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Some growth analysis of entire functions in the form of vector valued Dirichlet series on the basis of their relative Ritt L^*-order and relative Ritt L^*-lower order

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Abstract: In this paper we study some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of relative Ritt L^*-order and relative Ritt L^*-lower order.

Keywords: Vector valued Dirichlet series (VVDS), relative Ritt L^*-order and relative Ritt L^*-lower order, growth.

1 Introduction, definitions and notations

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent *vector valued Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)\,, \lambda_n \to \infty$ as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\limsup_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty\,.$$

If σ_a and σ_c denote respectively the abscissa of convergence and absolute convergence of (1), then in this case clearly $\sigma_a = \sigma_c = \infty$.

The function $M_f(\sigma)$ known as *maximum modulus* function corresponding to an entire function $f(s)$ defined by (1) is written as follows.

$$M_f(\sigma) = \underset{-\infty<t<\infty}{l.u.b.} |f(\sigma + it)|\,.$$

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In the sequel the following two notations are used:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1,2,3,\cdots;$$
$$\log^{[0]} x = x$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1,2,3,\cdots;$$
$$\exp^{[0]} x = x.$$

Taking this into account, the *Ritt order* (See [1]) of $f(s)$, denoted by ρ_f, which is generally used in computational purpose, is defined in terms of the growth of $f(s)$ with respect to the $\exp\exp z$ function as follows:

$$\rho_f = \limsup_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$

Similarly, one can define the *Ritt lower order* of $f(s)$, denoted by λ_f in the following manner.

$$\lambda_f = \liminf_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$

Further an entire function $f(s)$ defined by (1) is said to be of *regular Ritt growth* if its *Ritt order* coincides with its *Ritt lower order.* Otherwise $f(s)$ is said to be of *irregular Ritt-growth*.

During the past decades, several authors {e.g., cf., [1],[2],[3],[5],[7]} have made intensive investigations on the properties of entire Dirichlet series related to *Ritt order*. Further, Srivastava [6] defined different growth parameters such as *order* and *lower order* of entire functions represented by *vector valued Dirichlet series*. He also obtained the results for coefficient characterization of *order*.

Somasundaram and Thamizharasi [8] introduced the notions of L-order (L-lower order) for entire functions where $L \equiv L(\sigma)$ is a positive continuous function increasing slowly i.e., $L(a\sigma) \sim L(\sigma)$ as $\sigma \to \infty$ for every positive constant 'a'. In the line of Somasundaram and Thamizharasi [8], one may introduce the notion of *Ritt L-order* for an entire function represented by *vector valued Dirichlet series* in the following manner.

Definition 1. *Let f be an entire function represented by* vector valued Dirichlet series. *Then the* Ritt L-order ρ_f^L *of f is defined as*

$$\rho_f^L = \limsup_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma L(\sigma)}.$$

Similarly one may define λ_f^L, the Ritt L-lower order *of f in the following way.*

$$\lambda_f^L = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma L(\sigma)}.$$

Further one may introduce more generalized concept of *Ritt L-order* and *Ritt L-lower order* of an entire function represented by *vector valued Dirichlet series* in the following way.

Definition 2. *The* Ritt L^*-order *and* Ritt L^*-lower order *of an entire function* f *represented by* vector valued Dirichlet series *are defined as*

$$\rho_f^{L^*} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma e^{L(\sigma)}} \text{ and } \lambda_f^{L^*} = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma e^{L(\sigma)}} \text{ respectively.}$$

Srivastava [4] introduced the *relative Ritt order* between two entire functions represented by *vector valued Dirichlet series* to avoid comparing growth just with $\exp\exp z$ as follows.

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(\sigma) < M_g(\sigma\mu) \text{ for all } \sigma > \sigma_0(\mu)\}\\ &= \limsup_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma}.\end{aligned}$$

Similarly, one can define the *relative Ritt lower order* of $f(s)$ with respect to $g(s)$, denoted by $\lambda_g(f)$ in the following manner.

$$\lambda_g(f) = \liminf_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma}.$$

Extending the notion of *relative Ritt order* as introduced by Srivastava [4], next in this paper we introduce *relative Ritt* L^**-order* between two entire functions represented by *vector valued Dirichlet series* as follows.

$$\begin{aligned}\rho_g^{L^*}(f) &= \inf\left\{\mu > 0 : M_f(\sigma) < M_g\left(\sigma e^{L(\sigma)}\mu\right) \text{ for all } \sigma > \sigma_0(\mu)\right\}\\ &= \limsup_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma e^{L(\sigma)}}.\end{aligned}$$

