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On the property between theta functions and Dedekind's function according to the changes in the value periods

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Abstract: In the study, two relations between the Dedekind's η-function and θ- function were established by using characteristic values $(\varepsilon,\varepsilon^{'})\equiv(0,0),(0,1)(mod 2)$ for θ-function according to $\frac{1}{8}$ coefficient of period pair(u,τ). The transformations among the theta functions according to the periods have been given and a Jacobian style elliptic functions has been set up the theta function by the help of a defined function.

Keywords: Theta functions,Dedekinds function, periods pair, charasteristics.

1 Introduction

It is of advantage to introduce function, defined by θ(u,τ), which has a rapidly convergent expansion in infinite series, and which is directly connected whith the sigma function of Weierstrass.Let τ be a complex variable, with τ=$\frac{\omega_2}{\omega_1}\neq$ real, Imτ>0, and ω= m ω_1+nω_2 with (m,n) $\neq(0,0)$, m,n are integers and u is a complex variable. We define the function $\theta(u,\tau)$ by the series,

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)=\sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})\right\} \tag{1}$$

where $(\varepsilon,\varepsilon^{'})$ is $\begin{bmatrix}\varepsilon\\ \varepsilon^{'}\end{bmatrix}$ and $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\bmod 2)$, ε and ε' are integers, and n ranges over all the integers (-∞ to ∞). The series in [1] converges absolutely, and uniformly in compact sets of the u-complex plane, and therefore represents an entire function of u[2]. We can see the following alternative theta functions.

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\mu\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}$$

where μ=exp{-$\frac{1}{4^r}(\tau+2)\pi$i- $\frac{1}{2^r}$(2u+ε')πi} and $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\bmod 2)$, and ε' are integers in [2].

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)=\sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})\right\}$$

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where $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix} (\bmod 2)$, ε and ε' are integers n ranges over all the integers (-∞ to ∞) in [2].

$$\theta \begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau) = \sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2i(n+\frac{\varepsilon}{2})(u-\frac{\varepsilon'}{2}\pi)\right\}$$

where $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix} (\bmod 2)$, ε and ε' are integers n ranges over all the integers (-∞ to ∞) in [4]. If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\1\end{bmatrix} (\bmod 2)$ then $\theta \begin{bmatrix}1\\1\end{bmatrix}(u,\tau) = -i\sum_n (-1)^n \exp\left\{(n+\frac{1}{2})^2 \pi i\tau + (2n+1)\pi iu\right\}$ This function $\theta \begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$ is an alternative formula in [4].

If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}0\\0\end{bmatrix} (\bmod 2)$ and $u=0$ then the function $\theta \begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$ is an alternative formula in [4].

If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}0\\0\end{bmatrix} (\bmod 2)$ and u=0 then $\theta \begin{bmatrix}0\\0\end{bmatrix}(0,\tau) = \sum_n \exp(n^2\pi i\tau)$ The function$\theta \begin{bmatrix}0\\0\end{bmatrix}(0,\tau)$ is an alternative formula in [4].

Definition 1. *A period, denoted* $\begin{Bmatrix}a\\b\end{Bmatrix}$, *is* $b+a\tau$.

if r = 3 then $\frac{1}{8}\begin{Bmatrix}a\\b\end{Bmatrix} = \frac{b}{8} + \frac{a\tau}{8}$.

A reduced quarter-period is a quarter-period in which a,b equal 0 or 1 [4]. With the help of this alternativa formula in [2]. Above, we can get the following equalities according to quarter-periods. If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\1\end{bmatrix} (\bmod 2)$ then

$$\begin{aligned}\theta \begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau) &= \sum_n \exp\left\{(n+\frac{1}{2})^2 \pi i\tau + 2\pi i(n+\frac{1}{2})(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix}+\frac{1}{4}\right\}\\ &= ie^{-\frac{\pi i\tau}{4}} \sum_n (-1)^n \exp\left\{(n+\frac{1}{2})^2 \pi i\tau + (2n+1)\pi iu + \frac{n\pi i}{2} + \frac{n\pi i\tau}{2} + \frac{\pi i\tau}{4} + \frac{\pi i}{4}\right\}\end{aligned}$$

