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Some generalized inequalities of Hermite-Hadamard type for strongly s-convex functions

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Abstract: In this paper, some new generalized results related to the left-hand and the right-hand of the Hermite-Hadamard inequalities for the class of functions whose derivatives are strongly s-convex functions in the second sense are established. Some previous results are also recaptured as a special case.

Keywords: Hadamard type inequalities, Hölder inequality, strongly s-convex functions.

1 Introduction

In this section, we firstly list several definitions and some known results.

Definition 1. *A function $f : I \to \mathbb{R}$, $\varnothing \neq I \subset \mathbb{R}$, is said to be convex on I if the inequality*

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

holds for all $x.y \in I$ and $t \in [0,1]$.

Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamards inequality, due to its rich geometrical significance and applications([4], [12, p.137]). These inequalities state that if $f : I \subseteq R \to R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamards inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensens inequality. Hadamards inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [2],[5],[11],[16],[18]) and the references cited therein.

In [6], Hudzik and Maligranda considered, among others, the class of functions which are s- convex in the second sense. This class is defined in the following way: a function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

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holds for $x,y\in[0,\infty)$, $t\in[0,1]$ and for some fixed $s\in[0,1]$. This class of s-convex functions in the second sense is usually by K_S^2.

It can be easily see that for $s=1$ s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Definition 2. [13] *A function $f:I\to\mathbb{R}$ is called strongly s-convex with modulus c if*

$$f(tx+(1-t)y)\le t^s f(x)+(1-t)^s f(y)-ct(1-t)(b-a)^2.$$

In [1], Angulo et al. proved the following Hermite-Hadamard type inequality for strongly $h-$convex function.

Theorem 1. *Let $h:(0,1)\to(0,\infty)$ be a given function. If a function $f:I\subset\mathbb{R}\to\mathbb{R}$ is Lebesgue integrable and strongly $h-$convex with modulus $c>0$, then*

$$\frac{1}{h\left(\frac{1}{2}\right)}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\right]\le\frac{1}{b-a}\int_a^b f(x)\,dx\le(f(a)+f(b))\int_0^1 h(t)dt-\frac{c}{6}(b-a)^2 \tag{2}$$

for all $a,b\in I$, $a<b$.

Corollary 1. *Suppose that $f:[0,\infty)\to\mathbb{R}$ is a strongly $s-$convex function in the second sense with modulus $c>0$, where $s\in(0,1)$ (i.e $h(t)=t^s$ in (2)), then following inequalities hold;*

$$2^{s-1}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\right]\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{s+1}-\frac{c}{6}(b-a)^2. \tag{3}$$

For more information and recent developments on inequalities for strongly convex function, please refer to ([1],[3],[8],[9],[10],[15],[17],[19],[20]).

To prove our main results, we consider the following Lemmas given by Sarikaya et al. in [14] and Kiriş and Sarikaya in [7], respectively:

Lemma 1. *Let $f:I\to\mathbb{R}$ be a differantiable mapping on I^0 where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then we have*

$$\int_a^b p(x)f'(x)\,dx=(m-n)f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx \tag{4}$$

where

$$p(x)=\left\{\begin{array}{l} x-n,\ x\in\left[a,\frac{a+b}{2}\right] \\ \\ x-m,\ x\in\left(\frac{a+b}{2},b\right] \end{array}\right\}$$

for $n\in\left[a,\frac{a+b}{2}\right]$ and $m\in\left[\frac{a+b}{2},b\right]$.

Lemma 2. *Let $f:I\to\mathbb{R}$ be a differantiable mapping on I^0 where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then we have*

$$\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)=(b-a)wf \tag{5}$$

where $wf = \left[\int_0^{\frac{1}{2}} mt f'((ta)+(1-t)b)\,dt + \int_{\frac{1}{2}}^{1} n(1-t) f'((ta)+(1-t)b)\,dt\right]$, $n,m>0$

The aim of the paper is to establish some new generalized Hermite-Hadamard inequalities for function whose derivatives absolute values are strongly $s-$convex.

