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On the Hermite-Hadamard's and Ostrowski's inequalities for the co-ordinated convex functions

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Abstract: In this paper, we give new some inequalities of Hermite-Hadamard's and Ostrowski's type for convex functions on the co-ordinates defined in a rectangle from the plane. Our established results generalize some recent results for functions whose partial derivatives in absolute value are convex on the co-ordinates on the rectangle from the plane.

Keywords: Convex function, co-ordinated convex mapping, Ostrowski inequality, Hermite-Hadamard inequality.

1 Introduction

In 1938, the classical integral inequality established by Ostrowski [12] as follows.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds.*

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty \tag{1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence inequality (1) has attracted considerable attention and interest from mathematicans and researchers. For some results which generalize, improve and extend the inequality (1) see ([3], [4], [9] and [19]) and the references therein. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one.

Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a,b\in I$, with $a<b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [7].

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}.$$

Let us now consider a bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx+(1-t)z,ty+(1-t)w)\leq tf(x,y)+(1-t)f(z,w)$$

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holds, for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$. A function $f:\Delta\to\mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b]$ and $y\in[c,d]$ (see,[6]).

A formal definition for co-ordinated convex function may be stated as follows.

Definition 1. *A function $f:\Delta\to\mathbb{R}$ will be called co-ordinated canvex on Δ, for all $t,s\in[0,1]$ and $(x,y),(u,v)\in\Delta$, if the following inequality holds.*

$$f(tx+(1-t)y,su+(1-s)v)\le tsf(x,u)+s(1-t)f(y,u)+t(1-s)f(x,v)+(1-t)(1-s)f(y,v).$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [6]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1], [2], [9]-[11] and [16]).

Also, in [6], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 2. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities.*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (2)\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

In [16], Sarikaya et al. proved some new inequalities that give estimate of the deference between the middle and the right most terms in (2) for differentiable co-ordinated convex functions on rectangele from the plane $\mathbb{R}^2$. For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$; we refer the reader to ([1], [2], [6]-[11], [14]-[18]).

In [20], Sarikaya and Yildiz proved the following Lemma for double integrals. The lemma is necessary and plays an important role in establishing our main results. Firstly, the $S_\lambda(f;g,h)$ operator that we will use throughout the article may be defined as follow.

$$\begin{aligned}
S_\lambda(f;g,h) &= \left(\int_a^b g(u)du\right)\left(\int_c^d h(u)du\right)\times\left[(1-\lambda)^2 f(x,y)+\lambda(1-\lambda)f(b,y)+\lambda(1-\lambda)f(x,d)+\lambda^2 f(b,d)\right]\\
&-\left(\int_a^x g(u)du\right)\int_c^d h(s)\left[(1-\lambda)f(x,s)+\lambda f(a,s)\right]-\left(\int_x^b g(u)du\right)\int_c^d h(s)\left[(1-\lambda)f(x,s)+\lambda f(b,s)\right]ds \qquad (3)\\
&-\left(\int_c^y h(u)du\right)\int_a^b g(t)\left[(1-\lambda)f(t,y)+\lambda f(t,c)\right]dt-\left(\int_y^d h(u)du\right)\int_a^b g(t)\left[(1-\lambda)f(t,y)+\lambda f(t,d)\right]dt\\
&+\int_a^b\int_c^d g(t)h(s)f(t,s)\,dsdt.
\end{aligned}$$

Lemma 1. *Let* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ *be an absolutely continuous function such that the partial derivative of order 2 exists for all* $(t,s)\in[a,b]\times[c,d]=:\Delta$*, and the functions* $g:[a,b]\to[0,\infty)$ *and* $h:[c,d]\to[0,\infty)$ *are integrable. If* $f_{ts}(t,s)\in L(\Delta)$*, then the following equality holds.*

$$\int_a^b\int_c^d P_\lambda(x,t)\,Q_\lambda(y,s)\,f_{ts}(t,s)\,dsdt=S_\lambda(f;g,h)$$

where

$$P_\lambda(x,t):=\begin{cases}(1-\lambda)\int_a^t g(u)\,du+\lambda\int_x^t g(u)\,du\ ,a\le t<x\\[2ex](1-\lambda)\int_b^t g(u)\,du+\lambda\int_x^t g(u)\,du\ ,x\le t\le b.\end{cases}$$

and

$$Q_\lambda(y,s):=\begin{cases}(1-\lambda)\int_c^s h(u)\,du+\lambda\int_y^s h(u)\,du\ ,a\le t<x\\[2ex](1-\lambda)\int_d^s h(u)\,du+\lambda\int_y^s h(u)\,du\ ,x\le t\le b.\end{cases}$$

for $\lambda\in[0,1]$.

