s^2}]=-y^3+3x^2y-y.
\end{aligned}$$

Hence

$$w=u+iv=x^3+x-3xy^2+i\left(-y^3+3x^2y-y\right)=z^3+\bar{z}.$$

Example 2. Solve the following problem

$$2\frac{\partial w}{\partial z}-\frac{\partial w}{\partial \bar{z}}=4z+1$$

with the condition

$$w(x,0)=x^2+5x.$$

Solution 1. Coefficients of equation are $A=2, B=-1, C=0$ and $F(z,\bar{z})=4z+1$.

$$\begin{aligned}
F_1^*(x,s) &= L[F_1(x,y)]=(4x+1)/s \\
F_2^*(x,s) &= L[F_2(x,y)]=4/s^2
\end{aligned}$$

$$
\begin{aligned}
u(x,y) &= L^{-1}[\frac{\frac{\partial}{\partial x}((8x+2)/s)-3s(8/s^2-3\left(x^2+5x\right))}{D^2+s^2}] \\
&= L^{-1}[\frac{\frac{8}{s}-\frac{24}{s}+9s\left(x^2+5x\right)}{D^2+9s^2}] \\
&= L^{-1}[\frac{1}{9s^2(1+\frac{D^2}{9s^2})}(\frac{-16}{s}+9sx^2+45sx)] \\
&= L^{-1}[\frac{1}{9s^2}\left(1-\frac{D^2}{9s^2}+\frac{D^4}{81s^4}-\frac{D^6}{729s^6}+...\right)(\frac{-16}{s}+9sx^2+45sx)] \\
&= L^{-1}[\frac{1}{9s^2}\left(\frac{-16}{s}+9sx^2+45sx-\frac{18}{9s}\right)] \\
&= L^{-1}\left[-\frac{2}{s^3}+\frac{x^2+5x}{s}\right]=x^2+5x-y^2.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
v(x,y) &= L^{-1}[\frac{\frac{\partial}{\partial x}(\frac{8}{s^2}-3\left(x^2+5x\right))+3s(8x+2)/s)}{D^2+9s^2}] \\
&= L^{-1}\left[\frac{1}{9s^2(1+\frac{D^2}{9s^2})}(-6x-15+24x+6)\right] \\
&= L^{-1}[\frac{1}{9s^2}\left(1-\frac{D^2}{9s^2}+\frac{D^4}{81s^4}-\frac{D^6}{729s^6}+...\right)(18x-9) \\
&= L^{-1}\left[\frac{1}{9s^2}(18x-9)\right] \\
&= 2xy-y.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
w &= u+iv \\
&= x^2+5x-y^2+i(2xy-y) \\
&= x^2+2ixy-y^2+3(x-iy)+2(x+iy) \\
&= z^2+3\overline{z}+2z.
\end{aligned}
$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Yin, F. K., Han, W. Y.and Song, J. Q, "Modified Laplace decomposition method for Lane-Emden Type differential equations", International Jaurnal of Applied Physics and Matematics, 3 (2): 98-102 (2013).

[2] Mohamed, M. A. and Torky, M. S., “Numerical solution of nonlinear system of parial differential equations by the Laplace decomposition method and the padeapproximation”, American Journal of Computational Mathematics,3 (3): 175-184 (2013).

[3] Yusufoglu, E., “Numerical Solution of duffing equation by the Laplace decomposition algorithm”, Applied Mathematics and Computation,177 (2): 572-580 (2006).

[4] Kazem, S.,”Exact Solution of Some Linear Fractional Differential Equations by Laplace Transform”, International Journal of Nonlinear Science, 16 (2013) No:1 pp. 3-11

[5] Kexue, L., Jigen, P. ”Laplace transform and fractional differential equations”, Applied Mathematics Letters, 24 (2011) 2019–2023

[6] Gupta, S., Kumar, D., Singh, J.”Numerical study for systems of fractional differential equations via Laplace transform ”, Journal of the Egyptian Mathematical Society, (2015) 23, 256–262