Similarly, one can define the *relative Ritt* L^**-lower order* of $f(s)$ with respect to $g(s)$, denoted by $\lambda_g^{L^*}(f)$ in the following manner.

$$\lambda_g^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma e^{L(\sigma)}} .$$

For entire functions, the notions of their growth indicators such as *Ritt order* is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in different directions using the classical growth indicators. But at that time, the concepts of *relative Ritt order* of entire functions and as well as their technical advantages of not comparing with the growths of $\exp\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of entire functions in the light of their relative growth indicators are the prime concern of this paper. Actually in this paper we establish some newly developed results related to the growth rates of entire functions on the basis of their *relative Ritt* L^**-order* and *relative Ritt* L^**-lower order.*

2 Main results

In this section we present the main results of the paper.

Theorem 1. *If* f, g,h *and* k *be any four entire functions represented by vector valued Dirichlet series such that* $0 < \lambda_h^{L^*}(f) \le \rho_h^{L^*}(f) < \infty$ *and* $0 < \lambda_k^{L^*}(g) \le \rho_k^{L^*}(g) < \infty$ *then*

$$\frac{\lambda_h^{L^*}(f)}{\rho_k^{L^*}(g)} \le \liminf_{\sigma\to\infty} \frac{M_h^{-1} M_f(\sigma)}{M_k^{-1} M_g(\sigma)} \le \frac{\lambda_h^{L^*}(f)}{\lambda_k^{L^*}(g)} \le \limsup_{\sigma\to\infty} \frac{M_h^{-1} M_f(\sigma)}{M_k^{-1} M_g(\sigma)} \le \frac{\rho_h^{L^*}(f)}{\lambda_k^{L^*}(g)} .$$

Proof. From the definition of $\rho_k^{L^*}(g)$ and $\lambda_h^{L^*}(f)$, we have for arbitrary positive ε and for all sufficiently large values of σ that

$$M_h^{-1}M_f(\sigma) \geqslant \left(\lambda_h^{L^*}(f)-\varepsilon\right)\sigma e^{L(\sigma)} \tag{2}$$

and

$$M_k^{-1}M_g(\sigma) \leq \left(\rho_k^{L^*}(g)+\varepsilon\right)\sigma e^{L(\sigma)} . \tag{3}$$

Now from (2) and (3), it follows for all sufficiently large values of σ that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geqslant \frac{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\sigma e^{L(\sigma)}}{\left(\rho_k^{L^*}(g)+\varepsilon\right)\sigma e^{L(\sigma)}} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geqslant \frac{\lambda_h^{L^*}(f)}{\rho_k^{L^*}(g)}. \tag{4}$$

Again for a sequence of values of σ tending to infinity,

$$M_h^{-1}M_f(\sigma) \leq \left(\lambda_h^{L^*}(f)+\varepsilon\right)\sigma e^{L(\sigma)} \tag{5}$$

and for all sufficiently large values of σ,

$$M_k^{-1}M_g(\sigma) \geqslant \left(\lambda_k^{L^*}(g)-\varepsilon\right)\sigma e^{L(\sigma)}. \tag{6}$$

Combining (5) and (6), we get for a sequence of values of σ tending to infinity that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\left(\lambda_h^{L^*}(f)+\varepsilon\right)\sigma e^{L(\sigma)}}{\left(\lambda_k^{L^*}(g)-\varepsilon\right)\sigma e^{L(\sigma)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\lambda_h^{L^*}(f)}{\lambda_k^{L^*}(g)}. \tag{7}$$

Also for a sequence of values of σ tending to infinity that

$$M_k^{-1}M_g(\sigma) \leq \left(\lambda_k^{L^*}(g)+\varepsilon\right)\sigma e^{L(\sigma)}. \tag{8}$$

Now from (2) and (8), we obtain for a sequence of values of σ tending to infinity that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geq \frac{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\sigma e^{L(\sigma)}}{\left(\lambda_k^{L^*}(g)+\varepsilon\right)\sigma e^{L(\sigma)}}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geq \frac{\lambda_h^{L^*}(f)}{\lambda_k^{L^*}(g)}. \tag{9}$$

Also for all sufficiently large values of σ,

$$M_h^{-1}M_f(\sigma) \leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\sigma e^{L(\sigma)}. \tag{10}$$

Now it follows from (6) and (10), for all sufficiently large values of σ that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)\sigma e^{L(\sigma)}}{\left(\lambda_k^{L^*}(g)-\varepsilon\right)\sigma e^{L(\sigma)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\rho_h^{L^*}(f)}{\lambda_k^{L^*}(g)}. \tag{11}$$

Thus the theorem follows from (4), (7), (9) and (11).

