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On spherical indicatrices partially null curves in R_2^4

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Abstract: In this study, we investigate spherical indicatrix of partially null curves in Semi-Riemann space R_2^4 . First, we calculate Frenet apparatus of tangent, normal, first and second binormal indicatrices. Moreover, we characterized spacelike and timelike null curves in R_2^4 and we give necessary condition the curve lie on pseudo hypresphere.

Keywords: Minkowski 4-space, null curves, spherical indicatrices, partially null curves, timelike curves, spacelike curves.

1 Introduction

The local theory of space curves are mainly developed by Serret-Frenet theorem which expresses the derivative of a geometrically chosen basis of $\mathbb{R}^3$ by the aid of itself is proved. Then it is observed that by the solution of some special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, Mannheim curves, constant breadth curves, slant helix, general helix, involutes and evolutes are investigated. One of the mentioned works is spherical images of a regular curve in the Euclidian space. It is a well-known concept in the local differential geometry of the curves. Such curves are obtained in terms of the Serret-Frenet vector fields [1].

Spherical indicatrices are investigated in four dimensional S. Yilmaz studied spherical indicatrices of curves in Euclidean 4 space and Lorenzian 4 space [9,10]. Moreover, Yılmaz and Turgut studied spherical images according to Type-1 and Type-2 Bishop frame in Euclidean and Minkowski space [12,13]. Later Yımaz and Savcı given characterizations of spherical images according to Type-2 Bishop frame in Minkowski 3-space. Additionally, Yılmaz characterized spherical images according to Type-1 Bishop frame in Minkowski 3-space [7,8].

In analogy with the Euclidean curves and their Frenet equations, the Frenet equations of a spacelike and a timelike curves, lying fully in the semi–Euclidean space E_2^4, are given in [4]. For the moving Frenet frames along such curves, it is assumed to be orthonormal, consisting of four non–null vector fields $\{T, N, B_1, B_2\}$, which are called respectively the tangent, the principal normal, the first binormal and the second binormal vector field. In particular, when the Frenet frame along a spacelike or a timelike curve contains a null vectors, such curve is said to be a pseudo null curve or a partially null curve [6].

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2 Preliminaries

The semi–Euclidean space E_2^4 is the Euclidean 4-space E^4 equipped with an indefinite flat metric given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^3. Since g is an indefinite metric recall that a vector $v \in E_2^4$ can be one of three causal characters; it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_2^4 can locally be space-like, time-like or null (light-like) if all of its velocity vector $\alpha^{'}$ is respectively space-like, time-like or null (light-like) for every $s \in J \in \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in E_2^4$ is given by $\|a\| = \sqrt{|< a, a >|}$. The curve α is called a unit speed curve if its velocity vector $\alpha^{'}$ satisfies $\|\alpha^{'}\| = \mp 1$. For any vectors $u, w \in E_2^4$, they are said to be orthogonal if and only if $< u, w >= 0$. The Lorentzian pseudo hyper sphere of center $m = (m_1, m_2, m_3, m_4)$ and radius $r \in R^+$ in the space E_2^4 defined by

$$S_2^3(m,r) = \left\{ \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E_2^4 \mid g(\alpha - m, \alpha - m) = r^2 \right\}.$$

Denote by $\{T, N, B_1, B_2\}$ the moving Frenet frame along curve α in the space E_2^4. Let α be a space-like curve in the space E_2^4, the Frenet formulae are given as

$$\begin{bmatrix} T^{'} \\ N^{'} \\ B_1^{'} \\ B_2^{'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{1}$$

where k_1, k_2 and k_3 are the first, second and third curvatures of the curve α, respectively and

$$g(T,T) = 1,\ g(N,N) = -1,\ g(B_1,B_1) = g(B_2,B_2) = 0,\ g(B_1,B_2) = 1.$$

Let α be a time-like curve in the space E_2^4, the Frenet formulae are given as

$$\begin{bmatrix} T^{'} \\ N^{'} \\ B_1^{'} \\ B_2^{'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2}$$

where k_1, k_2 and k_3 are the first, second and third curvatures of the curve α, respectively and [6]

$$g(T,T) = -1,\ g(N,N) = 1,\ g(B_1,B_1) = g(B_2,B_2) = 0,\ g(B_1,B_2) = 1.$$

Theorem 1. *Let α unit speed curve in four dimensional Lorentzian space. In this case*

$$\begin{aligned} T &= \frac{\alpha^{'}}{\|\alpha^{'}\|} \\ N &= \frac{\|\alpha^{'}\|^2 \alpha^{''} - < \alpha^{'}, \alpha^{''} > \alpha^{'}}{\left\| \|\alpha^{'}\|^2 \alpha^{''} - < \alpha^{'}, \alpha^{''} > \alpha^{'} \right\|} \\ B_1 &= \sigma(B_2 \wedge T \wedge N) \\ B_2 &= \frac{T \wedge N \wedge \alpha^{'''}}{\|T \wedge N \wedge \alpha^{'''}\|} \end{aligned} \tag{3}$$