The main purpose of this paper is to establish new Hadamard-type and Ostrowski-type inequalities of convex functions of 2-variables on the co-ordinates by using Lemma 1 and elementary analysis.

2 Main results

For convenience, we give the following notations used to simplify the details of presentation,

$$\begin{aligned}A_\lambda(x)&=(x-a)^2[(1-\lambda)(3b-a-2x)+\lambda(3b-2a-x)]+(2-\lambda)(b-x)^3\\&=(1-\lambda)\left\{(b-a)(x-a)^2+4(b-a)^2(b-x)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\right\}\\&+2\lambda\left\{(b-a)(x-a)^2+(b-a)^2(b-x)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\right\},\end{aligned}$$

$$\begin{aligned}B_\lambda(x)&=(b-x)^2[(1-\lambda)(b-3a+2x)+\lambda(2b-3a+x)]+(2-\lambda)(x-a)^3\\&=(1-\lambda)\left\{(b-a)(b-x)^2+4(b-a)^2(x-a)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\right\}\\&+2\lambda\left\{(b-a)(b-x)^2+(b-a)^2(x-a)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\right\},\end{aligned}$$

$$\begin{aligned}C_\lambda(y)&=(y-c)^2[(1-\lambda)(3d-c-2y)+\lambda(3d-2c-y)]+(2-\lambda)(d-y)^3\\&=(1-\lambda)\left\{(d-c)(y-c)^2+4(d-c)^2(d-y)\left[\frac{1}{4}+\frac{\left(y-\frac{c+d}{2}\right)^2}{(d-c)^2}\right]\right\}\\&+2\lambda\left\{(d-c)(y-c)^2+(d-c)^2(d-y)\left[\frac{1}{4}+\frac{\left(y-\frac{c+d}{2}\right)^2}{(d-c)^2}\right]\right\},\end{aligned}$$

$$
\begin{aligned}
D_\lambda(y) &= (d-y)^2\left[(1-\lambda)(d-3c+2y)+\lambda(2d-3c+y)\right]+(2-\lambda)(y-c)^3 \\
&= (1-\lambda)\left\{(d-c)(d-y)^2+4(d-c)^2(y-c)\left[\frac{1}{4}+\frac{\left(y-\frac{c+d}{2}\right)^2}{(d-c)^2}\right]\right\} \\
&\quad +2\lambda\left\{(d-c)(d-y)^2+(d-c)^2(y-c)\left[\frac{1}{4}+\frac{\left(y-\frac{c+d}{2}\right)^2}{(d-c)^2}\right]\right\}.
\end{aligned}
$$

Using the Lemma 1, we can obtain the following general integral inequalities.

Theorem 3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$, and the functions $g:[a,b]\to[0,\infty)$ and $h:[c,d]\to[0,\infty)$ are integrable. If $|f_{ts}(t,s)|$ is a convex function on the co-ordinates on Δ, then the following inequality holds.*

$$
\begin{aligned}
|S_\lambda(f;g,h)| \le \frac{\|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}}{36(b-a)(d-c)} &\times\{|f_{ts}(a,c)|A_\lambda(x)C_\lambda(y)+|f_{ts}(a,d)|A_\lambda(x)D_\lambda(y) \\
&+|f_{ts}(b,c)|B_\lambda(x)C_\lambda(y)+|f_{ts}(b,d)|B_\lambda(x)D_\lambda(y)\}
\end{aligned}
\tag{4}
$$

where $\lambda\in[0,1]$, $\|g\|_{[a,b],\infty}=\sup_{u\in[a,b]}|g(u)|$ and $\|h\|_{[c,d],\infty}=\sup_{u\in[c,d]}|h(u)|$.

Proof. We take absolute of (3). Using bounded of the mappings g and h, we find that