2 Main results

Firstly, we will give some calculated integrals which used our main results:

$$\int_a^{\frac{a+b}{2}} |x-n|(b-x)^s\,dx = \frac{2(b-n)^{s+2}}{(s+1)(s+2)} - \frac{(b-n)(b-a)^{s+1}\left[2^{s+1}+1\right]}{2^{s+1}(s+1)} + \frac{(b-a)^{s+2}\left[2^{s+2}+1\right]}{2^{s+2}(s+2)}, \tag{6}$$

$$\int_{\frac{a+b}{2}}^{b} |x-m|(b-x)^s = \frac{2(b-m)^{s+2}}{(s+1)(s+2)} + \frac{(b-a)^{s+2}}{2^{s+2}(s+2)} - \frac{(b-m)(b-a)^{s+1}}{2^{s+1}(s+1)}, \tag{7}$$

$$\int_a^{\frac{a+b}{2}} |x-n|(x-a)^s\,dx = \frac{2(n-a)^{s+2}}{(s+1)(s+2)} + \frac{(b-a)^{s+2}}{2^{s+2}(s+2)} - \frac{(n-a)(b-a)^{s+1}}{2^{s+1}(s+1)}, \tag{8}$$

$$\int_{\frac{a+b}{2}}^{b} |x-m|(x-a)^s\,dx = \frac{2(m-a)^{s+2}}{(s+1)(s+2)} - \frac{(m-a)(b-a)^{s+1}\left[2^{s+1}+1\right]}{2^{s+1}(s+1)} + \frac{(b-a)^{s+2}\left[2^{s+2}+1\right]}{2^{s+2}(s+2)}, \tag{9}$$

$$\int_a^{\frac{a+b}{2}} |x-n|(b-x)(x-a)\,dx = \frac{(b-a)(n-a)^3}{3} - \frac{(n-a)^4}{6} + \frac{5(b-a)^4}{192} - \frac{(n-a)(b-a)^3}{12} \tag{10}$$

and

$$\int_{\frac{a+b}{2}}^{b} |x-m|(b-x)(x-a)\,dx = \frac{(b-a)(b-m)^3}{3} - \frac{(b-m)^4}{6} + \frac{5(b-a)^4}{192} - \frac{(b-m)(b-a)^3}{12}. \tag{11}$$

Theorem 2. *Let $f: I \to \mathbb{R}$ be a differantiable mapping on I^0 where $a,b \in I$ with $a<b$. If $|f'|$ is strongly s-convex on $[a,b]$, for some $s \in (0,1]$ with modulus $c>0$, then following inequality holds :*

$$\begin{aligned}\left|(m-n)f\left(\frac{a+b}{2}\right) + (b-m)f(b) + (n-a)f(a) - \int_a^b f(x)\,dx\right| &\le \frac{|f'(a)|}{(b-a)^s}\left[\frac{2\left[(b-m)^{s+2}+(b-n)^{s+2}\right]}{(s+1)(s+2)}\right. \\ &\left. - \frac{(2b-n-m)(b-a)^{s+1}}{2^{s+1}(s+1)} - \frac{(b-n)(b-a)^{s+1}}{(s+1)} + \frac{(b-a)^{s+2}\left[2^{s+1}+1\right]}{2^{s+1}(s+2)}\right] + \frac{|f'(b)|}{(b-a)^s}\left[\frac{2\left[(b-m)^{s+2}+(b-n)^{s+2}\right]}{(s+1)(s+2)}\right. \\ &\left. - \frac{(m+n-2a)(b-a)^{s+1}}{2^{s+1}(s+1)} - \frac{(m-a)(b-a)^{s+1}}{(s+1)} + \frac{(b-a)^{s+2}\left[2^{s+1}+1\right]}{2^{s+1}(s+2)}\right] \\ &- c\left[(b-a)\frac{(n-a)^3+(b-m)^3}{3} - \frac{(n-a)^4+(b-m)^4}{6} + \frac{5(b-a)^4}{96} - \frac{((b-a)-(m-n))(b-a)^3}{12}\right]\end{aligned} \tag{12}$$

for $n \in \left[a, \frac{a+b}{2}\right]$ and $m \in \left[\frac{a+b}{2}, b\right]$.

