New Trends in Mathematical Sciences

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Growth properties of composite analytic functions of several complex variables in unit polydisc under the treatment of slowly changing functions

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Abstract: In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative L^*-order and relative L^*-lower order as compared to their corresponding left and right factors.

Keywords: Entire function, meromorphic function, composition, growth, relative L^*-order, relative L^*-lower order, slowly changing function.

1 Introduction, definitions and notations

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$, is said to be of finite Nevanlinna order [4] if there exist a number μ such that Nevanlinna characteristic function

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta$$

satisfies $T_f(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order ρ_f of f is given by

$$\rho_f = \limsup_{r\to 1} \frac{\log T_f(r)}{-\log(1-r)} .$$

Similarly, the Nevanlinna lower order λ_f of f is given by

$$\lambda_f = \liminf_{r\to 1} \frac{\log T_f(r)}{-\log(1-r)} .$$

Somasundaram and Thamizharasi [6] introduced the notions of L-order (L-lower order) for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. In the line of Somasundaram and Thamizharasi [6], one may introduce the notion of Nevanlinna L-order for an analytic function

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f in the unit disc $U = \{z : |z| < 1\}$ where $L \equiv L\left(\frac{1}{1-r}\right)$ is a positive continuous function in the unit disc U increasing slowly i.e., $L\left(\frac{a}{1-r}\right) \sim L\left(\frac{1}{1-r}\right)$ as $r \to 1$, for every positive constant a, in the following manner.

Definition 1. *If f be analytic in U, then the Nevanlinna L-order ρ_f^L of f is defined as*

$$\rho_f^L = \inf\left\{\mu > 0 : T_f(r) < \left[\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right]^{\mu} \textit{ for all } 0 < r_0(\mu) < r < 1\right\}.$$

Similarly one may define λ_f^L, the Nevanlinna L-lower order of f in the following way.

$$\lambda_f = \liminf_{r\to 1} \frac{\log T_f(r)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)}.$$

The more generalised concept of the Nevanlinna L-order and the Nevanlinna L-lower order an analytic function f in the unit disc U are the Nevanlinna L^*-order and the Nevanlinna L^*-lower order. Their definitions are as follows.

Definition 2. [2] *The Nevanlinna L^*-order $\rho_f^{L^*}$ and Nevanlinna L^*-lower order $\lambda_f^{L^*}$ of an analytic function f in the unit disc U are defined as*

$$\rho_f^{L^*} = \limsup_{r\to 1} \frac{\log T_f(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} \quad \textit{and}$$

$$\lambda_f^{L^*} = \liminf_{r\to 1} \frac{\log T_f(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} \quad \textit{respectively.}$$

Extending the notion of single variables to several variables, let $f(z_1, z_2, \cdots, z_n)$ be a non-constant analytic function of n complex variables $z_1, z_2, \cdots, z_{n-1}$ and z_n in the unit polydisc.

$$U = \left\{(z_1, z_2, \cdots, z_n) : \left|z_j\right| \le 1, j = 1, 2, \cdots, n; r_1 > 0, r_2 > 0, \cdots, r_n > 0\right\}.$$

Now in the line of Nevanlinna L^*-order and Nevanlinna L^*-lower order, in this paper we introduce the Nevanlinna n variables based L^*-order and the Nevanlinna n variables based L^*-lower order for functions of n complex variables analytic in a unit polydisc as follows.

$$_{v_n}\rho_f^{L^*} = \limsup_{r_1, r_2, \cdots, r_n \to 1} \frac{\log T_f(r_1, r_2, \cdots, r_n)}{\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}$$

and

$$_{v_n}\lambda_f^{L^*} = \liminf_{r_1, r_2, \cdots, r_n \to 1} \frac{\log T_f(r_1, r_2, \cdots, r_n)}{\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}$$

where $L \equiv L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)$ is a positive continuous function in the unit polydisc U increasing slowly i.e., $L\left(\frac{a}{1-r_1}, \frac{a}{1-r_2}, \cdots, \frac{a}{1-r_n}\right) \sim L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)$ as $r \to 1$, for every positive constant 'a'. In this paper we study some growth properties of Nevanlinna's Characteristic function relating to the composition of two analytic functions in the unit polydisc on the basis of Nevanlinna n variables based L^*-order and Nevanlinna n variables based L^*-lower order as compared to the growth of their corresponding left and right factors.. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [1], [3] and [5].