If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\0\end{bmatrix} (\bmod 2)$ then

$$\begin{aligned}\theta \begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau) &= \sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{1}{2})(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}\\ &= e^{-\frac{\pi i\tau}{4}} \sum_n \exp\left\{(n+\frac{1}{2})^2 \pi i\tau + (2n+1)\pi iu + \frac{n\pi i}{4} + \frac{n\pi i\tau}{4} + \frac{\pi i\tau}{8} + \frac{\pi i}{8}\right\}\end{aligned}$$

Using the equations we can get

$$\frac{\theta \begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta \begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)} = \frac{ie^{-\frac{\pi i\tau}{8}} \sum_n (-1)^n \exp\left\{(n+\frac{1}{2})^2 \pi i\tau + (2n+1)\pi iu + \frac{n\pi i}{4} + \frac{n\pi i\tau}{4} + \frac{\pi i\tau}{8} + \frac{\pi i}{8}\right\}}{e^{-\frac{\pi i\tau}{8}} \sum_n \exp\left\{(n+\frac{1}{2})^2 \pi i\tau + (2n+1)\pi iu + \frac{n\pi i}{4} + \frac{n\pi i\tau}{4} + \frac{\pi i\tau}{8} + \frac{\pi i}{8}\right\}}$$

(i) If n is 0 or even integer then, $\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=i\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$.

(ii) If n is odd integer then $\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=-i\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$.

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\1\end{bmatrix}\pmod 2$ then

$$\begin{aligned}\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi i(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix}+\frac{1}{4}\right\}\\&=\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{4}+\frac{n\pi i\tau}{4}\right\}\end{aligned}$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\0\end{bmatrix}\pmod 2$ then

$$\begin{aligned}\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi i(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}\\&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{4}+\frac{n\pi i\tau}{4}\right\}\end{aligned}$$

From the above equations we obtain

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}=\frac{\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{4}+\frac{n\pi i\tau}{4}\right\}}{\sum_n\exp\left\{n^2\pi i\tau(2n\pi iu+\frac{n\pi i}{4}+\frac{n\pi i\tau}{4}\right\}}$$

(iii) If n is 0 or even integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

(iv) If n is odd integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{8}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$$

Theorem 1. *The function $\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$ defined in [3] is odd function of u and it can be expressed by infinite product*

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)=ce^{\frac{\pi i\tau}{4}}2\sin\pi u\prod_{n=1}^{\infty}\left\{1-e^{2(n\tau+u)\pi i}\right\}\prod_{n=1}^{\infty}\left\{1-e^{2(n\tau-u)\pi i}\right\}$$

where $c=\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau})$, Imt >0 [2]. We consider the function $\phi(u,\tau)$ expressed by product

$$\varphi(u,\tau)=\prod_{n=1}^{\infty}\left\{1-e^{[(2n-1)\tau+2u]\pi i}\right\}\prod_{n=1}^{\infty}\left\{1-e^{[(2n-1)\tau-2u]\pi i}\right\}.$$

Theorem 2. *The function* $\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}$ *is a elliptic function with periods 1 and* τ.

Proof. Let $\psi(u,\tau)=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}$, then we write

$$\psi(u+1,\tau)=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+1,\tau)}{\varphi(u+1,\tau)}=\frac{-\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(-u,\tau)}{\varphi(u,\tau)}$$

where $\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)=-\theta\begin{bmatrix}1\\1\end{bmatrix}(-u,\tau)$ from theorem 2.

$$\psi(u+\tau,\tau)=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\tau,\tau)}{\varphi(u+\tau,\tau)}=\frac{-e^{-(2u+\tau)\pi i}\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{e^{-(2u+\tau)\pi i}\varphi(u,\tau)}=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}$$

since

$$\begin{aligned}
\phi(u+\tau,\tau)&=\prod_{n=1}^{\infty}\left\{1-e^{(2n-1)\tau\pi i+2\pi i(u+\tau)}\right\}\prod_{n=1}^{\infty}\left\{1-e^{(2n-1)\tau\pi i-2\pi i(u+\tau)}\right\}\\
&=\prod_{n=1}^{\infty}\left\{1-e^{(2n+1)\tau\pi i+2\pi iu}\right\}\prod_{n=1}^{\infty}\left\{1-e^{(2n-3)\tau\pi i-2\pi iu}\right\}\\
&=\prod_{n=1}^{\infty}\left\{1-e^{[2(n+1)-1]\tau\pi i+2\pi iu}\right\}\prod_{n=1}^{\infty}\left\{1-e^{[2(n-1)-1]\tau\pi i-2\pi iu}\right\}\\
&=\prod_{m=2}^{\infty}\left\{1-e^{(2n+1)\tau\pi i+2\pi iu}\right\}\prod_{n=0}^{\infty}\left\{1-e^{(/2n-3)\tau\tau i-2\tau ui}\right\}\\
&=\prod_{m=1}^{\infty}\left\{1-e^{(2n+1)\tau\pi i+2\pi iu}\right\}\prod_{m=1}^{\infty}\left\{1-e^{(2m-1)\tau\pi i-2\pi iu}\right\}\left\{1-e^{-(\pi i\tau+2\pi iu)}\right\}\left\{1-e^{(\pi i\tau+2\pi iu)}\right\}^{-1}\\
&=-e^{-(2u+\tau)\pi i}\phi(u,\tau)
\end{aligned}$$

