

A fast algorithm of two-level banded Toeplitz systems of linear equations with application to image restoration

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Abstract: A fast algorithm is proposed for the solution of $nm \times nm$ block banded Toeplitz systems with banded Toeplitz blocks (BBTBTB). The algorithm consists of using the idea of embedding a BBTBTB matrix in a $m \times (n+k)$ block-triangular Toeplitz matrix. The computational complexity of the proposed algorithm is $O(m^2 n \log n + m^3 n + k^3 m^3)$. The validity of the approach is illustrated by an application to image restoration.

Keywords: Toeplitz matrices, Block-tridiagonal Toeplitz matrices, Block-triangular Toeplitz matrices.

1 Introduction

We study the linear system

$$Tx = f$$

where T is an $N \times N$ block banded Toeplitz matrix with banded Toeplitz blocks (BBTBTB) of the form

$$T = \begin{pmatrix} T_0 & T_{-1} & \dots & \dots & T_{-k} & 0 & \dots & 0 \\ T_1 & T_0 & \ddots & \ddots & \ddots & T_{-k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ T_k & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & T_{-k} \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & T_{-1} \\ 0 & \dots & 0 & T_k & \dots & \dots & T_1 & T_0 \end{pmatrix} \quad (1)$$

in which all T_j , $j = -k, \dots, 0, \dots, k$ are m -by- m matrices, $N = nm$, x is the unknown vector and f is the right hand side. This kind of matrix is encountered, in fields like signal and image processing [8,9]. Very often these matrices are accompanied by a Toeplitz structure of the blocks (See [6,7,12] and reference therein). Among the best known methods, the block-reduction cyclic reduction is the most popular algorithm for solving a BBTBTB system [4]. Another fast method based on the non-standard QR factorization of the matrix T is given in [15]. A very extensive study for the problem of solving linear system with banded Toeplitz structure can be founded in [10]. In the scalar case, this problem was studied recently via the technique of embedding the matrix T as a triangular Toeplitz matrix [2] and by using a fast

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inversion of a triangular Toeplitz matrix [1].

The main aim of this paper is to propose a method for solving BBTBTB systems based on the idea of embedding block banded Toeplitz matrix with banded Toeplitz blocks (1) in a larger block-triangular Toeplitz matrix [3]. From this point, we shall present some nice properties for a block triangular Toeplitz matrix. Let

$$L = \begin{pmatrix} L_0 & & & \\ L_1 & L_0 & & \\ \vdots & \ddots & \ddots & \\ L_{n-1} & & L_1 & L_0 \end{pmatrix} \quad (2)$$

in which all $L_j, j = 0, \dots, n$, are m -by- m matrices.

If $m = 1$ the inversion of L by forward substitution method [13] can be computed by $O(m^2n^2 + m^3)$ ops. By the use of FFT (Fast Fourier Transformation), a fast exact inversion method based on the divide-and-conquer method [11], requires $O(n \log n)$. If we use the technique of divide-and-conquer to solve block-triangular Toeplitz systems, we need about $O(m^2n \log n + m^3n)$ ops and $O(m^2n)$ of the storage. When m is very large, this technique is not better than the block forward substitution method which costs $O(mn^2)$ ops and $O(mn)$ storage requirement; for more details see [5].

Below a superfast algorithm which computes the inverse of a block lower triangular Toeplitz matrix.

Algorithm 1.1: Block lower triangular Toeplitz matrix inversion

Input: k integer and $m \times m$ block $L_0, \dots, L_{n-1}, n = 2^k$.

Output: The first block column v_n of L_n^{-1} .

1. Set $v_1 = L_0^{-1}$.

2. For $i = 0, \dots, k-1$

(a). $h = 2^i$.

(b). Compute the block Toeplitz matrix-vector products $w = S_h v_h$ and $u = -L(v_h)w$ where S_h are $h \times h$ Toeplitz matrices and $L(v_h)$ is the lower triangular Toeplitz matrix whose first column is v_h (See [5, p. 36]).

(c). Set

$$v_{2h} = \begin{bmatrix} v_h \\ u \end{bmatrix}.$$

So, the inversion of the block triangular Toeplitz system L_n^{-1} can be solved in $O(m^2n \log n + m^3n)$. In fact, in the Step 2 which appears in both the block Toeplitz matrix-vector products can be computed with the block-DFT (Block Discrete Fourier Transform) in $O(m^2n \log n + m^3n)$ (For more details, see [5]).

Remark.[5]

- (i) Let B_n and A_n be two block lower (upper) triangular Toeplitz matrices. Then $B_n A_n$ is also a block lower (upper) triangular Toeplitz matrix, and $B_n A_n = A_n B_n$.
- (ii) If A_n is an invertible lower (upper) triangular Toeplitz matrix, then T^{-1} is also a lower (upper) triangular Toeplitz matrix. Therefore, to obtain T^{-1} we only require to compute the entries of its first block-column (block-row).
- (iii) If the block Toeplitz matrix A_n is not triangular, its inverse is not block Toeplitz.

In this paper, we give a new algorithm for solving BBTBTB systems and motivate our approach by an application to image restoration. The computational complexity in the case one uses fast block-triangular Toeplitz techniques is equal to $O(m^2n \log n + m^3n)$ where the blocks are m -by- m Toeplitz matrices and n is the order of the BBTBTB matrix.

be the matrix inversion of M defined by its first block-column (V_0, V_1, \dots, V_n) , where V_i is a m -by- m matrix not necessary Toeplitz and $i = 0, \dots, n$.