Theorem 2. *If f, g, h and k be any four entire functions represented by vector valued Dirichlet series such that $0<\rho_h^{L^*}(f)<\infty$ and $0<\rho_k^{L^*}(g)<\infty$ then*

$$\liminf_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\rho_h^{L^*}(f)}{\rho_k^{L^*}(g)} \leq \limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)}.$$

Proof. From the definition of $\rho_k^{L^*}(g)$, we get for a sequence of values of σ tending to infinity that

$$M_k^{-1}M_g(\sigma) \geqslant \left(\rho_k^{L^*}(g)-\varepsilon\right)\sigma e^{L(\sigma)}. \tag{12}$$

Now from (10) and (12), it follows for a sequence of values of σ tending to infinity that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)\sigma e^{L(\sigma)}}{\left(\rho_k^{L^*}(g)-\varepsilon\right)\sigma e^{L(\sigma)}}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\rho_h^{L^*}(f)}{\rho_k^{L^*}(g)}. \tag{13}$$

Again for a sequence of values of σ tending to infinity,

$$M_h^{-1}M_f(\sigma) \geqslant \left(\rho_h^{L^*}(f)-\varepsilon\right)\sigma e^{L(\sigma)}. \tag{14}$$

So combining (3) and (14), we get for a sequence of values of σ tending to infinity that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geqslant \frac{\left(\rho_h^{L^*}(f)-\varepsilon\right)\sigma e^{L(\sigma)}}{\left(\rho_k^{L^*}(g)+\varepsilon\right)\sigma e^{L(\sigma)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geqslant \frac{\rho_h^{L^*}(f)}{\rho_k^{L^*}(g)}. \tag{15}$$

Thus the theorem follows from (13) and (15).

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

Theorem 3. *If f, g, h and k be any four entire functions represented by vector valued Dirichlet series such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $0 < \lambda_k^{L^*}(g) \leq \rho_k^{L^*}(g) < \infty$ then*

$$\liminf_{\sigma\to\infty} \frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \min\left\{\frac{\lambda_h^{L^*}(f)}{\lambda_k^{L^*}(g)}, \frac{\rho_h^{L^*}(f)}{\rho_k^{L^*}(g)}\right\} \leq \max\left\{\frac{\lambda_h^{L^*}(f)}{\lambda_k^{L^*}(g)}, \frac{\rho_h^{L^*}(f)}{\rho_k^{L^*}(g)}\right\} \leq \limsup_{\sigma\to\infty} \frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} .$$

The proof is omitted.

Theorem 4. *Let f, g and h be any three entire functions represented by vector valued Dirichlet series such that $\rho_k^{L^*}(g) < \infty$. If $\lambda_h^{L^*}(f) = \infty$ then*

$$\lim_{r\to\sigma} \frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} = \infty .$$

Proof. Let us suppose that the conclusion of the theorem do not hold.Then we can find a constant $\beta > 0$ such that for a sequence of values of σ tending to infinity

$$M_h^{-1}M_f(\sigma) \leq \beta M_k^{-1}M_g(\sigma) . \tag{16}$$

Again from the definition of $\rho_k^{L^*}(g)$, it follows that for all sufficiently large values of σ that

$$M_k^{-1}M_g(\sigma) \leq \left(\rho_k^{L^*}(g) + \varepsilon\right) \sigma e^{L(\sigma)} . \tag{17}$$

Thus from (16) and (17), we have for a sequence of values of σ tending to infinity that

$$M_h^{-1}M_f(\sigma) \leq \beta\left(\rho_k^{L^*}(g) + \varepsilon\right) \sigma e^{L(\sigma)},$$

that is,

$$\frac{M_h^{-1}M_f(\sigma)}{\sigma e^{L(\sigma)}} \leq \frac{\beta\left(\rho_k^{L^*}(g) + \varepsilon\right) \sigma e^{L(\sigma)}}{\sigma e^{L(\sigma)}}.$$

Therefore,

$$\liminf_{r\to\sigma} \frac{M_h^{-1}M_f(\sigma)}{\sigma e^{L(\sigma)}} = \lambda_h^{L^*}(f) < \infty .$$

This is a contradiction. This proves the theorem.

Remark. Theorem 4 is also valid with “limit superior” instead of “limit” if $\lambda_h^{L^*}(f) = \infty$ is replaced by $\rho_h^{L^*}(f) = \infty$ and the other conditions remaining the same.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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