where $\sigma = +1$ if curve is spacelike, $\sigma = -1$ if curve is timelike. First, second and third curvature of curve α respectively

$$k_1 = \frac{\left\| \|\alpha^{'}\|^2 \alpha^{''} - <\alpha^{'}, \alpha^{''}> \alpha^{'} \right\|}{\|\alpha^{'}\|^4}$$
$$k_2 = \frac{\|T \wedge N \wedge \alpha^{'''}\| \, \|\alpha^{'}\|}{\left\| \|\alpha^{'}\|^2 \alpha^{''} - <\alpha^{'}, \alpha^{''}> \alpha^{'} \right\|} \tag{4}$$
$$k_3 = \frac{<\alpha^{(IV)}, B_2>}{\|T \wedge N \wedge \alpha^{'''}\|}$$

Definition 1. *Let $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$ and $z = (z_1, z_2, z_3, z_4)$ be vectors in E_2^4. The vectors product in E_2^4 is defined with the determinant*

$$x \wedge y \wedge z = -\det \begin{bmatrix} -e_1 & -e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

where e_1, e_2, e_3 and e_4 are mutually orthogonal vectors satisfying equations [9]

$$e_1 \wedge e_2 \wedge e_3 = e_4, \; e_2 \wedge e_3 \wedge e_4 = e_1, \; e_3 \wedge e_4 \wedge e_1 = e_2, \; e_4 \wedge e_1 \wedge e_2 = e_3.$$

3 Spherical indicatrices of spacelike partially null curve in E_2^4

In this section, we examine spacelike curve with normal is timelike, first and second binormal are null vectors.

Theorem 2. *Let $\alpha = \alpha(s)$ be spacelike partially null unit speed curve with normal is timelike, first and second binormal are null vectors and $k_1(s)$, $k_2(s)$ and $k_3(s)$ first, second and third curvature of $\alpha(s)$ respectively in E_2^4. If the curve $\alpha(s)$ is pseudo hyper spherical curve then*

$$k_1^{''} = \frac{k_1^{'}}{k_1 k_2}[(k_1 k_2)^{'} + k_2(k_1^{'} + k_1 k_3)]$$

where $k_1 \neq 0$, $k_2 \neq 0$ and $k_3 \neq 0$.

Proof. Equation of pseudo hyper sphere is defined by

$$(\overrightarrow{C} - \overrightarrow{\alpha}(s)) \cdot (\overrightarrow{C} - \overrightarrow{\alpha}(s)) = r^2$$

where r is radius and C is centre of pseudo hyper sphere. If the curve $\alpha(s)$ is pseudo hyper spherical curve then

$$f^{'}(s) = f^{''}(s) = f^{'''}(s) = f^{(\mathrm{IV})}(s) = 0$$

where $f(s) = (\overrightarrow{\alpha}(s) - \overrightarrow{C})^2 - r^2 = 0$ then we can write

$$\begin{aligned} &\text{if } f^{'}(s) = 0, &&\text{then } (\overrightarrow{\alpha} - \overrightarrow{C}) \cdot \overrightarrow{T} = 0 \\ &\text{if } f^{''}(s) = 0, &&\text{then } (\overrightarrow{\alpha} - \overrightarrow{C}) \cdot k_1 \overrightarrow{N} - 1 = 0 \\ &\text{if } f^{'''}(s) = 0, &&\text{then } (\overrightarrow{\alpha} - \overrightarrow{C}) \cdot [k_1 k_2 \overrightarrow{B}_1 + k_1^{'} \overrightarrow{N}] = 0 \\ &\text{if } f^{(\mathrm{IV})}(s) = 0, &&\text{then } (\overrightarrow{\alpha} - \overrightarrow{C}) \cdot [k_1^{''} \overrightarrow{N} + \{(k_1 k_2)^{'} + k_1^{'} k_2 + k_1 k_2 k_3\} \overrightarrow{B}_1] = 0. \end{aligned} \tag{5}$$

Using equations (1) and (5), we get

$$(\overrightarrow{\alpha}-\overrightarrow{C})\cdot\overrightarrow{N}=\tfrac{1}{k_1},\qquad (\overrightarrow{\alpha}-\overrightarrow{C})\cdot\overrightarrow{B}_1=\tfrac{k_1'}{k_1^2k_2}, \tag{6}$$

substituting equations (6) into equation $(5)_4$ we obtain

$$k_1''=\frac{k_1'}{k_1k_2}[(k_1k_2)'+k_2(k_1'+k_1k_3)].$$

Let $\overrightarrow{C}=\theta_1(s)T+\theta_2(s)N+\theta_3(s)B_1+\theta_4(s)B_2$ is vectorial equation of center of pseudo hyper sphere. Using equations (5) and (6), we obtain

$$\overrightarrow{C}=\overrightarrow{\alpha}+\frac{1}{k_1}\overrightarrow{N}-\frac{k_1'}{k_1^2k_2}\overrightarrow{B}_2.$$

