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# On relative Ritt $L^*$-type and relative Ritt $L^*$-weak type of entire functions presented in the form of vector valued Dirichlet series

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**Abstract:** In this paper we introduce the idea of relative Ritt $L^*$-type and relative Ritt$L^*$-weak type of entire functions represented by vector valued dirichlet series. Further we wish to study some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of relative Ritt $L^*$-type and relative Ritt $L^*$-weak type.

**Keywords:** Vector valued Dirichlet series (VVDS), relative Ritt $L^*$-order, relative Ritt $L^*$-lower order, relative Ritt $L^*$-type, relative Ritt $L^*$-weak type, growth.

## 1 Introduction, definitions and notations

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ ($\sigma$ and $t$ are real variables) defined by everywhere absolutely convergent *vector valued Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1}$$

where $a_n$'s belong to a Banach space $(E, \|.\|)$ and $\lambda_n$'s are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)\,, \lambda_n \to \infty$ as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\limsup_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty.$$

If $\sigma_a$ and $\sigma_c$ denote respectively the abscissa of convergence and absolute convergence of (1), then in this case clearly $\sigma_a = \sigma_c = \infty$.

The function $M_f(\sigma)$ known as *maximum modulus* function corresponding to an entire function $f(s)$ defined by (1) is written as follows.

$$M_f(\sigma) = \underset{-\infty<t<\infty}{l.u.b.} |f(\sigma + it)|\,.$$


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In the sequel the following two notations are used:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1,2,3,\cdots;$$
$$\log^{[0]} x = x$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1,2,3,\cdots;$$
$$\exp^{[0]} x = x.$$

Taking this into account, the *Ritt order* (See [1]) of $f(s)$, denoted by $\rho_f$, which is generally used in computational purpose, is defined in terms of the growth of $f(s)$ with respect to the $\exp\exp z$ function as follows.

$$\rho_f = \limsup_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$

Similarly, one can define the *Ritt lower order* of $f(s)$, denoted by $\lambda_f$ in the following manner:

$$\lambda_f = \liminf_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$

Further an entire function $f(s)$ defined by (1) is said to be of *regular Ritt growth* if its *Ritt order* coincides with its *Ritt lower order.* Otherwise $f(s)$ is said to be of *irregular Ritt-growth*.

During the past decades, several authors {e.g., cf., [1],[2],[3],[5],[7]} have made intensive investigations on the properties of entire Dirichlet series related to *Ritt order*. Further, Srivastava [6] defined different growth parameters such as *order* and *lower order* of entire functions represented by *vector valued Dirichlet series*. He also obtained the results for coefficient characterization of *order*.

Somasundaram and Thamizharasi [8] introduced the notions of $L$-order ($L$-lower order ) for entire functions where $L \equiv L(\sigma)$ is a positive continuous function increasing slowly i.e., $L(a\sigma) \sim L(\sigma)$ as $\sigma \to \infty$ for every positive constant '$a$'. In the line of Somasundaram and Thamizharasi [8], one may introduce the notion of *Ritt $L$-order* for an entire functions represented by *vector valued Dirichlet series* in the following manner.

**Definition 1.** *Let $f$ be an entire function represented by* vector valued Dirichlet series. *Then the* Ritt $L$-order $\rho_f^L$ *of $f$ is defined as*

$$\rho_f^L = \limsup_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma L(\sigma)}.$$

*Similarly one may define $\lambda_f^L$, the* Ritt $L$-lower order *of $f$ in the following way.*

$$\lambda_f^L = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma L(\sigma)}.$$

Further one may introduce more generalized concept of *Ritt $L$-order* and *Ritt $L$-lower order* of an entire functions represented by *vector valued Dirichlet series* in the following way.