$$
\begin{aligned}
|S_\lambda(f;g,h)| &\le \int_a^b\int_c^d |P_\lambda(x,t)||Q_\lambda(y,s)||f_{ts}(t,s)|\,dsdt \\
&= \int_a^x\int_c^y \left[(1-\lambda)\left|\int_a^t g(u)\,du\right|+\lambda\left|\int_x^t g(u)\,du\right|\right]\times\left[(1-\lambda)\left|\int_c^s h(u)\,du\right|+\lambda\left|\int_y^s h(u)\,du\right|\right]|f_{ts}(t,s)|\,dsdt \\
&+\int_a^x\int_y^d \left[(1-\lambda)\left|\int_a^t g(u)\,du\right|+\lambda\left|\int_x^t g(u)\,du\right|\right]\times\left[(1-\lambda)\left|\int_d^s h(u)\,du\right|+\lambda\left|\int_y^s h(u)\,du\right|\right]|f_{ts}(t,s)|\,dsdt \quad (5)\\
&+\int_x^b\int_c^y \left[(1-\lambda)\left|\int_b^t g(u)\,du\right|+\lambda\left|\int_x^t g(u)\,du\right|\right]\times\left[(1-\lambda)\left|\int_c^s h(u)\,du\right|+\lambda\left|\int_y^s h(u)\,du\right|\right]|f_{ts}(t,s)|\,dsdt \\
&+\int_x^b\int_y^d \left[(1-\lambda)\left|\int_b^t g(u)\,du\right|+\lambda\left|\int_x^t g(u)\,du\right|\right]\times\left[(1-\lambda)\left|\int_d^s h(u)\,du\right|+\lambda\left|\int_y^s h(u)\,du\right|\right]|f_{ts}(t,s)|\,dsdt \\
&= I_1+I_2+I_3+I_4. \quad (6)
\end{aligned}
$$

Because $|f_{ts}(t,s)|$ is a convex function on the co-ordinates on Δ, we have

$$
\begin{aligned}
\left|f_{ts}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b,\frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right| &\le \frac{(b-t)(d-s)}{(b-a)(d-c)}|f_{ts}(a,c)|+\frac{(b-t)(s-c)}{(b-a)(d-c)}|f_{ts}(a,d)| \\
&+\frac{(t-a)(d-s)}{(b-a)(d-c)}|f_{ts}(b,c)|+\frac{(t-a)(s-c)}{(b-a)(d-c)}|f_{ts}(b,d)|.
\end{aligned}
\tag{7}
$$

From (7), it follows that

$$I_1 \leq \|g\|_{[a,x],\infty}\|h\|_{[c,y],\infty}\int_a^x\int_c^y [(1-\lambda)(t-a)+\lambda(x-t)][(1-\lambda)(s-c)+\lambda(y-s)]$$
$$\times\left(\frac{(b-t)(d-s)}{(b-a)(d-c)}|f_{ts}(a,c)|+\frac{(b-t)(s-c)}{(b-a)(d-c)}|f_{ts}(a,d)|+\frac{(t-a)(d-s)}{(b-a)(d-c)}|f_{ts}(b,c)|+\frac{(t-a)(s-c)}{(b-a)(d-c)}|f_{ts}(b,d)|\right)dsdt$$
$$=\|g\|_{[a,x],\infty}\|h\|_{[c,y],\infty}\left\{\frac{|f_{ts}(a,c)|}{36(b-a)(d-c)}\times(x-a)^2[(1-\lambda)(3b-a-2x)+\lambda(3b-2a-x)]\right.$$
$$\times(y-c)^2[(1-\lambda)(3d-c-2y)+\lambda(3d-2c-y)]$$
$$+\frac{|f_{ts}(a,d)|(2-\lambda)(y-c)^3}{36(b-a)(d-c)}(x-a)^2[(1-\lambda)(3b-a-2x)+\lambda(3b-2a-x)]$$
$$\left.+\frac{|f_{ts}(b,c)|(2-\lambda)(x-a)^3}{36(b-a)(d-c)}(y-c)^2[(1-\lambda)(3d-c-2y)+\lambda(3d-2c-y)]+\frac{|f_{ts}(b,d)|(2-\lambda)^2(x-a)^3(y-c)^3}{36(b-a)(d-c)}\right\}.$$

If we calculate the other integrals in a similar way, then we obtain

$$I_2 \leq \|g\|_{[a,x],\infty}\|h\|_{[y,d],\infty}\left\{\frac{|f_{ts}(a,c)|(2-\lambda)(d-y)^3}{36(b-a)(d-c)}\times(x-a)^2[(1-\lambda)(3b-a-2x)+\lambda(3b-2a-x)]\right.$$
$$+\frac{|f_{ts}(a,d)|}{36(b-a)(d-c)}\times(x-a)^2[(1-\lambda)(3b-a-2x)+\lambda(3b-2a-x)]\times(d-y)^2[(1-\lambda)(d-3c+2y)$$
$$+\lambda(2d-3c+y)]+\frac{|f_{ts}(b,c)|(2-\lambda)^2(x-a)^3(d-y)^3}{36(b-a)(d-c)}+\frac{|f_{ts}(b,d)|(2-\lambda)(x-a)^3}{36(b-a)(d-c)}$$
$$\left.\times(d-y)^2[(1-\lambda)(d-3c+2y)+\lambda(2d-3c+y)]\right\},$$