Proof. Taking modulus in Lemma 1 and using the strongly $s-$convexity of $|f'|$, we have

$$\begin{aligned}
&\left|(m-n)f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx\right| \\
&\leq \int_a^{\frac{a+b}{2}}|x-n|\left|f'(x)\right|dx+\int_{\frac{a+b}{2}}^b |x-m|\left|f'(x)\right|dx \qquad (13)\\
&=\int_a^{\frac{a+b}{2}}|x-n|\,f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-b}b\right)dx+\int_{\frac{a+b}{2}}^b |x-m|\,f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-b}b\right)dx \\
&\leq \int_a^{\frac{a+b}{2}}|x-n|\left[\left(\frac{b-x}{b-a}\right)^s\left|f'(a)\right|+\left(\frac{x-a}{b-a}\right)^s\left|f'(b)\right|-c(b-x)(x-a)\right]dx \\
&+\int_{\frac{a+b}{2}}^b |x-m|\left[\left(\frac{b-x}{b-a}\right)^s\left|f'(a)\right|+\left(\frac{x-a}{b-a}\right)^s\left|f'(b)\right|-c(b-x)(x-a)\right]dx \\
&=\frac{|f'(a)|}{(b-a)^s}\left[\int_a^{\frac{a+b}{2}}|x-n|(b-x)^s\,dx+\int_{\frac{a+b}{2}}^b |x-m|(b-x)^s\,dx\right] \\
&+\frac{|f'(b)|}{(b-a)^s}\left[\int_a^{\frac{a+b}{2}}|x-n|(x-a)^s\,dx+\int_{\frac{a+b}{2}}^b |x-m|(x-a)^s\,dx\right] \\
&-c\left[\int_a^{\frac{a+b}{2}}|x-n|(b-x)(x-a)\,dx+\int_{\frac{a+b}{2}}^b |x-m|(b-x)(x-a)\,dx\right].
\end{aligned}$$

If we substitute the equalities (6)-(11) in (13), then we obtain required result (12).

Remark. If we choose $m=b$, $n=a$ in Theorem 2, then we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\frac{2^{s+1}-1}{2^s(s+1)(s+2)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]-\frac{5c}{96}(b-a)^3$$

Remark. If we choose $m=n=\frac{a+b}{2}$ in Theorem 2, then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\frac{s2^s+1}{2^s(s+1)(s+2)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]-\frac{c}{32}(b-a)$$

Remark. If we choose $m=\frac{a+5b}{6}$, $n=\frac{5a+b}{6}$ in Theorem 2, then we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\frac{2\times 5^{s+2}-(4-s)6^{s+1}-2\times 3^{s+12}+2}{6^{s+2}(s+1)(s+2)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$
$$-\frac{67}{2\times 6^4}c(b-a)^3.$$

For $c=0$ it reduces to the Hermite–Hadamard-type inequalities for s-convex functions proved by Sarikaya et al. in [21].

Theorem 3. *Let $f:I\to\mathbb{R}$ be a differantiable mapping on I^0 where $a,b\in I$ with $a<b$. If $|f'|^q$ is strongly $s-$convex on $[a,b]$ for some $s\in(0,1]$ with modulus $c>0$, then following inequality holds.*

$$\left|(m-n)f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}\left\{\left[(n-a)^{p+1}+\left(\frac{a+b}{2}-n\right)^{p+1}\right]^{\frac{1}{p}}\right.$$
$$\times\left(\frac{1}{2^{s+1}(s+1)}\left[\left[2^{s+1}-1\right]\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}+\left[(b-m)^{p+1}+\left(m-\frac{a+b}{2}\right)^{p+1}\right]^{\frac{1}{p}}\tag{14}$$
$$\left.\times\left(\frac{1}{2^{s+1}(s+1)}\left[\left|f'(a)\right|^q+\left[2^{s+1}-1\right]\left|f'(b)\right|^q\right]-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}\right\}$$