**Definition 2.** *The* Ritt $L^*$-order *and* Ritt $L^*$-lower order *of an entire function* $f$ *represented by* vector valued Dirichlet series *are defined as*

$$\rho_f^{L^*} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma e^{L(\sigma)}} \quad \textit{and} \quad \lambda_f^{L^*} = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma e^{L(\sigma)}} \quad \textit{respectively.}$$

Srivastava [4] introduced the *relative Ritt order* between two entire functions represented by *vector valued Dirichlet series* to avoid comparing growth just with $\exp\exp z$ as follows.

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(\sigma) < M_g(\sigma\mu) \text{ for all } \sigma > \sigma_0(\mu)\}\\ &= \limsup_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma}.\end{aligned}$$

Similarly, one can define the *relative Ritt lower order* of $f(s)$ with respect to $g(s)$, denoted by $\lambda_g(f)$ in the following manner.

$$\lambda_g(f) == \liminf_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma}.$$

Extending the notion of *relative Ritt order* as introduced by Srivastava [4], next in this paper we introduce *relative Ritt $L^*$-order* between two entire functions represented by *vector valued Dirichlet series* as follows.

$$\begin{aligned}\rho_g^{L^*}(f) &= \inf\left\{\mu > 0 : M_f(\sigma) < M_g\left(\sigma e^{L(\sigma)}\mu\right) \text{ for all } \sigma > \sigma_0(\mu)\right\}\\ &= \limsup_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma e^{L(\sigma)}}.\end{aligned}$$

Similarly, one can define the *relative Ritt $L^*$-lower order* of $f(s)$ with respect to $g(s)$, denoted by $\lambda_g^{L^*}(f)$ in the following manner.

$$\lambda_g^{L^*}(f) == \liminf_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma e^{L(\sigma)}}.$$

Further to compare the relative growth of two entire functions represented by *vector valued Dirichlet series* having same non zero finite *relative Ritt $L^*$-order* with respect to another entire function represented by *vector valued Dirichlet series*, one may introduce the definitions of *relative Ritt-type* and*relative Ritt-lower type* in the following manner.

**Definition 3.** *The relative Ritt $L^*$-type and relative Ritt $L^*$-lower type denoted respectively by $\Delta_g^{L^*}(f)$ and $\overline{\Delta}_g^{L^*}(f)$ of an entire function $f$ with respect to another entire function $g$ both represented by vector valued Dirichlet series are respectively defined as follows.*

$$\Delta_g^{L^*}(f) = \limsup_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\rho_g^{L^*}(f)\cdot\sigma e^{L(\sigma)}\right]}$$

*and*

$$\overline{\Delta}_g^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\rho_g^{L^*}(f)\cdot\sigma e^{L(\sigma)}\right]},\ 0 < \rho_g^{L^*}(f) < \infty.$$

Further to determine the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite *relative Ritt $L^*$-lower order* with respect to another entire function represented by *vector valued Dirichlet series* one may also introduce the definition of *relative Ritt $L^*$-weak type* in the following way.

**Definition 4.** *[8] The relative Ritt $L^*$-weak type denoted by $\tau_g^{L^*}(f)$ of an entire function $f$ with respect to another entire function g both represented by vector valued Dirichlet series is defined as follows.*

$$\tau_g^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\lambda_g^{L^*}(f)\cdot \sigma e^{L(\sigma)}\right]}, \; 0 < \lambda_g^{L^*}(f) < \infty.$$

*Also one may define the growth indicator $\overline{\tau}_g^{L^*}(f)$ of an entire function $f$ with respect to another entire function g both represented by vector valued Dirichlet series in the following manner.*

$$\overline{\tau}_g^{L^*}(f) = \limsup_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\lambda_g^{L^*}(f)\cdot \sigma e^{L(\sigma)}\right]}, \; 0 < \lambda_g^{L^*}(f) < \infty .$$

In the paper we study some relative growth properties of entire functions represented by *vector valued Dirichlet series* using *relative Ritt $L^*$-order, relative Ritt $L^*$- type* and *relative Ritt $L^*$-weak type* .

## 2 Main results

In this section we present the main results of the paper.