$$I_3 \leq \|g\|_{[x,b],\infty}\|h\|_{[c,y],\infty}\left\{\frac{|f_{ts}(a,c)|(2-\lambda)(b-x)^3}{36(b-a)(d-c)}\times(y-c)^2[(1-\lambda)(3d-c-2y)+\lambda(3d-2c-y)]\right.$$
$$+\frac{|f_{ts}(a,d)|(2-\lambda)^2(b-x)^3(y-c)^3}{36(b-a)(d-c)}+\frac{|f_{ts}(b,c)|}{36(b-a)(d-c)}$$
$$\times(b-x)^2[(1-\lambda)(b-3a+2x)+\lambda(2b-3a+x)]\times(y-c)^2[(1-\lambda)(3d-c-2y)+\lambda(3d-2c-y)]$$
$$\left.+\frac{|f_{ts}(b,d)|(2-\lambda)(y-c)^3}{36(b-a)(d-c)}\times(b-x)^2[(1-\lambda)(b-3a+2x)+\lambda(2b-3a+x)]\right\},$$

$$I_4 \leq \|g\|_{[x,b],\infty}\|h\|_{[y,d],\infty}\left\{\frac{|f_{ts}(a,c)|(2-\lambda)^2(b-x)^3(d-y)^3}{36(b-a)(d-c)}+\frac{|f_{ts}(a,d)|(2-\lambda)(b-x)^3}{36(b-a)(d-c)}\right.$$
$$\times(d-y)^2[(1-\lambda)(d-3c+2y)+\lambda(2d-3c+y)]$$
$$+\frac{|f_{ts}(b,c)|(2-\lambda)(d-y)^3}{36(b-a)(d-c)}\times(b-x)^2[(1-\lambda)(b-3a+2x)+\lambda(2b-3a+x)]$$
$$+\frac{|f_{ts}(b,d)|}{36(b-a)(d-c)}(b-x)^2[(1-\lambda)(b-3a+2x)+\lambda(2b-3a+x)]$$
$$\left.\times(y-c)^2[(1-\lambda)(d-3c+2y)+\lambda(2d-3c+y)]\right\}.$$

Substituting the integrals I_1, I_2, I_3, I_4 in (5) and triangle inequality for the moduls, because of $\|g\|_{[a,x],\infty}, \|g\|_{[x,b],\infty} \leq \|g\|_{[a,b],\infty}$ and $\|h\|_{[c,y],\infty}, \|h\|_{[y,d],\infty} \leq \|h\|_{[c,d],\infty}$, we easily deduce required inequality (4) which completes the proof.

Corollary 1. *Under the same assumptions of Theorem 3 with $\lambda = 1$; then the following inequality holds.*

$$\begin{aligned}|S_1(f;g,h)| \leq & \frac{\|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}}{36(b-a)(d-c)}\{|f_{ts}(a,c)|A_1(x)C_1(y)+|f_{ts}(a,d)|A_1(x)D_1(y) \\ & +|f_{ts}(b,c)|B_1(x)C_1(y)+|f_{ts}(b,d)|B_1(x)D_1(y)\}.\end{aligned}$$

Corollary 2. *Under the same assumptions of Theorem 3 with $\lambda = 0$, then the following identity holds.*

$$\begin{aligned}|S_0(f;g,h)| \leq & \frac{\|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}}{36(b-a)(d-c)}\{|f_{ts}(a,c)|A_0(x)C_0(y)+|f_{ts}(a,d)|A_0(x)D_0(y) \\ & +|f_{ts}(b,c)|B_0(x)C_0(y)+|f_{ts}(b,d)|B_0(x)D_0(y)\}.\end{aligned}$$

Corollary 3. *If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Theorem 3, then we get*

$$\left|S_\lambda\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right);g,h\right)\right| \leq \frac{\|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}(b-a)^2(d-c)^2}{16}\times\left\{\frac{|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|}{4}\right\}.$$

Corollary 4. *If we choose $g(u) = h(u) = 1$ and $\lambda = 1$ in Theorem 3, then the following inequality holds.*

$$\begin{aligned}&\Bigg|(b-a)(d-c)f(b,d)-(x-a)\int_c^d f(a,s)\,ds-(b-x)\int_c^d f(b,s)\,ds-(y-c)\int_a^b f(t,c)\,dt \\ &-(d-y)\int_a^b f(t,d)\,dt+\int_a^b\int_c^d f(t,s)\,dsdt\Bigg| \leq \frac{1}{36(b-a)(d-c)}\{|f_{ts}(a,c)|A_1(x)C_1(y)+|f_{ts}(a,d)|A_1(x)D_1(y) \\ &+|f_{ts}(b,c)|B_1(x)C_1(y)+|f_{ts}(b,d)|B_1(x)D_1(y)\}.\end{aligned} \tag{8}$$