for $n\in\left[a,\frac{a+b}{2}\right]$ and $m\in\left[\frac{a+b}{2},b\right]$ where $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and by using the Hölder inequality , then we have

$$\left|(m-n)f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx\right|$$
$$=\int_a^{\frac{a+b}{2}}|x-n|\left|f'(x)\right|dx+\int_{\frac{a+b}{2}}^b|x-m|\left|f'(x)\right|dx\tag{15}$$
$$\leq\left(\int_a^{\frac{a+b}{2}}|x-n|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}\left|f'(x)\right|^q dx\right)^{\frac{1}{q}}+\left(\int_{\frac{a+b}{2}}^b|x-m|^p\,dx\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b\left|f'(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$=\frac{1}{(p+1)^{\frac{1}{p}}}\left[(n-a)^{p+1}+\left(\frac{a+b}{2}-n\right)^{p+1}\right]^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}\left|f'(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$+\frac{1}{(p+1)^{\frac{1}{p}}}\left[(b-m)^{p+1}+\left(m-\frac{a+b}{2}\right)^{p+1}\right]^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b\left|f'(x)\right|^q dx\right)^{\frac{1}{q}}.$$

Using the strongly s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|(m-n)f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx\right| \\
&\leq \frac{1}{(p+1)^{\frac{1}{p}}}\left\{\left[(n-a)^{p+1}+\left(\frac{a+b}{2}-n\right)^{p+1}\right]^{\frac{1}{p}}\right. \\
&\times\left(\int_a^{\frac{a+b}{2}}\left[\left(\frac{b-x}{b-a}\right)^s|f'(a)|^q+\left(\frac{x-a}{b-a}\right)^s|f'(b)|^q-c(b-x)(x-a)\right]dx\right)^{\frac{1}{q}} \\
&+\left[(b-m)^{p+1}+\left(m-\frac{a+b}{2}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\left.\times\left(\int_{\frac{a+b}{2}}^b\left[\left(\frac{b-x}{b-a}\right)^s|f'(a)|^q+\left(\frac{x-a}{b-a}\right)^s|f'(b)|^q-c(b-x)(x-a)\right]dx\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{16}$$

By simple computation, we have

$$\int_a^{\frac{a+b}{2}}(b-x)^s\,dx=\int_{\frac{a+b}{2}}^b(x-a)^s\,dx=\frac{(b-a)^{s+1}\left[2^{s+1}-1\right]}{2^{s+1}(s+1)} \tag{17}$$

$$\int_a^{\frac{a+b}{2}}(x-a)^s\,dx=\int_{\frac{a+b}{2}}^b(b-x)^s\,dx=\frac{(b-a)^{s+1}}{2^{s+1}(s+1)} \tag{18}$$

$$\int_a^{\frac{a+b}{2}}(b-x)(x-a)\,dx=\int_{\frac{a+b}{2}}^b(b-x)(x-a)\,dx=\frac{(b-a)^3}{12}. \tag{19}$$

If we substitute the equalities (17)-(19) in (16), then we obtain result (14).

Corollary 2. *If we choose $m=b$ and $n=a$ in Theorem 3, then we have*

$$\begin{aligned}
\left|f\left(\frac{a+b}{2}\right)-\int_a^b f(x)\,dx\right| \leq & \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{\left[2^{s+1}-1\right]|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}-c\frac{(b-a)^2}{6}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{|f'(a)|^q+\left[2^{s+1}-1\right]|f'(b)|^q}{2^s(s+1)}-c\frac{(b-a)^2}{6}\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{20}$$

Remark. Choosing $s=1$ in Corollary 2, we obtain the inequality

$$\left|f\left(\frac{a+b}{2}\right)-\int_a^b f(x)\,dx\right|\leq\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}\right\}.$$

Corollary 3. *If we choose $m=n=\frac{a+b}{2}$ in Theorem 3, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b f(x)\,dx\right|\leq\frac{b-a}{2\times 2^{\frac{1}{p}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{\left[2^{s+1}-1\right]\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2^{s+1}(s+1)}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+\left[2^{s+1}-1\right]\left|f'(b)\right|^q}{2^{s+1}(s+1)}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}\right\}. \tag{21}$$