**Theorem 1.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series such that $0 < \overline{\Delta}_h^{L^*}(f) \le \Delta_h^{L^*}(f) < \infty$, $0 < \overline{\Delta}_k^{L^*}(g) \le \Delta_k^{L^*}(g) < \infty$ and $\rho_h^{L^*}(f) = \rho_k^{L^*}(g)$, then*

$$\frac{\overline{\Delta}_h^{L^*}(f)}{\Delta_k^{L^*}(g)} \le \liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \le \frac{\overline{\Delta}_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)} \le \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \le \frac{\Delta_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)} .$$

*Proof.* From the definition of $\Delta_k^{L^*}(g)$ and $\overline{\Delta}_h^{L^*}(f)$, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $\sigma$ that

$$\exp M_h^{-1} M_f(\sigma) \ge \left(\overline{\Delta}_h^{L^*}(f) - \varepsilon\right) \exp\left[\rho_h^{L^*}(f)\cdot \sigma e^{L(\sigma)}\right], \tag{2}$$

and

$$\exp M_k^{-1} M_g(\sigma) \le \left(\Delta_k^{L^*}(g) + \varepsilon\right) \exp\left[\rho L_k^*(g)\cdot \sigma e^{L(\sigma)}\right] . \tag{3}$$

Now from (2), (3) and the condition $\rho_h^{L^*}(f) = \rho_k^{L^*}(g)$, it follows that for all sufficiently large values of $\sigma$,

$$\frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \geqslant \frac{\overline{\Delta}_h^{L^*}(f) - \varepsilon}{\Delta_k^{L^*}(g) + \varepsilon} .$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \geqslant \frac{\overline{\Delta}_h^{L^*}(f)}{\Delta_k^{L^*}(g)} . \tag{4}$$

Again for a sequence of values of $\sigma$ tending to infinity,

$$\exp M_h^{-1} M_f(\sigma) \le \left(\overline{\Delta}_h^{L^*}(f) + \varepsilon\right) \exp\left[\rho_h^{L^*}(f)\cdot \sigma e^{L(\sigma)}\right] \tag{5}$$

and for all sufficiently large values of $\sigma$,

$$\exp M_k^{-1} M_g(\sigma) \geq \left(\overline{\Delta}_k^{L^*}(g) - \varepsilon\right) \exp\left[\rho_k^{L^*}(g) \cdot \sigma e^{L(\sigma)}\right]. \tag{6}$$

Combining (5) and (6) and the condition $\rho_h^{L^*}(f) = \rho_k^{L^*}(g)$, we get for a sequence of values of $\sigma$ tending to infinity that

$$\frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\Delta}_h^{L^*}(f) + \varepsilon}{\overline{\Delta}_k^{L^*}(g) - \varepsilon}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\liminf_{\sigma \to \infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\Delta}_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)}. \tag{7}$$

Also for a sequence of values of $r$ tending to infinity it follows that

$$\exp M_k^{-1} M_g(\sigma) \leq \left(\overline{\Delta}_k^{L^*}(g) + \varepsilon\right) \exp\left[\rho_k^{L^*}(g) \cdot \sigma e^{L(\sigma)}\right]. \tag{8}$$

Now from (2), (8) and the condition $\rho_h^{L^*}(f) = \rho_k^{L^*}(g)$, we obtain for a sequence of values of $\sigma$ tending to infinity that

$$\frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \geq \frac{\overline{\Delta}_h^{L^*}(f) - \varepsilon}{\overline{\Delta}_k^{L^*}(g) + \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\limsup_{\sigma \to \infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \geq \frac{\overline{\Delta}_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)}. \tag{9}$$

Also for all sufficiently large values of $\sigma$ ,

$$\exp M_h^{-1} M_f(\sigma) \leq \left(\Delta_h^{L^*}(f) + \varepsilon\right) \exp\left[\rho_h^{L^*}(f) \cdot \sigma e^{L(\sigma)}\right]. \tag{10}$$

In view of the condition $\rho_h^{L^*}(f) = \rho_k^{L^*}(g)$, it follows from (6) and (10) for all sufficiently large values of $\sigma$ that

$$\frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{L^*}(f) + \varepsilon}{\overline{\Delta}_k^{L^*}(g) - \varepsilon}.$$

Since $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\limsup_{\sigma \to \infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)}. \tag{11}$$

Thus the theorem follows from (4), (7), (9) and (11).