Remark. If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (8), we get

$$\begin{aligned}&\Bigg|f(b,d)-\frac{1}{2(d-c)}\left[\int_c^d f(a,s)\,ds+\int_c^d f(b,s)\,ds\right]-\frac{1}{2(b-a)}\left[\int_a^b f(t,c)\,dt+\int_a^b f(t,d)\,dt\right] \\ &+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\Bigg| \leq \frac{(b-a)(d-c)}{16}\left\{\frac{|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|}{4}\right\}.\end{aligned} \tag{9}$$

Remark. If we take $f(a,c) = f(a,d) = f(b,c) = f(b,d)$ in (9), then we have

$$\begin{aligned}&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{2(d-c)}\left[\int_c^d f(a,s)\,ds+\int_c^d f(b,s)\,ds\right] \\ &-\frac{1}{2(b-a)}\left[\int_a^b f(t,c)\,dt+\int_a^b f(t,d)\,dt\right]+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\Bigg| \\ &\leq \frac{(b-a)(d-c)}{16}\left\{\frac{|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|}{4}\right\}.\end{aligned}$$

which is proved by Sarikaya et al. in [16].

In [8], Latif et. al proved an Ostrowski type inequality by accepting bounded and co-ordinated convex of $|f_{ts}(t,s)|$, and we give a new Ostrowski type inequality whose left hand side is same as the left hand side of the inequality of Latif et. al by accepting condition of the Theorem 3 in the following corollary.

Corollary 5. *If we choose $g(u) = h(u) = 1$ and $\lambda = 0$ in Theorem 3, then the following inequality holds.*

$$\left| f(x,y) - \frac{1}{(d-c)}\int_c^d f(x,s)\,ds - \frac{1}{(b-a)}\int_a^b f(t,y)\,dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt \right| \le \frac{1}{36(b-a)^2(d-c)^2} \tag{10}$$
$$\{|f_{ts}(a,c)|A_0(x)C_0(y) + |f_{ts}(a,d)|A_0(x)D_0(y) + |f_{ts}(b,c)|B_0(x)C_0(y) + |f_{ts}(b,d)|B_0(x)D_0(y)\}.$$

Remark. If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (10), we obtain

$$\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},s\right) ds - \frac{1}{(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt \right|$$
$$\le \frac{(b-a)(d-c)}{16}\left\{\frac{|f_{ts}(a,c)| + |f_{ts}(a,d)| + |f_{ts}(b,c)| + |f_{ts}(b,d)|}{4}\right\}.$$

which is proved by Latif and Dragomir in [11].

Corollary 6. *If we choose $g(u) = h(u) = 1$ and $\lambda = \frac{1}{2}$ in Theorem 3, then the following inequality holds.*

$$\left|(b-a)(d-c)\frac{f(x,y)+f(b,y)+f(x,d)+f(b,d)}{4} - \frac{(x-a)}{2}\int_c^d [f(x,s)+f(a,s)]\,ds - \frac{(b-x)}{2}\int_c^d [f(x,s)+f(b,s)]\,ds\right.$$
$$\left. -\frac{(y-c)}{2}\int_a^b [f(t,y)+f(t,c)]\,dt - \frac{(d-y)}{2}\int_c^d [f(t,y)+f(t,d)]\,dt + \int_a^b\int_c^d f(t,s)\,dsdt\right| \le \frac{1}{36(b-a)(d-c)} \tag{11}$$
$$\left\{|f_{ts}(a,c)|A_{\frac{1}{2}}(x)C_{\frac{1}{2}}(y) + |f_{ts}(a,d)|A_{\frac{1}{2}}(x)D_{\frac{1}{2}}(y) + |f_{ts}(b,c)|B_{\frac{1}{2}}(x)C_{\frac{1}{2}}(y) + |f_{ts}(b,d)|B_{\frac{1}{2}}(x)D_{\frac{1}{2}}(y)\right\}.$$

Remark. If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (11), we get

$$\left|\frac{f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + f\left(b,\frac{c+d}{2}\right) + f\left(\frac{a+b}{2},d\right) + f(b,d)}{4} - \frac{1}{4(d-c)}\left[\int_c^d \left[f\left(\frac{a+b}{2},s\right) + f(a,s)\right] ds\right.\right.$$
$$\left. + \int_c^d \left[f\left(\frac{a+b}{2},s\right) + f(b,s)\right] ds\right] - \frac{1}{4(b-a)}\left[\int_a^b \left[f\left(t,\frac{c+d}{2}\right) + f(t,c)\right] dt + \int_c^d \left[f\left(t,\frac{c+d}{2}\right) + f(t,d)\right] ds\right]$$
$$\left. + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \le \frac{(b-a)(d-c)}{16}\left\{\frac{|f_{ts}(a,c)| + |f_{ts}(a,d)| + |f_{ts}(b,c)| + |f_{ts}(b,d)|}{4}\right\}.$$