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2 Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0 < {}_{v_n}\lambda_{f\circ g}^{L^*} \leq {}_{v_n}\rho_{f\circ g}^{L^*} < \infty$ and $0 < {}_{v_n}\lambda_f^{L^*} \leq {}_{v_n}\rho_f^{L^*} < \infty$. If $L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right) = o\left\{\log T_f(r_1, r_2, \cdots, r_n)\right\}$ as $r_1, r_2, \cdots, r_n \to 1$ then*

$$\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\rho_f^{L^*}} \leq \liminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log T_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)} \leq \frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_f^{L^*}}$$

$$\leq \limsup_{r_1,r_2,\cdots,r_n\to 1} \frac{\log T_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)} \leq \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\lambda_f^{L^*}}.$$

Proof. From the definition of Nevanlinna n variables based L^*-order and Nevanlinna n variables based L^*-lower order of analytic functions in the unit polydisc U, we have for arbitrary positive ε and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots,$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\log T_{f\circ g}(r_1, r_2, \cdots, r_n) \geq \left({}_{v_n}\lambda_{f\circ g}^{L^*} - \varepsilon\right) \log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right], \tag{1}$$

$$\log T_{f\circ g}(r_1, r_2, \cdots, r_n) \geq \left({}_{v_n}\lambda_{f\circ g}^{[m]L^*} - \varepsilon\right)\left[\log\left(\frac{1}{1-r_1}\right) + \log\left(\frac{1}{1-r_2}\right), \ldots, \left(\frac{1}{1-r_n}\right) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right]$$

and

$$\log T_f(r_1, r_2, \cdots, r_n) \leq \left({}_{v_n}\rho_f^{L^*} + \varepsilon\right) \log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]$$

$$\log T_f(r_1, r_2, \cdots, r_n) \leq \left({}_{v_n}\rho_f^{L^*} + \varepsilon\right)\left[\log\left(\frac{1}{1-r_1}\right) + \log\left(\frac{1}{1-r_2}\right), \left(\frac{1}{1-r_n}\right) + L\left(\frac{1}{1-r_1}, \ldots, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right].$$

$$\frac{\log T_f(r_1, r_2, \cdots, r_n)}{\left({}_{v_n}\rho_f^{L^*} + \varepsilon\right)} \leq \log\left(\frac{1}{1-r_1}\right) + \log\left(\frac{1}{1-r_2}\right), \cdots, \log\left(\frac{1}{1-r_n}\right) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right). \tag{2}$$

Now from (1) and (2), it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\log T_{f\circ g}(r_1, r_2, \cdots, r_n) \geq \frac{\left({}_{v_n}\lambda_{f\circ g}^{L^*} - \varepsilon\right)}{\left({}_{v_n}\rho_f^{L^*} + \varepsilon\right)} \log T_f(r_1, r_2, \cdots, r_n)$$

that is,

$$\frac{\log T_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)} \geq \frac{\left({}_{v_n}\lambda_{f\circ g}^{L^*} - \varepsilon\right)}{\left({}_{v_n}\rho_f^{L^*} + \varepsilon\right)} \cdot \frac{\log T_f(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)}$$

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that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}-\varepsilon}{{}_{v_n}\rho_f^{L^*}+\varepsilon}}{1+\frac{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}{\log T_f(r_1,r_2,\cdots,r_n)}}. \tag{3}$$

