

Some bounds on local connective Chromatic number

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Abstract: Graph coloring is one of the most important concept in graph theory. Many practical problems can be formulated as graph coloring problems. In this paper, we define a new coloring concept called local connective coloring. A local connective k -coloring of a graph G is a proper vertex coloring, which assigns colors from $\{1, 2, \dots, k\}$ to the vertices $V(G)$ in a such way that any two non-adjacent vertices u and v of a color i satisfies $\kappa(u, v) \geq i$, where $\kappa(u, v)$ is the maximum number of internally disjoint paths between u and v . Adjacent vertices are colored with different colors as in the proper coloring. The smallest integer k for which there exists a local connective k - coloring of G is called the *local connective chromatic number* of G , and it is denoted by $\chi_{lc}(G)$. We study this coloring on several classes of graphs and give some general bounds. We also compare local connective chromatic number of a graph with chromatic number and packing chromatic number of it.

Keywords: Graph coloring, packing chromatic number, internally disjoint path, local connective chromatic number.

1 Introduction

Graph coloring is a special case of graph labeling; it is an assignment of labels called "colors" to vertices or edges of a graph to certain constraints. Vertex coloring is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a proper coloring. The *chromatic number* $\chi(G)$ of G is the minimum number of colors needed in a proper coloring of a graph. There are several graph coloring applications such as scheduling, timetabling, frequency allocation, wavelength routing and many more [1, 11, 12, 13].

In this paper, we introduce and study a new graph coloring concept called local connective coloring. For this coloring we are inspired by the notion of packing chromatic number. In 2008, Goddard et al. [14] established the notion of *packing chromatic number* under the name *broadcast chromatic number*. The term packing chromatic number was introduced by Brešar et al. [4]. The concept of packing coloring comes from the area of frequency planning in wireless networks.

A *packing k -coloring* of a graph G is a mapping $c : V(G) \rightarrow \{1, 2, \dots, k\}$ such that any two vertices of color i are at a distance of at least $i + 1$. Thus, the vertices of G are partitioned into different color classes X_1, X_2, \dots, X_k , where every X_i is an i -packing of G . The i -packing number of G , denoted by $\rho_i(G)$, is the maximum cardinality of an i -packing that occurs in G . The *packing chromatic number* $\chi_p(G)$ of G is the smallest integer k for which G has packing k -coloring [3, 4, 10, 14].

A set of paths from a vertex u to a vertex v is said to be *internally vertex-disjoint* (abbreviated as *internally disjoint*) if no two paths share a common vertex except u and v . Internally disjoint paths contribute to transfer information quickly and safely between any two vertices in an interconnection network and provide alternative routes in the event of vertex or edge failures. Then the more internally disjoint paths between two vertices are the better for a network [5, 6]. Thus we use the term internally disjoint path in our coloring and color the vertices depending on the number of internally disjoint

paths between two vertices. A graph G which has a *local connective k -coloring* can be partitioned into disjoint color classes X_1, X_2, \dots, X_k and can be drawn as a k -partite graph. Thereby, the graph is partitioned into the subsets which have disjoint paths. Looking for a secure disjoint path between two vertices u and v in any color class X_i , we make this search with the vertices in the other color classes. This indicates that we look for disjoint paths starting from u and ending to v using the vertices in the other color classes. Thus, this search can be made with $V(G) - (|X_i| - 2)$ vertices. Local connective coloring provides to facilitate the routing of non-adjacent vertices to communicate with each other.

Throughout this paper, we consider undirected, finite and simple (no loops and multiple edges) graphs. For the notations and terminology not defined here, see [7]. For a graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$ are the sets of vertices and edges of G , respectively. For two vertices $u, v \in V(G)$, u and v are neighbors if u and v are adjacent, that is, if there is an edge $e = uv$. The *open neighborhood* of a vertex v in G , denoted by $N(v)$, is the set of all vertices adjacent to v . The numbers $|V(G)| = n$, $|E(G)| = m$ and $|N(v)| = \deg v$ are called the *order*, the *size* of G and the *degree* of v , respectively. The minimum degree $\delta(G) = \delta$ of a graph G is the smallest degree of all vertices in G .