Remark. If we take $x = a$ and $y = c$ in (11), we have

$$\left|\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} - \frac{1}{2(d-c)}\int_c^d [f(a,s)+f(b,s)]\,ds - \frac{1}{2(b-a)}\int_c^d [f(t,c)+f(t,d)]\,dt\right.$$
$$\left. + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \le \frac{(b-a)(d-c)}{4}\left\{\frac{|f_{ts}(a,c)| + |f_{ts}(a,d)| + |f_{ts}(b,c)| + |f_{ts}(b,d)|}{4}\right\}.$$

Theorem 4. *Let $f:\Delta\subset\mathbb{R}^2\rightarrow\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ and the functions $g:[a,b]\rightarrow[0,\infty)$ and $h:[c,d]\rightarrow[0,\infty)$ are integrable on Δ. If $|f_{ts}(t,s)|^q$, $q>1$, is a convex function on the co-ordinates on Δ, then the following inequalities hold: for $\lambda\in[0,1]\setminus\left\{\frac{1}{2}\right\}$*

$$|S_\lambda(f;g,h)|\le\|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}\left(\frac{(1-\lambda)^{p+1}-\lambda^{p+1}}{(p+1)(1-2\lambda)}\right)^{\frac{2}{p}}(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}} \tag{12}$$
$$\times\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left[(y-c)^{p+1}+(d-y)^{p+1}\right]^{\frac{1}{p}}$$
$$\times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}$$

and for $\lambda=\frac{1}{2}$

$$\left|S_{\frac{1}{2}}(f;g,h)\right|\le\frac{\|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}}{4}(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}\times\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left[(y-c)^{p+1}+(d-y)^{p+1}\right]^{\frac{1}{p}}$$
$$\times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\|g\|_{[a,b],\infty}=\sup\limits_{u\in[a,b]}|g(u)|$, *and* $\|h\|_{[c,d]\infty}=\sup\limits_{u\in[c,d]}|h(u)|$.

Proof. We take absolute value of (3). Using Hölder's inequality for double integrals, we find that

$$|S_\lambda(f;g,h)|\le\int_a^b\int_c^d|P_\lambda(x,t)||Q_\lambda(y,s)||f_{ts}(t,s)|\,dsdt \tag{13}$$
$$\le\left(\int_a^b\int_c^d|P_\lambda(x,t)|^p|Q_\lambda(y,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_c^d|f_{ts}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}$$

Using bounded of the mappings g and h, we calculate respectively above integrals that is in multiplication:

$$\int_a^b\int_c^d|P_\lambda(x,t)|^p|Q_\lambda(y,s)|^p\,dsdt\le\|g\|^p_{[a,x],\infty}\|h\|^p_{[c,y],\infty}\int_a^x\int_c^y[(1-\lambda)(t-a)+\lambda(x-t)]^p[(1-\lambda)(s-c)+\lambda(y-s)]^p\,dsdt$$
$$+\|g\|^p_{[a,x],\infty}\|h\|^p_{[y,d],\infty}\int_a^x\int_y^d[(1-\lambda)(t-a)+\lambda(x-t)]^p[(1-\lambda)(d-s)+\lambda(s-y)]^p\,dsdt$$
$$+\|g\|^p_{[x,b],\infty}\|h\|^p_{[c,y],\infty}\int_x^b\int_c^y[(1-\lambda)(b-t)+\lambda(t-x)]^p[(1-\lambda)(s-c)+\lambda(y-s)]^p\,dsdt$$
$$+\|g\|^p_{[x,b],\infty}\|h\|^p_{[y,d],\infty}\int_x^b\int_y^d[(1-\lambda)(b-t)+\lambda(t-x)]^p[(1-\lambda)(d-s)+\lambda(s-y)]^p\,dsdt.$$

Using the change of the variable

$$(1-\lambda)(t-a)+\lambda(x-t)=u\quad dt=\tfrac{du}{1-2\lambda}$$

$$(1-\lambda)(s-c)+\lambda(y-s)=v\quad dt=\tfrac{dv}{2\lambda-1},$$

we obtain

$$\begin{aligned}&\int_a^x\int_c^y[(1-\lambda)(t-a)+\lambda(x-t)]^p[(1-\lambda)(s-c)+\lambda(y-s)]^p\,dsdt\\&=\left(\int_a^x[(1-\lambda)(t-a)+\lambda(x-t)]^p\,dt\right)\left(\int_c^y[(1-\lambda)(s-c)+\lambda(y-s)]^p\,ds\right)\\&=\left(\frac{(1-\lambda)^{p+1}-\lambda^{p+1}}{(p+1)(1-2\lambda)}\right)^2(x-a)^{p+1}(y-c)^{p+1}.\end{aligned}$$