Remark. Choosing $s=1$ in Corollary 3, we obtain the inequality

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b f(x)\,dx\right|\leq\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}\right\}.$$

Corollary 4. *If we choose $m=\frac{a+5b}{6}$ and $n=\frac{5a+b}{6}$ in Theorem 3, then we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(x)\,dx\right|\leq\frac{b-a}{6}\left(\frac{2^{p+1}+1}{6(p+1)}\right)^{\frac{1}{p}} \tag{22}$$
$$\times\left\{\left(\frac{\left[2^{s+1}-1\right]\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2^{s+1}(s+1)}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+\left[2^{s+1}-1\right]\left|f'(b)\right|^q}{2^{s+1}(s+1)}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}\right\}.$$

Remark. Choosing $s=1$ in Corollary 3, we obtain the inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(x)\,dx\right|\leq\frac{b-a}{12}\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}\right\}.$$

Theorem 4. *Let $f: I\to\mathbb{R}$ be a differantiable mapping on I^0 where $a,b\in I$ with $a<b$. If $|f'|$ is strongly $s-$convex on $[a,b]$ for some $s\in(0,1]$ with modulus $c>0$, then following inequality holds.*

$$\left|\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right|\le(b-a)\left\{\left[\frac{m}{2^{s+2}(s+2)}+n\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}\right]|f'(a)|\right.$$
$$\left.+\left[\frac{m}{2^{s+2}(s+2)}+n\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}\right]|f'(b)|-\frac{5c(m+n)}{192}(b-a)^2\right\} \tag{23}$$

where $n\in\left[a,\frac{a+b}{2}\right], m\in\left[\frac{a+b}{2},b\right]$.

Proof. From Lemma 2 and using strongly s-convex of $|f'|$, we have

$$\left|\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right|\le(b-a)\left\{\int_0^{\frac12} mt\,|f'((ta)+(1-t)b)|\,dt\right.$$
$$\left.+\int_{\frac12}^1 n(1-t)\,|f'((ta)+(1-t)b)|\,dt\right\}\le(b-a)\left\{\int_0^{\frac12} mt\left[t^s|f'(a)|+(1-t)^s|f'(b)|-ct(1-t)(b-a)^2\right]\right.$$
$$\left.+\int_{\frac12}^1 n(1-t)\left[t^s|f'(a)|+(1-t)^s|f'(b)|-ct(1-t)(b-a)^2\right]\right\} \tag{24}$$
$$=(b-a)\left\{m|f'(a)|\int_0^{\frac12}t^{s+1}dt+m|f'(b)|\int_0^{\frac12}t(1-t)^s dt-mc(b-a)^2\int_0^{\frac12}t^2(1-t)\,dt\right.$$
$$\left.+n|f'(a)|\int_{\frac12}^1(1-t)t^s dt+n|f'(b)|\int_{\frac12}^1(1-t)^{s+1}dt-nc(b-a)^2\int_{\frac12}^1 t(1-t)^2 dt\right\}.$$

Using the facts that

$$\int_0^{\frac12}t^{s+1}dt=\int_{\frac12}^1(1-t)^{s+1}dt=\frac{1}{2^{s+2}(s+2)},$$
$$\int_0^{\frac12}t(1-t)^s dt=\int_{\frac12}^1(1-t)t^s dt=\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)},$$

and

$$\int_0^{\frac12}t^2(1-t)\,dt=\int_{\frac12}^1 t(1-t)^2 dt=\frac{5}{192},$$

one can obtain required result.

Remark. If we choose $m=n$ in Theorem 4, then Theorem 4 reduces to Remark 2.