Theorem 1. Let T be a nonsingular block banded Toeplitz matrix with banded Toeplitz blocks (BBTBTB) and M the associate block triangular embedding matrix. Suppose that M is partitioned as follows

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (6)$$

where A, B, C , and D are block matrices of size $nm \times km, nm \times nm, km \times km$ et $km \times nm$, respectively. Then, the inverse of T , defined by the Schur complement of the block C , is given explicitly by

$$T^{-1} = -AC^{-1}D + B. \quad (7)$$

Proof. We may inductively apply the same proof in [2].

Thus, the algorithm for solving the BBTBTB system is given as follows :

Algorithm 1: BTT matrix inversion

1. Choose f and compute the inverse of L .
2. Compute $e_1 = Df$.
3. Solve $Ce_2 = e_1$.
4. Compute $x_1 = Ae_2$.
5. Compute $x_2 = Bf$.
6. Compute $x = -x_1 + x_2$.

Proposition 2. The complexity to solve a TBBTB system (1) is about $O(m^2n \log n + m^3n + k^3m^3)$ ops.

Proof. The inverse of the triangular matrix M requires $O(m^2n \log n + m^3n)$. To recover e_2 , we need $O(k^3m^3)$ ops where k is the size of the matrix M . x_2 is provided by $O(m^2n \log n + m^3n)$ ops. Thus, the complexity of the system (1) is about $O(m^2n \log n + m^3n + k^3m^3)$ ops.

3 Numerical tests and application to image restoration

We have run our numerical tests in MATLAB on the Intel processor E8500 (3.16 GHz). The results given in Tables 1 and 2 show the computational time and the relative error between the computed \tilde{x} by Algorithm 1 and x the exact one (here, the right-hand vector of the system has been chosen so that the system has the solution $x = (1, \dots, 1)^T$).

Example 1. This example is chosen so that $n = 2^{12} - 1$, $T_{-1} = T_1 = \text{circulant}([20 \text{ } -8 \text{ } 1 \text{ } \text{zeros}(1, m-5) \text{ } -1 \text{ } 8], -1)$ (A circulant matrix), $T_0 = I_m$.

m	$\ x - \tilde{x}\ /\ x\ $	Times(s)
5	$6.37 \cdot 10^{-14}$	0.025
10	$4.26 \cdot 10^{-14}$	0.032
20	$1.82 \cdot 10^{-13}$	0.077
30	$1.45 \cdot 10^{-13}$	0.150

Example 1. This example is chosen so that $n = 2^{10} - 1$, $T_0 = \text{toeplitz}([0.5, \text{ones}(1, m - 1)])$ (A Toeplitz matrix), $T_1 = T_{-1} = \text{toeplitz}([\text{ones}(1, m/2), \text{zero}(1, m/2)])$ (A Toeplitz matrix).

m	$\ x - \tilde{x}\ /\ x\ $	Times(s)
10	$5.64 \cdot 10^{-12}$	0.126
20	$5.08 \cdot 10^{-12}$	0.309
30	$1.94 \cdot 10^{-11}$	0.580

3.1 Application to image restoration

The problem of image restoration has a specific structure which depends on the imposed boundary conditions. In this section, we consider the Dirichlet zero boundary conditions. Let $X = (x_{i,j})_{i=1,\dots,m,j=1,\dots,n}$ is the original image and B is the blurred image.

Definition 1. The Point-Speared-Function (PSF) array P is the image of a single white pixel, and its dimensions are usually much more than those of B and X , if the blurring is local and spatially invariant, then P contains all information about the blurring throughout image.

Definition 2.[16] The zero boundary condition implies a block boundary, so that the pixels outside the borders of the image X are all zeros. That is, we assume that the image X has been extracted from a larger image which looks like

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The blurred image B can be written by a linear system as $b = Ax$ where A is a block banded matrix with $m \times m$ banded Toeplitz blocks when the block Toeplitz is derived by proprieties of the Point-Speared-Function (PSF), $b = \text{vec}(B)$ and $x = \text{vec}(X)$. A is a block $(2M + 1)$ -diagonal matrix with $(2M + 1)$ -diagonal blocks [4, 14]. When choosing a PSF as 3×3 matrix, we get A a BTT matrix.



(a) Original image



(b) Blurred image



(c) Restored image

Fig. 1: Deblurring gray image: Cameraman of size 256×256 .



(a) Original image

(b) Blurred image

(c) Restored image

Fig. 2: Deblurring gray image: Phantom of size 256×256 .

Algorithm 1 has been implemented and tested for a problem of image restoration of $N \times N$ pixels images, with the PSF defined by the following mask

$$\text{PSF} = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

which generates a 3-block diagonal block Toeplitz matrix with blocks of size N ; the off diagonal blocks are tridiagonal Toeplitz matrices with all the nonzero entries equal to $1/9$, the diagonal blocks are tridiagonal Toeplitz matrices having $1/9$ on the main diagonal and $1/9$ on the remaining nonzero diagonals. The matrix obtained in this way is reblocked into a block tridiagonal form, thus obtaining (1). Here, we consider the cameraman in figure 1 with size 256×256 and the phantom displayed in figure 2 cropped down to size 256×256 .

4 Concluding remarks

In this paper a fast algorithm, in $O(m^2 n \log n + m^3 n + k^3 m^3)$ flops, was proposed for the solution of $nm \times nm$ block banded Toeplitz systems with banded Toeplitz blocks (BBTB). An application to image restoration was also introduced when the imposed boundary condition is zero. In the future, we tend to focus our attention to solve the problem of image restoration with other boundary conditions such that periodic, reflective (also called Neumann or symmetric) and anti-reflective.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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