Since $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)=o\left\{\log T_f(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n\to 1$, it follows from (3) that

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{{}_{v_n}\lambda_{f\circ g}^{L^*}-\varepsilon}{{}_{v_n}\rho_f^{L^*}+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\rho_f^{L^*}}. \tag{4}$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n)\leq \left({}_{v_n}\lambda_{f\circ g}^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]$$

that is,

$$\begin{aligned}\log T_{f\circ g}(r_1,r_2,\cdots,r_n)\leq \left({}_{v_n}\lambda_{f\circ g}^{L^*}+\varepsilon\right)\Bigg[&\log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots\log\left(\frac{1}{1-r_n}\right)\\ &+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\Bigg]\end{aligned} \tag{5}$$

and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$,

$$\log T_f(r_1,r_2,\cdots,r_n)\geq \left({}_{v_n}\lambda_f^{L^*}-\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]$$

that is,

$$\log T_f(r_1,r_2,\cdots,r_n)\geq \left({}_{v_n}\lambda_f^{L^*}-\varepsilon\right)\left[\log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots,\log\left(\frac{1}{1-r_n}\right)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right]$$

that is,

$$\frac{\log T_f(r_1,r_2,\cdots,r_n)}{\left({}_{v_n}\lambda_f^{L^*}-\varepsilon\right)}\geq \log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots,\log\left(\frac{1}{1-r_n}\right)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right). \tag{6}$$

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Combining (5) and (6), we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \le \frac{\left(v_n\lambda_{f\circ g}^{L^*}+\varepsilon\right)}{\left(v_n\lambda_{f}^{L^*}-\varepsilon\right)}\log T_f(r_1,r_2,\cdots,r_n)$$

that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{\left(v_n\lambda_{f\circ g}^{L^*}+\varepsilon\right)}{\left(v_n\lambda_{f}^{L^*}-\varepsilon\right)}\cdot\frac{\log T_f(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}$$

that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{\frac{v_n\lambda_{f\circ g}^{L^*}+\varepsilon}{v_n\lambda_{f}^{L^*}-\varepsilon}}{1+\frac{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}{\log T_f(r_1,r_2,\cdots,r_n)}}. \tag{7}$$

As $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)=o\left\{\log T_f(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n\to 1$ we get from (7) that

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{v_n\lambda_{f\circ g}^{L^*}+\varepsilon}{v_n\lambda_{f}^{L^*}-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{v_n\lambda_{f\circ g}^{L^*}}{v_n\lambda_{f}^{L^*}}. \tag{8}$$

Also for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log T_f(r_1,r_2,\cdots,r_n) \le \left(v_n\lambda_{f}^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]$$

that is,

$$\log T_f(r_1,r_2,\cdots,r_n) \le \left(v_n\lambda_{f}^{L^*}+\varepsilon\right)\left[\log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots,\log\left(\frac{1}{1-r_n}\right)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right]$$

that is

$$\frac{\log T_f(r_1,r_2,\cdots,r_n)}{\left(v_n\lambda_{f}^{L^*}+\varepsilon\right)} \le \log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots\log,\left(\frac{1}{1-r_n}\right)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right). \tag{9}$$

Now from (1) and (9), we obtain for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \ge \frac{\left(v_n\lambda_{f\circ g}^{L^*}-\varepsilon\right)}{\left(v_n\lambda_{f}^{L^*}+\varepsilon\right)}\log T_f(r_1,r_2,\cdots,r_n)$$

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that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{\left({}_{v_n}\lambda^{L^*}_{f\circ g}-\varepsilon\right)}{\left({}_{v_n}\lambda^{L^*}_{f}+\varepsilon\right)}\cdot\frac{\log T_f(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}$$

that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{\frac{{}_{v_n}\lambda^{L^*}_{f\circ g}-\varepsilon}{{}_{v_n}\lambda^{L^*}_{f}+\varepsilon}}{1+\frac{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}{\log T_f(r_1,r_2,\cdots,r_n)}}. \tag{10}$$