The *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. The *local connectivity* $\kappa_G(u, v) = \kappa(u, v)$ between two distinct vertices u and v of a graph G is defined as the smallest number of vertices whose removal separates u and v . By Menger's Theorem [8], $\kappa(u, v)$ equals the maximum number of internally disjoint $u - v$ paths in G and $\kappa(G) = \min\{\kappa(u, v) : u, v \in V(G)\}$. It is straightforward to verify that $\kappa(G) \leq \delta(G)$ and $\kappa(u, v) \leq \min\{\deg u, \deg v\}$.

For a connected graph G , a *clique* of G is a complete subgraph of G . The *clique number* of a graph G , denoted $\omega(G)$, is the size of a largest clique. Two vertices that are not adjacent in a graph G are said to be *independent*. The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G . The diameter $\text{diam}(G)$ of a graph G is the maximum distance between any two vertices of G .

2 Some bounds on local connective Chromatic number

Definition 1. A *local connective k -coloring* of a graph G is a mapping $c : V(G) \rightarrow \{1, 2, \dots, k\}$ such that

- (i) If $uv \in E(G)$, then $c(u) \neq c(v)$, and
- (ii) If $uv \notin E(G)$ and $c(u) = c(v) = i$, then $\kappa(u, v) \geq i$, where $\kappa(u, v)$ is the maximum number of internally disjoint paths between u and v .

The smallest integer k for which there exists a local connective k -coloring of G is called the *local connective chromatic number* of G , and it is denoted by $\chi_{lc}(G)$.

The first condition characterizes proper coloring. Thus, every local connective coloring is a proper coloring. The vertices of G are partitioned into disjoint color classes X_1, X_2, \dots, X_k , where each color class X_i consists of distinct vertices $u, v \in X_i$ such that $\kappa(u, v) \geq i$ and $\bigcup_{i=1}^n X_i = V(G)$. The maximum cardinality of X_i in G is denoted by k_i .

Since all vertices of complete graph K_n are pairwise adjacent, by definition of local connective coloring we easily say $\chi_{lc}(K_n) = n$.

Lemma 1. If G is a connected graph of order n and independence number α , then $\chi_{lc}(G) \leq 1 + n - \alpha(G)$ and equality holds if $\kappa(u, v) = 1$ for all $u, v \in V(G)$.

Proof. Let S be a maximum independent set of G and then $|S| = \alpha(G)$. Since S is independent set, the vertices in S can be colored with the same color (color 1). Now, consider the remaining graph H with $n - \alpha(G)$ vertices. If H is complete

graph, then $\chi_{lc}(H) = n - \alpha(G)$. Otherwise $\chi_{lc}(H) < n - \alpha(G)$. Then, we have

$$\chi_{lc}(G) = 1 + \chi_{lc}(H) \leq 1 + n - \alpha(G).$$

If $\kappa(u, v) = 1$ for all $u, v \in V(G)$, then no two vertices can receive the same color i for $i \geq 2$. Furthermore, the maximum number of vertices which receives color 1 is $\alpha(G)$. Then, the proof is completed.

Lemma 2. *Let G be a graph of order n . Then $\chi_{lc}(G) \geq \omega(G)$.*

Proof. Let H be a subgraph of G which has maximum clique and $|V(H)| = p \leq n$. Since all vertices of H are pairwise adjacent, these vertices must be colored with p different colors. Then, the graph G is colored with at least p colors. Since $n \geq p$, we obtain $\chi_{lc}(G) \geq \chi_{lc}(H) = \omega(G)$.

Theorem 1. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$2 \leq \chi_{lc}(G) \leq n.$$

Proof. For any graph G with $n \geq 2$ vertices, since $\alpha(G) \geq 1$ and $\omega(G) \geq 2$, the proof follows from Lemma 1 and Lemma 2.

Theorem 2. *Let G be a graph of order n with $n \geq 2$. Then, $\chi(G) \leq \chi_{lc}(G)$.*

Proof. For $n \geq 2$, we know that $\chi(G) \geq 2$ and $\chi_{lc}(G) \geq 2$. Let p be the total number of vertices colored with color 1 and color 2. Then $(n - p)$ vertices remain uncolored. We consider the following three cases with respect to these $(n - p)$ vertices.

Case 1. Let $n - p = 0$. Then, $\chi(G) = \chi_{lc}(G) = 2$.