**Theorem 2.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series such that* $0 < \Delta_h^{L^*}(f) < \infty$, $0 < \Delta_k^{L^*}(g) < \infty$ *and* $\rho_h^{L^*}(f) = \rho_k^{L^*}(g)$, *then*

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{L^*}(f)}{\Delta_k^{L^*}(g)} \leq \limsup_{r\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)}.$$

*Proof.* From the definition of $\Delta_k^{L^*}(g)$, we get for a sequence of values of $\sigma$ tending to infinity that

$$\exp M_k^{-1} M_g(\sigma) \geq \left(\Delta_k^{L^*}(g) - \varepsilon\right) \exp\left[\rho_k^{L^*}(g) \cdot \sigma e^{L(\sigma)}\right]. \tag{12}$$

Now from (10), (12) and the condition $\rho_h^{L^*}(f) = \rho_k^{L^*}(g)$, it follows for a sequence of values of $\sigma$ tending to infinity that

$$\frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{L^*}(f) + \varepsilon}{\Delta_k^{L^*}(g) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{L^*}(f)}{\Delta_k^{L^*}(g)}. \tag{13}$$

Again for a sequence of values of $\sigma$ tending to infinity that

$$\exp M_h^{-1} M_f(\sigma) \geqslant \left(\Delta_h^{L^*}(f) - \varepsilon\right) \exp\left[\rho_h^{L^*}(f) \cdot \sigma e^{L(\sigma)}\right]. \tag{14}$$

So combining (3) and (14) and in view of the condition $\rho_h(f) = \rho_k(g)$, we get for a sequence of values of $\sigma$ tending to infinity that

$$\frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \geqslant \frac{\Delta_h^{L^*}(f) - \varepsilon}{\Delta_k^{L^*}(g) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \geqslant \frac{\Delta_h^{L^*}(f)}{\Delta_k^{L^*}(g)}. \tag{15}$$

Thus the theorem follows from (13) and (15).

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

**Theorem 3.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series such that* $0 < \overline{\Delta}_h^{L^*}(f) \leq \Delta_h^{L^*}(f) < \infty$, $0 < \overline{\Delta}_k^{L^*}(g) \leq \Delta_k^{L^*}(g) < \infty$ *and* $\rho_h^{L^*}(f) = \rho_k^{L^*}(g)$, *then*

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \min\left\{\frac{\overline{\Delta}_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)}, \frac{\Delta_h^{L^*}(f)}{\Delta_k^{L^*}(g)}\right\} \leq \max L^*\left\{\frac{\overline{\Delta}_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)}, \frac{\Delta_h^{L^*}(f)}{\Delta_k^{L^*}(g)}\right\} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)}.$$

Now in the line of Theorem 1, Theorem 2 and Theorem 3 respectively one can easily prove the following six theorems using the notion of *relative Ritt $L^*$-weak type* and therefore their proofs are omitted.

**Theorem 4.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series such that* $0 < \tau_h^{L^*}(f) \leq \overline{\tau}_h^{L^*}(f) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f) = \lambda_k^{L^*}(g)$, *then*

$$\frac{\tau_h^{L^*}(f)}{\overline{\tau}_k^{L^*}(g)} \leq \liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\tau_h^{L^*}(f)}{\tau_k^{L^*}(g)} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\tau}_h^{L^*}(f)}{\tau_k^{L^*}(g)}.$$

**Theorem 5.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series with* $0 < \overline{\tau}_h^{L^*}(f) < \infty$, $0 < \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f) = \lambda_k^{L^*}(g)$, *then*