The function $\psi(u,\tau)$ is therefore a doubly periodic with periods 1 and τhaving neither zeros nor poles on account of the fact that $\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$ possesses the same periodicity factors as ϕ(u,τ). Hence the function $\psi(u,\tau)$ is an elliptic function since the set of all meromorphic functions form a field and $\psi(u,\tau)$ is meromorphic and periodic with periods 1 and τ. [2], [4].

We first define the Riemann's θ-function by the series

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+a)^2\pi i\tau+2\pi i(n+a)(u+b)\right\}$$

with a given complex number u ,and complex number τ satisfying Im$(\tau=\frac{\omega_1}{\omega_2}\neq real)>0$ and characteristic $\begin{bmatrix}a\\b\end{bmatrix}$ where a,b are rational numbers. Let us recall the transformation $n^2+n=n(n+1)$ is congruent to zero module 2 $(^2|n(n+1)$ where $n\equiv 0(\bmod 2)$ and $^2|n(n+1)$ from $n+1\equiv 0(\bmod 2)$if $n\equiv 1(\bmod 2))$ [4].

In this article, we treat the special value of θ- function with characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (u,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{(n+\tfrac{\varepsilon}{2})^2 \pi i \tau + 2\pi i (n+\tfrac{\varepsilon}{2})(u+\tfrac{\varepsilon'}{2})\right\}$$

where $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\bmod 2)$,and $\varepsilon, \varepsilon'$are integers [6]. Thus, we have the following relations

$$\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \exp(n^2 \pi i \tau + 2n\pi i u).$$

Now , let us observe that

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u,\tau) = \sum_{n} \exp(n^2 \pi i \tau + 2n\pi i u).$$

Then we see

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u,\tau) = \sum_{n=-\infty}^{\infty} (\exp(n^2 \pi i \tau + 2n\pi i u).$$

that the function $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0,\tau)$ defined by the series in [1] is a alternative formula of $\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (u,\tau)$.The formulas $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u,\tau)$ and $\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u,\tau)$ were used in this article where $u \neq 0$. At first we see the infinite products

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u,\tau) = \prod_{n=1}^{\infty} (1-e^{2n\pi i \tau}). \prod_{n=1}^{\infty} (1+e^{(2n-1)\pi i \tau + 2\pi i u}). \prod_{n=1}^{\infty} (1+e^{(2n-1)\pi i \tau - 2\pi i u}).$$

which it converges absolutely [2] .

Theorem 3. *We have the relations*

(i) $\eta(u) = e^{\frac{\pi i u}{12}}.\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\frac{u+6}{4},\tau).$

(ii) $\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\frac{u+6}{4},\tau) = e^{-\frac{\pi i u}{12}} \eta(u). \prod_{n=1}^{\infty} (1-e^{(2n-1)\pi i u})$ *between the functions* $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u,\tau)$, $\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u,\tau)$ *and Dedekind's* η*-function which defined by the infinite product*

$$\eta(u) = e^{\frac{\pi i u}{12}}. \prod_{n=1}^{\infty} (1-e^{2n\pi i u})$$

where $\operatorname{Im}\tau > 0$ *and k is a integer [6].*

Proof.