Case 2. Let $n - p = 1$. Then, we color remaining one vertex with color 3. Thus, $\chi(G) = \chi_{lc}(G) = 3$.

Case 3. Let $n - p \geq 2$ and u, v be any two remaining vertices.

Case 3.1. If u and v are adjacent vertices, then by definition of proper coloring and local connective coloring we must color these vertices with different colors. Then, $\chi(G) = \chi_{lc}(G)$.

Case 3.2. Let u and v be non-adjacent vertices. For proper coloring, we color these vertices with the same color. For local connective coloring, consider the following cases.

Case 3.2.1. If $\kappa(u, v) \geq i$ for $i \geq 3$, we color these vertices with the same color i . Then, $\chi(G) = \chi_{lc}(G)$.

Case 3.2.2. If $\kappa(u, v) < i$ for $i \geq 3$, then we color these vertices with different colors. Then, this case increases local connective chromatic number. Hence, we have $\chi(G) < \chi_{lc}(G)$.

The proof is completed from these cases.

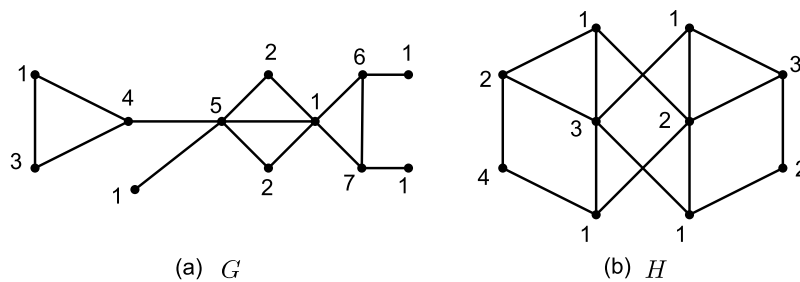


Fig. 1: Local connective coloring of two graphs

Fig. 1 (a) shows the graph G with $\chi_{lc}(G) > \chi_p(G)$ and Fig. 1 (b) shows the graph H with $\chi_{lc}(H) < \chi_p(H)$.

Theorem 3. Let G be a graph of order n with $diam(G) \geq 2$. Then,

$$\chi_{lc}(G) \leq \chi_p(G) + 2 - diam(G) + \sum_{i=2}^{diam(G)-1} \rho_i.$$

Proof. Let G be a graph of order n , diameter $diam(G) \geq 2$ and for $i \geq 1$ let c_i be the number of vertices which is colored with packing coloring.

Since ρ_i is the maximum number of vertices which is colored with packing coloring for $i \geq 1$ and $diam(G) \geq 2$, we have $\rho_i(G) \leq 1$ for $i \geq diam(G)$. Thereby, the graph is colored at least $diam(G) - 1$ colors with respect to packing coloring. For the lower bound, we give color 1 to every vertex in a maximum independent set in G . Then we have $c_1 = \rho_1(G) = \alpha(G)$ and

$$\begin{aligned} \chi_p(G) &\geq diam(G) - 1 + n - \sum_{i=1}^{diam(G)-1} c_i \\ &= diam(G) - 1 + n - \alpha(G) - \sum_{i=2}^{diam(G)-1} c_i. \end{aligned} \quad (1)$$

Since $c_i \leq \rho_i(G)$ for all $i \geq 1$, by inequality (1) we have

$$\chi_p(G) \geq diam(G) - 1 + n - \alpha(G) - \sum_{i=2}^{diam(G)-1} \rho_i. \quad (2)$$

By Lemma 1 and inequality (2), we get

$$\chi_{lc}(G) \leq \chi_p(G) + 2 - diam(G) + \sum_{i=2}^{diam(G)-1} \rho_i.$$

Corollary 1. Let G be a graph of order n with $diam(G) \geq 2$ and independence number α . Then,

$$\chi_{lc}(G) \leq \chi_p(G) + (2 - diam(G))(1 - \alpha(G)).$$

Proof. Since $\rho_i(G) \leq \alpha(G)$ for $i \geq 2$, by Theorem 3,

$$\begin{aligned} \chi_p(G) &\geq \chi_{lc}(G) + diam(G) - 2 - \sum_{i=2}^{diam(G)-1} \alpha(G) \\ &= \chi_{lc}(G) + (diam(G) - 2) - (diam(G) - 2)\alpha(G) \\ &= \chi_{lc}(G) + (diam(G) - 2)(1 - \alpha(G)). \end{aligned}$$

Corollary 2. If G is a graph of diameter two, then $\chi_{lc}(G) \leq \chi_p(G)$.