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\tau}_h^{L^*}(f)}{\overline{\tau}_k^{L^*}(g)} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)}.$$

**Theorem 6.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series such that* $0 < \tau_h^{L^*}(f) \leq \overline{\tau}_h^{L^*}(f) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f) = \lambda_k^{L^*}(g)$, *then*

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \min\left\{ \frac{\tau_h^{L^*}(f)}{\tau_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f)}{\overline{\tau}_k^{L^*}(g)} \right\} \leq \max\left\{ \frac{\tau_h^{L^*}(f)}{\tau_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f)}{\overline{\tau}_k^{L^*}(g)} \right\} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)}.$$

We may now state the following theorems without their proofs based on *relative Ritt $L^*$-type* and *relative Ritt $L^*$-weak type*.

**Theorem 7.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series such that* $0 < \overline{\Delta}_h^{L^*}(f) \leq \Delta_h^{L^*}(f) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\rho_h^{L^*}(f) = \lambda_k^{L^*}(g)$, *then*

$$\frac{\overline{\Delta}_h^{L^*}(f)}{\overline{\tau}_k^{L^*}(g)} \leq \liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\Delta}_h^{L^*}(f)}{\tau_k^{L^*}(g)} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{L^*}(f)}{\tau_k^{L^*}(g)}.$$

**Theorem 8.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series with* $0 < \Delta_h^{L^*}(f) < \infty$, $0 < \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\rho_h^{L^*}(f) = \lambda_k^{L^*}(g)$, *then*

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{L^*}(f)}{\overline{\tau}_k^{L^*}(g)} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)}.$$

**Theorem 9.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series such that* $0 < \overline{\Delta}_h^{L^*}(f) \leq \Delta_h^{L^*}(f) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\rho_h^{L^*}(f) = \lambda_k^{L^*}(g)$, *then*

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \min\left\{ \frac{\overline{\Delta}_h^{L^*}(f)}{\tau_k^{L^*}(g)}, \frac{\Delta_h^{L^*}(f)}{\overline{\tau}_k^{L^*}(g)} \right\} \leq \max\left\{ \frac{\overline{\Delta}_h^{L^*}(f)}{\tau_k^{L^*}(g)}, \frac{\Delta_h^{L^*}(f)}{\overline{\tau}_k^{L^*}(g)} \right\} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)}.$$

**Theorem 10.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series with* $0 < \tau_h^{L^*}(f) \leq \overline{\tau}_h^{L^*}(f) < \infty$, $0 < \overline{\Delta}_k^{L^*}(g) \leq \Delta_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f) = \rho_k^{L^*}(h)$, *then*

$$\frac{\tau_h^{L^*}(f)}{\Delta_k^{L^*}(g)} \leq \liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\tau_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\tau}_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)}.$$

**Theorem 11.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series such that* $0 < \overline{\tau}_h^{L^*}(f) < \infty$, $0 < \Delta_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f) = \rho_k^{L^*}(g)$, *then*

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\tau}_h^{L^*}(f)}{\Delta_k^{L^*}(g)} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)}.$$

**Theorem 12.** *If $f$, $g$, $h$ and $k$ be any four entire functions represented by vector valued Dirichlet series with* $0 < \tau_h^{L^*}(f) \leq \overline{\tau}_h^{L^*}(f) < \infty$, $0 < \overline{\Delta}_k^{L^*}(g) \leq \Delta_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f) = \rho_k^{L^*}(g)$, *then*

$$\liminf_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)} \leq \min\left\{\frac{\tau_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f)}{\Delta_k^{L^*}(g)}\right\} \leq \max\left\{\frac{\tau_h^{L^*}(f)}{\overline{\Delta}_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f)}{\Delta_k^{L^*}(g)}\right\} \leq \limsup_{\sigma\to\infty} \frac{\exp M_h^{-1} M_f(\sigma)}{\exp M_k^{-1} M_g(\sigma)}.$$

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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