If the other integrals are calculated in a similar way, then we get

$$\begin{aligned}\int_a^b\int_c^d|P_\lambda(x,t)|^p|Q_\lambda(y,s)|^p\,dsdt\le&\left(\frac{(1-\lambda)^{p+1}-\lambda^{p+1}}{(p+1)(1-2\lambda)}\right)^2\|g\|^p_{[a,b],\infty}\|h\|^p_{[c,d],\infty}\\&\times\left[(x-a)^{p+1}+(b-x)^{p+1}\right]\left[(y-c)^{p+1}+(d-y)^{p+1}\right].\end{aligned}\tag{14}$$

Since $|f_{ts}(t,s)|^q$ is convex function on the co-ordinates on Δ, we have

$$\begin{aligned}\left|f_{ts}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b,\frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right|^q\le&\frac{(b-t)(d-s)}{(b-a)(d-c)}|f_{ts}(a,c)|^q+\frac{(b-t)(s-c)}{(b-a)(d-c)}|f_{ts}(a,d)|^q\\&+\frac{(t-a)(d-s)}{(b-a)(d-c)}|f_{ts}(b,c)|^q+\frac{(t-a)(s-c)}{(b-a)(d-c)}|f_{ts}(b,d)|^q.\end{aligned}\tag{15}$$

From (15), we get

$$\int_a^b\int_c^d|f_{ts}(t,s)|^q\,dsdt\le(b-a)(d-c)\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}.\tag{16}$$

Substituting (14) and (16) in (13), we obtain the inequality (12).

For $\lambda=\frac{1}{2}$, then from Lemma 1 and using (16), we get

$$\begin{aligned}&\left|S_{\frac12}(f;g,h)\right|\le\int_a^b\int_c^d\left|P_{\frac12}(x,t)\right|\left|Q_{\frac12}(y,s)\right||f_{ts}(t,s)|\,dsdt\le\left(\int_a^b\int_c^d\left|P_{\frac12}(x,t)\right|^p\left|Q_{\frac12}(y,s)\right|^p dsdt\right)^{\frac1p}\left(\int_a^b\int_c^d|f_{ts}(t,s)|^q\,dsdt\right)^{\frac1q}\\&\le\left\{\frac{\|g\|^p_{[a,x],\infty}\|h\|^p_{[c,y],\infty}}{4^p}(x-a)^{p+1}(y-c)^{p+1}+\frac{\|g\|^p_{[a,x],\infty}\|h\|^p_{[y,d],\infty}}{4^p}(x-a)^{p+1}(d-y)^{p+1}\right.\\&\left.+\frac{\|g\|^p_{[x,b],\infty}\|h\|^p_{[c,y],\infty}}{4^p}(b-x)^{p+1}(y-c)^{p+1}+\frac{\|g\|^p_{[x,b],\infty}\|h\|^p_{[y,d],\infty}}{4^p}(b-x)^{p+1}(d-y)^{p+1}\right\}^{\frac1p}\\&\times(b-a)^{\frac1q}(d-c)^{\frac1q}\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac1q}.\end{aligned}$$

Hence, the proof is completed.

Corollary 7. *Under the same assumptions of Theorem 4 with $\lambda = 1$; then the following inequality holds.*

$$\left|S_1(f;g,h)\right| \leq \|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{(p+1)^{\frac{2}{p}}} \times \left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left[(y-c)^{p+1}+(d-y)^{p+1}\right]^{\frac{1}{p}} \times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}.$$

Corollary 8. *Under the same assumptions of Theorem 4 with $\lambda = 0$; then the following identity holds.*

$$\left|S_0(f;g,h)\right| \leq \|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{(p+1)^{\frac{2}{p}}} \times \left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left[(y-c)^{p+1}+(d-y)^{p+1}\right]^{\frac{1}{p}} \times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}.$$

Corollary 9. *If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Theorem 4, then we get*

$$\left|S_\lambda\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right);g,h\right)\right| \leq \|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}\left(\frac{(1-\lambda)^{p+1}-\lambda^{p+1}}{(p+1)(1-2\lambda)}\right)^{\frac{2}{p}}\frac{(b-a)^2(d-c)^2}{4} \times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}.$$