Proof. If we substitute $diam(G) = 2$ into Corollary 1, the proof is completed.

Corollary 3. If G is a bipartite graph of diameter 3, then $\chi_{lc}(G) \leq \chi_p(G) + 1$.

Proof. Since $diam(G) = 3$, for packing coloring each color at least 3 appears at most once and since the graph is bipartite graph of diameter 3, color 2 can be used only twice [14]. Substituting $diam(G) = 3$ and $c_2 = \rho_2(G) = 2$ into the inequality of Theorem 3, we get $\chi_{lc}(G) \leq \chi_p(G) + 1$.

3 Local connective Chromatic number of some graphs

Theorem 4. Let P_n be a path of order n with $n \geq 2$. Then,

$$\chi_{lc}(P_n) = 1 + \lfloor \frac{n}{2} \rfloor.$$

Proof. The number of internally disjoint paths between all pairs of vertices of P_n is 1. Then,

$$k_i(P_n) \leq 1 \text{ for } i \geq 2.$$

Let X_1 be a maximum independent set of P_n . Then, every vertex in X_1 has color 1 and $k_1(P_n) = \alpha(P_n) = \lceil \frac{n}{2} \rceil$. Since $k_i(P_n) \leq 1$ for $i \geq 2$, the other vertices receive different colors. Consequently, we have

$$\chi_{lc}(P_n) = 1 + n - \lceil \frac{n}{2} \rceil = 1 + \lfloor \frac{n}{2} \rfloor.$$

Theorem 5. Let C_n be a cycle of order n with $n \geq 3$. Then,

$$\chi_{lc}(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since cycle graphs are 2-regular graphs, $\kappa(u, v) = 2$ for any two vertices u and v in C_n . Thus, $k_i(C_n) \leq 1$ for $i \geq 3$. Let X_1 and X_2 be subsets of C_n containing the vertices labelled by color 1 and color 2, respectively. Since $k_i(C_n) \leq 1$ for $i \geq 3$, the other vertices receive different colors. Therefore,

$$k_1(C_n) = \lfloor \frac{n}{2} \rfloor, k_2(C_n) = \lfloor \frac{n}{2} \rfloor.$$

Then, we have

$$\begin{aligned} \chi_{lc}(C_n) &= 1 + 1 + n - \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor \\ &= 2 + n - 2 \lfloor \frac{n}{2} \rfloor. \end{aligned} \tag{3}$$

The proof is completed depending on n being odd and even.

Theorem 6. Let W_n be a wheel graph of order n . Then,

$$\chi_{lc}(W_n) = \begin{cases} 3, & \text{if } n \text{ is odd} \\ 4, & \text{if } n \text{ is even.} \end{cases}$$

Proof. In the wheel graph W_n , the center vertex is adjacent to each vertex of the cycle C_{n-1} . Then this vertex must be colored with a different color than the others. Therefore, we have

$$\chi_{lc}(W_n) = 1 + \chi_{lc}(C_{n-1}).$$

By equality (3),

$$\begin{aligned} \chi_{lc}(W_n) &= 1 + 2 + n - 1 - 2 \lfloor \frac{n-1}{2} \rfloor \\ &= 2 + n - 2 \lfloor \frac{n-1}{2} \rfloor. \end{aligned}$$

The proof is completed depending on n being odd and even.

Theorem 7. Let $K_{1,n}$, where $n > 1$, be a star. Then,

$$\chi_{lc}(K_{1,n}) = 2.$$

Proof. Since $\kappa(u, v) \leq \min\{\deg u, \deg v\} = 1$ for any two non-adjacent vertices u and v of graph $K_{1,n}$, we have $k_i(K_{1,n}) \leq 1$ for $i \geq 2$. Thus, n leaves of $K_{1,n}$ must be colored with color 1 and internal vertex receives a distinct color. Hence, we have $\chi_{lc}(K_{1,n}) = 1 + 1 = 2$.