(i) Let us recall the formula

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u,\tau) = \prod_{n=1}^{\infty} (1-e^{2n\pi i \tau}). \prod_{n=1}^{\infty} (1+e^{(2n-1)\pi i \tau + 2\pi i u}). \prod_{n=1}^{\infty} (1+e^{(2n-1)\pi i \tau - 2\pi i u}).$$

If k integer, then we have

$$\begin{aligned}\theta\begin{bmatrix}0\\0\end{bmatrix}(3u+2k,\tau) &= \prod_{n=1}^{\infty}(1-e^{2n\pi i(3u+2k)}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i(3u+2k)+2\pi i\tau}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i(3u+2k)-2\pi i\tau})\\ &= \prod_{n=1}^{\infty}(1-e^{6n\pi iu}).\prod_{n=1}^{\infty}(1+e^{6n\pi iu-2\pi iu-(2k-1)\pi i}).\prod_{n=1}^{\infty}(1+e^{6n\pi iu-4\pi iu-(2k+1)\pi i})\\ &= \prod_{n=1}^{\infty}(1-e^{6n\pi iu}).\prod_{n=1}^{\infty}(1-e^{6n\pi iu-2\pi iu}).\prod_{n=1}^{\infty}(1-e^{6n\pi iu-4\pi iu}).\end{aligned}$$

If we set $G=e^{2\pi iu}$, then we obtain

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(3u+2k,\tau)=\prod_{n=1}^{\infty}(1-G^{3n}).\prod_{n=1}^{\infty}(1-G^{3n-1}).\prod_{n=1}^{\infty}(1-G^{3n-2}).$$

On the other hand, we may set $n=m+1$, then

$$\begin{aligned}\theta\begin{bmatrix}0\\0\end{bmatrix}(3u+2k,\tau) &= \prod_{n'=1}^{\infty}(1-G^{3m+3}).\prod_{n'=1}^{\infty}(1-G^{3m+2}).\prod_{n'=1}^{\infty}(1-G^{3m+1})\\ &= (1-G)(1-G^2)(1-G^3)(1-G^4)=\cdots\\ &= \prod_{m=1}^{\infty}(1-G^m)=\prod_{m=1}^{\infty}(1-e^{2m\pi iu}).\end{aligned}$$

According to above equations, we have

$$\eta(u)=e^{\frac{\pi iu}{12}}.\theta\begin{bmatrix}0\\0\end{bmatrix}(3u+2k,\tau)$$

from the Dedekind's η-function defined by the infinite product

$$\eta(u)=e^{\frac{\pi iu}{12}}.\prod_{n=1}^{\infty}(1-e^{2n\pi iu})$$

(ii) According to the equation,

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}(-1)^n\exp(n^2\pi i\tau+2n\pi iu)$$

we have

$$\begin{aligned}\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+6}{4},\tau) &= \sum_{n=-\infty}^{\infty}(-1)^n\exp\left[\tfrac{1}{2}n(3n+1)\pi iu\right]\\ &= 1+\sum_{n=1}^{\infty}(-1)^n\left\{\exp\left[\tfrac{1}{2}n(3n-1)\pi iu\right]+\exp\left[\tfrac{1}{2}n(3n+1)\pi iu\right]\right\}\end{aligned}$$

Where $\frac{1}{2}n(3n+1)$ are the pentagonal numbers and n is negative integers [4].

This results play a role of key stone in the forthcoming work concerning relation between the θ-theta function and Dedekind's η-function. In fact , if the application of theorem.3(ii)on the relation obtained with the theorem.3.(i) which known as the equation between Dedekind's η-function and L.Euler's theorem on pentagonal numbers is done, we obtain

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(\frac{u+6}{4},\tau)}{\prod\limits_{n=1}^{\infty}(1-e^{(2n-1)\pi iu})}=\frac{\sum\limits_{n=-\infty}^{\infty}(-1)^n\exp(\frac{1}{2}n(3n+1)\pi iu)}{\prod\limits_{n=1}^{\infty}(1-e^{(2n-1)\pi iu})}$$

$$=\frac{\prod\limits_{n=1}^{\infty}(1-e^{n\pi iu})}{\prod\limits_{n=1}^{\infty}(1-e^{(2n-1)\pi iu})}=\prod_{n=1}^{\infty}(1-e^{2n\pi iu})=e^{-\frac{\pi iu}{12}}\eta(u).$$

As a result, the relation has been obtained between theta and Dedkind's -$\eta(u)$ functions by using the characteristic $\begin{bmatrix}0\\1\end{bmatrix}$ and the variable $\frac{u+6}{4}$ instead of the characteristic $\begin{bmatrix}0\\1\end{bmatrix}$ and the variable $\frac{u+4}{4}$ which were previously used by [4].

2 Conclusion

A new correlation was established between theta and Dedekind eta functions using 1/8 times of periods and an appropriate value of the variable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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