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Extremal functions for starlike functions and convex functions

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Abstract: In this paper, we obtain new extremal functions for starlike functions and convex functions on the range

$$0 \leq \alpha \leq \frac{1}{2^{r+1}}$$

defined on the unit disk using analytic and univalent functions.

Keywords: Starlike function, convex function, analytic function,extremal function.

1 Introduction

Definition 1. *Let $U = \{z \in \mathbb{C} : |z| < 1\}$. A function analytic $f(z) = z + \sum_2^\infty a_n z^n \in A$ is said to be starlike of order α if it satisfies*

$$Re\left(\frac{z.f'(z)}{f(z)}\right) > \alpha, (z \in U)$$

for some real $\alpha(0 \leq \alpha < 1)$. The class of starlike functions $f(z) \in A$ of order α is denoted by $S^*(\alpha)$.

Also, a function $f(z) \in A$ is said to be convex of order α if it satisfies

$$Re\left(1 + \frac{z.f''(z)}{f'(z)}\right) > \alpha,\ (z \in U)$$

for some real $\alpha(0 \leq \alpha < 1)$.The class of convex functions $f(z) \in A$ of order α is denoted by $K(\alpha)$. Where $f(0) = 0$ and $f'(0) = 1$. [1],[4],[5].

Remark.

$$f(z) \in K(\alpha) \Leftrightarrow z.f'(z) \in S^*(\alpha),$$

$$f(z) \in S^*(\alpha) \Leftrightarrow \int_0^z \frac{f(t)}{t} dt \in K(\alpha).$$

Definition 2. *Let $p(z)$ be analytic in U with $p(0) = 1$. If $p(z)$ satisfies*

$$Rep(z) > 0 \quad (z \in U)$$

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then $p(z)$ is said to be the Caratheodory function. We denote by P all Caratheodory functions.

Example 1. Let us define a function $p(z)$ by

$$p(z)=\frac{1+z}{1-z} \quad (z\in U)$$

Then $p(z)$ analytic in U with $p(0)=1$. Furthermore, for $z=re^{i\theta}$ $(0\le r<1, 0\le\theta\le 2\pi)$, we know that

$$Re p(z)=Re\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)\ge\frac{1-r}{1+r}>0.$$

Thus

$$p(z)=\frac{1+z}{1-z}\in P.$$

Lemma 1. [2] *If $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n\in P$ then*

$$|p_n|\le 2 \quad (n=1,2,3,...).$$

Equality is attended for

$$p(z)=\frac{1+z}{1-z}=1+\sum_{n=2}^{\infty}2.z^n.$$

Proof. We use the following fact that if $p(z)$ is analytic in U and $Re(z)>0$ $(z\in U)$, then $p(z)$ can be written by

$$p(z)=\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)+i\gamma,$$

where $\mu(t)$ is the probability measure such that

$$d\mu(t)\ \geqslant 0 \ \text{ and } \ \int_0^{2\pi}d\mu(t)=1$$

With above fact, if $p(0)=1$, then $\gamma=0$. Therefore, we can write the function

$$p(z)=\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)$$

we see that

$$\frac{e^{it}+z}{e^{it}-z}=\frac{1+e^{it}z}{1-e^{it}z}=1+2\sum\nolimits_{n=1}^{\infty}e^{-int}z^n.$$

This show that

$$\begin{aligned} p(z)&=\int_0^{2\pi}\left(1+2\sum\nolimits_{n=1}^{\infty}e^{-int}z^n\right)d\mu(t)\\ &=1+2\sum\nolimits_{n=1}^{\infty}\left(\int_0^{2\pi}e^{-int}d\mu(t)\right)z^n\\ &=1+\sum\nolimits_{n=1}^{\infty}p_nz^n \end{aligned}$$

where

$$p_n=2\int_0^{2\pi}e^{-int}d\mu(t).$$

It follows that

$$|p_n| = \left| 2\int_0^{2\pi} e^{-int} d\mu(t) \right| \leq 2\int_0^{2\pi} d\mu(t) = 2$$

Furthermore, if $|p_n| = 2$ then $t = 0$. Thus we have

$$p_n = \int_0^{2\pi} \frac{1+z}{1-z} d\mu(t) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

Theorem 1. *A function* $f(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$ *is an extremal function for the class K. A function* $f(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n.z^n$ *is an extremal function for the class* S^*.

Proof. Let $f(z) \in S^*$. Then a function $p(z)$ given by $p(z) = \frac{zf'(z)}{f(z)}$ is a Caratheodory function, so that $p(z) \in P$. Applying Lemma 1, we have that

$$p(z) = \frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}$$

is the an extremal function for the class P. This gives us that

$$\frac{f'(z)}{f(z)} = \frac{1+z}{z(1-z)} = \frac{1}{z} + \frac{2}{1-z}$$

which show that

$$\log f(z) = \log z - 2\log(1-z) = \log \frac{z}{(1-z)^2}.$$

Thus we obtain that

$$f(z) = \frac{z}{(1-z)^2}.$$

Next, we note that $f \in K$ if and only if $z.f\prime(z) \in S^*$. Thus, we consider

$$z.f'(z) = \frac{z}{(1-z)^2}$$

for an extremal function $f(z)$ for the class K. It is easy to get

$$f(z) = \frac{z}{1-z}$$

Theorem 2. [3] *Let f be analytic in U, with* $f(0) = 0$ *and* $f\prime(0) = 1$. *Then* $f \in C$ *if and only if* $z.f\prime(z) \in S^*$.