In view of the condition $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)=o\left\{\log T_f(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n\to 1$, we obtain from (10) that

$$\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{{}_{v_n}\lambda^{L^*}_{f\circ g}-\varepsilon}{{}_{v_n}\lambda^{L^*}_{f}+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{{}_{v_n}\lambda^{L^*}_{f\circ g}}{{}_{v_n}\lambda^{L^*}_{f}}. \tag{11}$$

Also for all sufficiently large values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$,

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n)\leq\left({}_{v_n}\rho^{L^*}_{f\circ g}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]$$

that is,

$$\begin{aligned}\log T_{f\circ g}(r_1,r_2,\cdots,r_n)\leq\left({}_{v_n}\rho^{L^*}_{f\circ g}+\varepsilon\right)&\left[\log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots,\log\left(\frac{1}{1-r_n}\right)\right.\\&\left.+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right]\end{aligned} \tag{12}$$

So from (6) and (12), it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n)\leq\frac{\left({}_{v_n}\rho^{L^*}_{f\circ g}+\varepsilon\right)}{\left({}_{v_n}\lambda^{L^*}_{f}-\varepsilon\right)}\log T_f(r_1,r_2,\cdots,r_n)$$

that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \leq \frac{\left({}_{v_n}\rho^{L^*}_{f\circ g}+\varepsilon\right)}{\left({}_{v_n}\lambda^{L^*}_{f}-\varepsilon\right)}\cdot\frac{\log T_f(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}$$

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that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{\frac{{}_{v_n}\rho_{f\circ g}^{L^*}+\varepsilon}{{}_{v_n}\lambda_f^{L^*}-\varepsilon}}{1+\frac{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}{\log T_f(r_1,r_2,\cdots,r_n)}}. \tag{13}$$

Using $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)=o\left\{\log T_f(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n\to 1$, we obtain from (13) that

$$\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{{}_{v_n}\rho_{f\circ g}^{L^*}+\varepsilon}{{}_{v_n}\lambda_f^{L^*}-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\lambda_f^{L^*}}. \tag{14}$$

Thus the theorem follows from (4), (8), (11) and (14).

Similarly in view of Theorem 1 , we may state the following theorem without its proof for the right factor g of the composite function $f\circ g$:

Theorem 2. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0<{}_{v_n}\lambda_{f\circ g}^{L^*}\le {}_{v_n}\rho_{f\circ g}^{L^*}<\infty$ and $0<{}_{v_n}\lambda_g^{L^*}\le {}_{v_n}\rho_g^{L^*}<\infty$. If $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)=o\left\{\log T_g(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n\to 1$ then*

$$\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\rho_g^{L^*}} \le \liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_g^{L^*}}$$
$$\le \limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\lambda_g^{L^*}}.$$

Theorem 3. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0<{}_{v_n}\rho_{f\circ g}^{L^*}<\infty$ and $0<{}_{v_n}\rho_f^{L^*}<\infty$. If $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)=o\left\{\log T_f(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n\to 1$ then*

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_f^{L^*}}$$
$$\le \limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}.$$

Proof. From the definition of ${}_{v_n}\rho_f^{L^*}$, the Nevanlinna n variables based L^*-order of analytic function f in the unit disc U, we get for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots,$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_f(r_1,r_2,\cdots,r_n)\ge\left({}_{v_n}\rho_f^{L^*}-\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]$$

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that is,

$$\log T_f(r_1,r_2,\cdots,r_n) \geq \left({}_{v_n}\rho_f^{L^*}-\varepsilon\right)\left[\log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots,\log\left(\frac{1}{1-r_n}\right)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right]$$

that is,

$$\frac{\log T_f(r_1,r_2,\cdots,r_n)}{\left({}_{v_n}\rho_f^{L^*}-\varepsilon\right)} \geq \log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots\log\left(\frac{1}{1-r_n}\right)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right). \tag{15}$$