Corollary 10. *If we choose $g(u) = h(u) = 1$ and $\lambda = 1$ in Theorem 4, then the following inequality holds.*

$$\left|(b-a)(d-c)f(b,d)-(x-a)\int_c^d f(a,s)\,ds-(b-x)\int_c^d f(b,s)\,ds -(y-c)\int_a^b f(t,c)\,dt-(d-y)\int_a^b f(t,d)\,dt+\int_a^b\int_c^d f(t,s)\,dsdt\right| \leq \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{(p+1)^{\frac{2}{p}}}\times\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left[(y-c)^{p+1}+(d-y)^{p+1}\right]^{\frac{1}{p}} \times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}. \quad (17)$$

Remark. If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (17), we get

$$\left|f(b,d)-\frac{1}{2(d-c)}\left[\int_c^d f(a,s)\,ds+\int_c^d f(b,s)\,ds\right]-\frac{1}{2(b-a)}\left[\int_a^b f(t,c)\,dt+\int_a^b f(t,d)\,dt\right] +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}. \quad (18)$$

Remark. If we take $f(a,c)=f(a,d)=f(b,c)=f(b,d)$ in (18), then we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{2(d-c)}\left[\int_c^d f(a,s)\,ds+\int_c^d f(b,s)\,ds\right]\right.$$
$$\left.-\frac{1}{2(b-a)}\left[\int_a^b f(t,c)\,dt+\int_a^b f(t,d)\,dt\right]+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|$$
$$\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}$$

which is proved by Sarikaya et al. in [16].

In [8], Latif et. al proved an Ostrowski type inequality by accepting bounded of $|f_{ts}(t,s)|$ and co-ordinated convex of $|f_{ts}(t,s)|^q$, and we give a new Ostrowski type inequality whose left hand side is same as the left hand side of the inequality of Latif et. al by accepting condition of the Theorem 4 in the following corollary.

Corollary 11. *If we choose $g(u)=h(u)=1$ and $\lambda=0$ in Theorem 4; then the following inequality holds.*

$$\left|f(x,y)-\frac{1}{(d-c)}\int_c^d f(x,s)\,ds-\frac{1}{(b-a)}\int_a^b f(t,y)\,dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|$$
$$\leq\frac{1}{(b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}(p+1)^{\frac{2}{p}}}\times\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left[(y-c)^{p+1}+(d-y)^{p+1}\right]^{\frac{1}{p}} \tag{19}$$
$$\times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}.$$

Remark. If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (19), we obtain

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|$$
$$\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}.$$

which is proved by Latif and Dragomir in [11].

Corollary 12. *If we choose $g(u)=h(u)=1$ and $\lambda=\frac{1}{2}$ in Theorem 3; then the following inequality holds.*

$$\left|(b-a)(d-c)\frac{f(x,y)+f(b,y)+f(x,d)+f(b,d)}{4}-\frac{(x-a)}{2}\int_c^d[f(x,s)+f(a,s)]\,ds-\frac{(b-x)}{2}\right.$$
$$\left.\int_c^d[f(x,s)+f(b,s)]\,ds-\frac{(y-c)}{2}\int_a^b[f(t,y)+f(t,c)]\,dt-\frac{(d-y)}{2}\int_c^d[f(t,y)+f(t,d)]\,dt+\int_a^b\int_c^d f(t,s)\,dsdt\right|$$
$$\leq\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left[(y-c)^{p+1}+(d-y)^{p+1}\right]^{\frac{1}{p}} \tag{20}$$
$$\times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}.$$

Remark. If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (20), we get

$$\left| \frac{f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, d\right) + f(b,d)}{4} - \frac{1}{4(d-c)} \left[\int_c^d \left[f\left(\frac{a+b}{2}, s\right) + f(a,s) \right] ds \right.\right.$$
$$\left. + \int_c^d \left[f\left(\frac{a+b}{2}, s\right) + f(b,s) \right] ds \right] - \frac{1}{4(b-a)} \left[\int_a^b \left[f\left(t, \frac{c+d}{2}\right) + f(t,c) \right] dt + \int_c^d \left[f\left(t, \frac{c+d}{2}\right) + f(t,d) \right] ds \right]$$
$$\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds dt \right| \le \frac{(b-a)(d-c)}{16} \left\{ \frac{|f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q + |f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q}{4} \right\}^{\frac{1}{q}}.$$

Remark. If we take $x = a$ and $y = c$ in (20), we have

$$\left| \frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{4} - \frac{1}{2(d-c)} \int_c^d [f(a,s) + f(b,s)]\, ds - \frac{1}{2(b-a)} \int_c^d [f(t,c) + f(t,d)]\, dt \right.$$
$$\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds dt \right| \le \frac{(b-a)(d-c)}{4} \left\{ \frac{|f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q + |f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q}{4} \right\}^{\frac{1}{q}}.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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