Theorem 8. *Let T be a tree of order n with $n \geq 2$. Then*

$$\chi_{lc}(T) = 1 + n - \alpha(T).$$

Proof. Let S be a maximum independent set in T . We can give color 1 to every vertex in S . Thus $k_1(T) = \alpha(T)$. Since the number of internally disjoint paths between all non-adjacent pairs of vertices of T is 1, we have $k_i(T) \leq 1$ for $i \geq 2$. Hence, the remaining $n - \alpha(T)$ vertices receive $n - \alpha(T)$ different colors other than color 1. Then the proof is completed.

Corollary 4. *Let T be a tree of order n with $n \geq 2$. Then*

$$\chi_{lc}(T) \leq 1 + \lfloor \frac{n}{2} \rfloor$$

and equality holds if T is a path.

Proof. If T is a tree with $n \geq 2$ vertices, $\alpha(T) \geq \lceil \frac{n}{2} \rceil$ [2]. By Theorem 8, we have

$$\chi_{lc}(T) = 1 + n - \alpha(T) \leq 1 + n - \lceil \frac{n}{2} \rceil = 1 + \lfloor \frac{n}{2} \rfloor.$$

Then the proof is completed.

The complete p -partite graph K_{n_1, n_2, \dots, n_p} has a vertex set V that can be partitioned into p parts V_1, V_2, \dots, V_p so that V_i has n_i vertices and two vertices are adjacent if and only if they are in distinct parts. If $n_1 = n_2 = \dots = n_p$, the graph is regular [7].

Theorem 9.[9] *Let $p \geq 2$ be an integer. If G is a p -partite graph of order n such that*

$$n \leq \delta(G) \frac{2p-1}{2p-3}$$

then $\kappa(u, v) = \min\{\deg u, \deg v\}$ for all pairs of distinct vertices u and v in G .

Theorem 10. *For any integer $p \geq 3$, let G be a regular complete p -partite graph of order n . Then $\chi_{lc}(G) = p$.*

Proof. Let $G = K_{n_1, n_2, \dots, n_p}$ be a regular complete p -partite graph with the partition $V(G) = V_1 \cup V_2 \cup \dots \cup V_p$ to the independent sets V_1, V_2, \dots, V_p , where $|V_i| = n_i$ for each i and $n_1 = n_2 = \dots = n_p = t$, $t \geq 2$. Thus, $n_1 + n_2 + \dots + n_p = n = tp$ and $\deg u = (p-1)t = \delta(G)$ for all $u \in V(G)$.

Since

$$\begin{aligned}
 \delta(G) \frac{2p-1}{2p-3} &= (p-1)t \frac{2p-1}{2p-3} \\
 &= (p-1)t \left(1 + \frac{2}{2p-3}\right) \\
 &> (p-1)t \left(1 + \frac{2}{2p}\right) \\
 &= (p-1)t \left(1 + \frac{1}{p}\right) \\
 &= \frac{p-1}{p}t(p+1) \\
 &= \left(1 - \frac{1}{p}\right)t(p+1) \\
 &> (p+1)t \\
 &> pt = n
 \end{aligned}$$

by Theorem 9, $\kappa(u, v) = \min\{\deg u, \deg v\} = (p-1)t$ for all pairs of distinct vertices u and v in G .

For $p \geq 3$ and $t \geq 2$,

$$\kappa(u, v) = (p-1)t > p. \quad (4)$$

Since every vertex in V_i is adjacent to each vertex in V_j , we have $\chi_{lc}(G) \geq p$, where $i, j \in \{1, 2, \dots, p\}$, $i \neq j$. Without loss of generality assume that $\chi_{lc}(G) \geq p+1$. Since G is colored with at least $p+1$ colors, $\kappa(u, v) < p$ for a vertex u which receives color $p+1$ and a vertex v which receives color p . This contradicts (4). Thus, we have $\chi_{lc}(G) = p$.

4 Conclusion

In this paper, we introduce and study a new coloring concept called local connective coloring. Since efficiency and reliability of routing can be achieved by using internally disjoint paths, we use the term internally disjoint path in our definition. We give some general bounds and study on several graph classes. Also, we give relation between local connective coloring and some other colorings.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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