Now from (12) and (15), it follows for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \leq \frac{\left({}_{v_n}\rho_{f\circ g}^{L^*}+\varepsilon\right)}{\left({}_{v_n}\rho_f^{L^*}-\varepsilon\right)}\log T_f(r_1,r_2,\cdots,r_n)$$

that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \leq \frac{\left({}_{v_n}\rho_{f\circ g}^{L^*}+\varepsilon\right)}{\left({}_{v_n}\rho_f^{L^*}-\varepsilon\right)}\cdot\frac{\log T_f(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}.$$

Thus,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \leq \frac{\frac{{}_{v_n}\rho_{f\circ g}^{L^*}+\varepsilon}{{}_{v_n}\rho_f^{L^*}-\varepsilon}}{1+\frac{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}{\log T_f(r_1,r_2,\cdots,r_n)}}. \tag{16}$$

Using $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)=o\left\{\log T_f(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n\to 1$, we obtain from (16) that

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \leq \frac{{}_{v_n}\rho_{f\circ g}^{L^*}+\varepsilon}{{}_{v_n}\rho_f^{L^*}-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \leq \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_f^{L^*}}. \tag{17}$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \geq \left({}_{v_n}\rho_{f\circ g}^{L^*}-\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]$$

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that is,

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \geq \left({}_{v_n}\rho_{f\circ g}^{[m]L^*} - \varepsilon\right)\left[\log\left(\frac{1}{1-r_1}\right)+\log\left(\frac{1}{1-r_2}\right),\cdots,\log\left(\frac{1}{1-r_n}\right)\right.$$
$$\left. +L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right] \tag{18}$$

So combining (2) and (18), we get for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \geq \frac{\left({}_{v_n}\rho_{f\circ g}^{L^*} - \varepsilon\right)}{\left({}_{v_n}\rho_{f}^{L^*} + \varepsilon\right)}\log T_f(r_1,r_2,\cdots,r_n)$$

that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{\left({}_{v_n}\rho_{f\circ g}^{L^*} - \varepsilon\right)}{\left({}_{v_n}\rho_{f}^{L^*} + \varepsilon\right)}\cdot\frac{\log T_f(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}.$$

Thus,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{\frac{{}_{v_n}\rho_{f\circ g}^{L^*}-\varepsilon}{{}_{v_n}\rho_{f}^{L^*}+\varepsilon}}{1+\frac{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}{\log T_f(r_1,r_2,\cdots,r_n)}}. \tag{19}$$

Since $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right) = o\left\{\log T_f(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n \to 1$, it follows from (19) that

$$\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{{}_{v_n}\rho_{f\circ g}^{L^*}-\varepsilon}{{}_{v_n}\rho_{f}^{L^*}+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \geq \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_{f}^{L^*}}. \tag{20}$$

Thus the theorem follows from (17) and (20).

Theorem 4. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0 < {}_{v_n}\rho_{f\circ g}^{L^*} < \infty$ and $0 < {}_{v_n}\rho_{g}^{L^*} < \infty$. If $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right) = o\left\{\log T_g(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n \to 1$ then*

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \leq \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_{g}^{L^*}}$$
$$\leq \limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}.$$

The proof is omitted . The following theorem is a natural consequence of Theorem 1 and Theorem 3.