Corollary 5. *Under assumption of Theorem 4 with $s=1$, we have*

$$\left|\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{24}\left[(m+2n)|f'(a)|+(2m+n)|f'(b)|\right]$$
$$-\frac{5c(m+n)}{192}(b-a)^3. \tag{25}$$

Theorem 5. *Let $f : I \to \mathbb{R}$ be a differantiable mapping on I^0 where $a,b \in I$ with $a < b$. If $|f'|^q$ is strongly $s-$convex on $[a,b]$ for some $s \in (0,1]$ with modulus $c > 0$, then following inequality holds :*

$$\left| \frac{n}{b-a} \int_a^{\frac{a+b}{2}} f(x)\,dx + \frac{m}{b-a} \int_{\frac{a+b}{2}}^b f(x)\,dx - \frac{n+m}{2} f\left(\frac{a+b}{2}\right) \right|$$
$$\le \frac{b-a}{4} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ m \left(\frac{|f'(a)|^q + |f'(b)|^q \left[2^{s+1}-1\right]}{2^s(s+1)} - \frac{c(b-a)^2}{6} \right)^{\frac{1}{q}} \right. \tag{26}$$
$$\left. +n \left(\frac{|f'(a)|^q \left[2^{s+1}-1\right] + |f'(b)|^q}{2^s(s+1)} - \frac{c(b-a)^2}{6} \right)^{\frac{1}{q}} \right\}$$

where $n \in \left[a, \frac{a+b}{2}\right], m \in \left[\frac{a+b}{2}, b\right]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2 and using Hölder Inequality, then we have

$$\left| \frac{n}{b-a} \int_a^{\frac{a+b}{2}} f(x)\,dx + \frac{m}{b-a} \int_{\frac{a+b}{2}}^b f(x)\,dx - \frac{n+m}{2} f\left(\frac{a+b}{2}\right) \right|$$
$$\le (b-a) \left\{ \int_0^{\frac{1}{2}} mt \left| f'(ta+(1-t)b) \right| dt + \int_{\frac{1}{2}}^1 n(1-t) \left| f'(ta+(1-t)b) \right| dt \right\}$$
$$\le (b-a) \left\{ \left(\int_0^{\frac{1}{2}} (mt)^p\,dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| f'(ta+(1-t)b) \right|^q dt \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_{\frac{1}{2}}^1 n^p (1-t)^p\,dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left| f'(ta+(1-t)b) \right|^q dt \right)^{\frac{1}{q}} \right\}$$

Using strongly s-convexity of $|f'|^q$;

$$\left| \frac{n}{b-a} \int_a^l f(x)\,dx + \frac{m}{b-a} \int_l^b f(x)\,dx - \frac{n+m}{2} f(l) \right| \le (b-a) \left\{ \left(\int_0^{\frac{1}{2}} (mt)^p\,dt \right)^{\frac{1}{p}} \theta_m + \left(\int_{\frac{1}{2}}^1 \left[n(1-t)^p\,dt \right] \right)^{\frac{1}{p}} \theta_m \right\}$$

where $\theta_m = \left(\int_{\frac{1}{2}}^1 \left[t^s |f'(a)|^q + (1-t)^s |f'(b)|^q - ct(1-t)(b-a)^2 \right] dt \right)^{\frac{1}{q}}$ and $l = \frac{a+b}{2}$. By simple computation, the desired inequality (26) can be easily established.

Remark. If we choose $m = n$ in Theorem 5, then Theorem 5 reduces to Remark 2.

Corollary 6. *Under assumption of Theorem 4 with $s = 1$, we have*

$$\left| \frac{n}{b-a} \int_a^{\frac{a+b}{2}} f(x)\,dx + \frac{m}{b-a} \int_{\frac{a+b}{2}}^b f(x)\,dx - \frac{n+m}{2} f\left(\frac{a+b}{2}\right) \right| \le \frac{b-a}{4} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ m(\theta_q)^{\frac{1}{q}} + n(\theta_p)^{\frac{1}{q}} \right\}, \tag{27}$$

where $\theta_p = \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} - \frac{c(b-a)^2}{6} \right)$ *and* $\theta_q = \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} - \frac{c(b-a)^2}{6} \right)$

3 Conclusions

In this study, we presented some generalized Hermite type inequalities for the mappings whose derivatives are strongly s-convex functions in the second sense are established. A further study could be assess weighted versions of these inequalities.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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