Corollary 1. $k(z)$ *is Koebe Function , for which*

$$\frac{z.k'(z)}{k(z)} = \frac{1+z}{1-z}$$

clearly offers equality.

Proof. The leading example of a function of class S is the Koebe function indeed;

$$k(z) = z + \sum_{2}^{\infty} m.z^m = z + 2z^2 + 3z^3 + 4.z^4 + \ldots$$

$$k'(z) = 1 + 4z + 9z^2 + 16z^3 + \ldots$$

$$z.k'(z) = z + 4z^2 + 9z^3 + 16z^4 + \ldots$$

$$\frac{z.k'(z)}{k(z)} = 1+2z+2z^2+2z^3+... = \frac{1+z}{1-z}$$

2 Main theorem

Theorem 3. *Let* $0 \leq \alpha \leq \frac{1}{2^{r+1}}$ *and* $z \in U$. *Then, an extremal function for* $S^*(\alpha)$ *is*

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha 2^r)}}.$$

an extremal function for $K(\alpha)$ *is*

$$f(z) = \frac{1-(1-z)^{\alpha 2^{r+1}}}{\alpha 2^{r+1}-1}, \alpha \neq \frac{1}{2^{r+1}} \log(\frac{1}{1-z}), \alpha = \frac{1}{2^{r+1}}.$$

Proof. Let us consider

$$F(z) = \frac{\frac{zf'(z)}{f(z)} - \alpha 2^r}{1-\alpha 2^r} \quad , f(z) \in S^*(\alpha)$$

Then $F(z) = 1+\sum_1^\infty b_m z^m$ is analytic in U and $ReF(z) > 0, (z \in U)$.Using Lemma 1,If $F(z)$ is an extremal function for Lemma 1, then

$$F(z) = \frac{\frac{zf'(z)}{f(z)} - \alpha 2^r}{1-\alpha 2^r} = \frac{1+z}{1-z}$$

which is equivalent to

$$\frac{zf'(z)}{f(z)} - \alpha 2^r = (1-\alpha 2^r)\left[\frac{1+z}{1-z}\right]$$

$$\frac{f'(z)}{f(z)} - \frac{\alpha 2^r}{z} = \frac{(1-\alpha 2^r)(1+z)}{z(1-z)}$$

thus we have

$$\frac{f'(z)}{f(z)} = \frac{\alpha 2^r}{z} + (1-\alpha 2^r)\left[\frac{1}{z} + \frac{2}{1-z}\right]$$

integrating both sides, we have that

$$\int_0^z \frac{f'(t)}{f(t)} dt = \alpha 2^r \int_0^z \frac{1}{t} dt + (1-\alpha 2^r) \int_0^z [\frac{1}{t} + \frac{2}{1-t}] dt$$

this show that

$$[\log f(t)]_0^z = \alpha 2^r [\log t]_0^z + (1-\alpha 2^r)[\log t]_0^z - 2(1-\alpha 2^r)[\log(1-t)]_0^z$$

that is, that

$$\log f(z) = \log \frac{z}{(1-z)^{2(1-\alpha 2^r)}}$$

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha 2^r)}}.$$

For $f(z) \in K(\alpha)$, we use $z.f'(z) \in S^*(\alpha)$. This means that

$$z.f'(z) = \frac{z}{(1-z)^{2(1-\alpha 2^r)}}$$

since,

$$f^{'}(z) = \frac{1}{(1-z)^{2(1-\alpha 2^r)}}.$$

If $\alpha = \frac{1}{2^{r+1}}$,then

$$f(z) = \int_0^t \frac{1}{1-t}dt = -\log(1-t)]_0^z = \log\frac{1}{1-z}$$

If $\alpha \neq \frac{1}{2^{r+1}}$,then

$$f(z) = \int_0^z \frac{1}{(1-t)^{2(1-\alpha 2^r)}}dt = \int_0^z (1-t)^{2(\alpha 2^r - 1)}dt$$

$$= \left[\frac{-(1-t)^{2\alpha 2^r}}{2^{r+1}}\right]_0^z$$

$$= \frac{1-(1-z)^{\alpha 2^{r+1}-1}}{\alpha 2^{r+1}-1}.$$

This completes the proof.

3 Conclusion

In this study we obtained extremal functions for starlike and convex functions according to $\alpha = \frac{1}{2^{r+1}}$ values changing to between $0 \leq \alpha < 1$ for $r = 0,1,2,\ldots$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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