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Theorem 5. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0 < {}_{v_n}\lambda_{f\circ g}^{L^*} \le {}_{v_n}\rho_{f\circ g}^{L^*} < \infty$ and $0 < {}_{v_n}\lambda_{f}^{L^*} \le {}_{v_n}\rho_{f}^{L^*} < \infty$. If $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right) = o\left\{\log T_f(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n \to 1$ then*

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \min\left\{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_{f}^{L^*}},\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_{f}^{L^*}}\right\}$$
$$\le \max\left\{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_{f}^{L^*}},\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_{f}^{L^*}}\right\} \le \limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}\,.$$

The proof is omitted. Combining Theorem 2 and Theorem 4, we may state the following theorem:

Theorem 6. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0 < {}_{v_n}\lambda_{f\circ g}^{L^*} \le {}_{v_n}\rho_{f\circ g}^{L^*} < \infty$ and $0 < {}_{v_n}\lambda_{g}^{L^*} \le {}_{v_n}\rho_{g}^{L^*} < \infty$. If $L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right) = o\left\{\log T_g(r_1,r_2,\cdots,r_n)\right\}$ as $r_1,r_2,\cdots,r_n \to 1$ then*

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)} \le \min\left\{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_{g}^{L^*}},\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_{g}^{L^*}}\right\}$$
$$\le \max\left\{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_{g}^{L^*}},\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_{g}^{L^*}}\right\} \le \limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)+L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}\,.$$

Theorem 7. *Let f be a non-constant analytic function of n complex variables in the unit polydisc U with ${}_{v_n}\rho_{f}^{L^*} < \infty$. Also let g be a non-constant analytic function of n complex variables in the unit polydisc U. If ${}_{v_n}\lambda_{f\circ g}^{L^*} = \infty$ then*

$$\lim_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold.Then we can find a constant $\beta > 0$ such that for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \le \beta \log T_f(r_1,r_2,\cdots,r_n)\ . \tag{21}$$

Again from the definition of ${}_{v_n}\rho_{f}^{L^*}$, it follows that for all sufficiently large values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\log T_f(r_1,r_2,\cdots,r_n) \le \left({}_{v_n}\rho_{f}^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]\ .$$

Thus from (21) and above, we have for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \le \beta\left({}_{v_n}\rho_{f}^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]$$

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that is,

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]} \leq \frac{\beta\left({}_{v_n}\rho_f^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}{\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}.$$

Therefore,

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]} = {}_{v_n}\lambda_{f\circ g}^{L^*} < \infty\,.$$

This is a contradiction. This proves the theorem.

Remark. Theorem 7 is also valid with “limit superior” instead of “limit” if ${}_{v_n}\lambda_{f\circ g}^{L^*}=\infty$ is replaced by ${}_{v_n}\rho_{f\circ g}^{L^*}=\infty$ and the other conditions remaining the same.

Corollary 1. *Under the assumptions of Theorem 7 or Remark 2,*

$$\lim_{r_1,r_2,\cdots,r_n\to 1}\frac{T_{f\circ g}(r_1,r_2,\cdots,r_n)}{T_f(r_1,r_2,\cdots,r_n)}=\infty\,.$$

Proof. From Theorem 7 or Remark 2, we obtain for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ and for $K>1$ that

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) > KT_f(r_1,r_2,\cdots,r_n)\,,$$
$$T_{f\circ g}(r_1,r_2,\cdots,r_n) > \left\{T_f(r_1,r_2,\cdots,r_n)\right\}^K,$$

from which the corollary follows.

Theorem 8. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that ${}_{v_n}\rho_g^{L^*}<\infty$ and ${}_{v_n}\lambda_{f\circ g}^{L^*}=\infty$. Then*

$$\lim_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)}=\infty\,.$$

We omit the proof of Theorem 8 because it can be carried out in the line of Theorem 7.

Remark. Theorem 8 is also valid with “limit superior” instead of “limit” if ${}_{v_n}\lambda_{f\circ g}^{L^*}=\infty$ is replaced by ${}_{v_n}\rho_{f\circ g}^{L^*}=\infty$ and the other conditions remaining the same.

In the line of Corollary 1, we may easily verify the following:

Corollary 2. *Under the assumptions of Theorem 8 or Remark 2,*

$$\lim_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)}=\infty\,.$$

The proof is omitted because it may be carried out in the line of Corollary 1.

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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