Theorem 3. *Let $\alpha=\alpha(s)$ be spacelike (or timelike) partially null unit speed curve with normal is timelike (or spacelike), first and second binormal are null vectors and $k_1(s)$, $k_2(s)$ and $k_3(s)$ first, second and third curvature of $\alpha(s)$ respectively in E_2^4. Then normal vector N of $\alpha(s)$ satisfies a vectorial differential equations as follow*

$$\frac{d^2y}{ds^2}+[-\frac{k_2'}{k_2}-k_3]\frac{dy}{ds}-k_1^2y+[\frac{k_1k_2'}{k_2}-k_1'-k_1k_3]\int_0^s k_1(s)y(s)ds=0$$

where $k_2\neq 0$ and $y(s)=(\dfrac{T'(s)}{k_1(s)})$.

Proof. Let $\alpha=\alpha(s)$ be spacelike from equations $(1)_1$ and $(1)_2$, we get

$$N=\tfrac{T'}{k_1}$$
$$B_1=\frac{1}{k_2}[(\tfrac{T'}{k_1})'-k_1T]$$

taking the derivative above equations and using equations $(1)_3$ we get

$$\left(\frac{1}{k_2}[(\frac{T'}{k_1})'-k_1T]\right)'=\frac{k_3}{k_2}[(\frac{T'}{k_1})'-k_1T]$$

we arrangement this equations and taking into consideration $y=(\frac{T'}{k_1})$ we have

$$\frac{d^2y}{ds^2}+[-\frac{k_2'}{k_2}-k_3]\frac{dy}{ds}-k_1^2y+[\frac{k_1k_2'}{k_2}-k_1'-k_1k_3]\int_0^s k_1(s)y(s)ds=0$$

where $T=\int_0^s k_1(s)y(s)ds$.

3.1 Tangent indicatrices of spacelike partially null curve in E_2^4

Let $\alpha = \alpha(s)$ be spacelike partially null unit speed curve in E_2^4. By means of Frenet equations, let us calculate following differentiations respect to s.

$$
\begin{aligned}
&\tfrac{d\alpha(s)}{ds} = T(s) = \beta(s) \\
&\beta^{'} = k_1 N \\
&\beta^{''} = k_1^2 T + k_1^{'} N + k_1 k_2 B_1 \\
&\beta^{'''} = 3k_1 k_1^{'} T + (k_1^3 + k_1^{''})N + (2k_1^{'} k_2 + k_1 k_2^{'} + k_1 k_2 k_3) B_1 .
\end{aligned} \tag{7}
$$

From the equations (1), (3) and (7) we have tangent and normal indicatrices

$$T_\beta = \frac{k_1 N}{k_1} = N$$

$$N_\beta = \frac{k_1^2 T + 2k_1^{'} N + k_1 k_2 B_1}{\sqrt{\left|k_1^4 - (2k_1^{'})^2\right|}}$$

using, the equations (1), (4) and (7), we can obtain first and second curvatures of curve β

$$k_{1\beta} = \frac{\sqrt{\left|k_1^2 - (2k_1^{'})^2\right|}}{k_1^2}$$

$$k_{2\beta} = 0.$$

Since $\|T \wedge N \wedge \beta^{'''}\| = 0$, $B_{1\beta}$, $B_{2\beta}$ and $k_{3\beta}$ are not defined.

3.2 Normal indicatrices of spacelike partially null curve in E_2^4

Let $\alpha = \alpha(s)$ be spacelike partially null unit speed curve in E_2^4. By means of Frenet equations, let us calculate following differentiations respect to s.

$$
\begin{aligned}
&\varphi(s) = N(s) \\
&\varphi^{'} = k_1 T + k_2 B_1 \\
&\varphi^{''} = k_1^{'} T + k_1^2 N + (k_2^{'} + k_2 k_3) B_1 \\
&\varphi^{'''} = (k_1^3 + k_1^{''})T + 3k_1 k_1^{'} N + (k_1^2 k_2 + k_2^{''} + (k_2 k_3)^{'} + k_2^{''} k_3 + k_2 k_3^2) B_1
\end{aligned} \tag{8}
$$

From the equations (1), (3) and (8), we have

$$T_\varphi = \frac{k_1 T + k_2 B_1}{k_1}$$

$$N_\varphi = \frac{k_1^3 N + (k_1 k_2^{'} - k_1^{'} k_2 + k_1 k_2 k_3) B_1}{k_1^3}$$

Using, the equations (1), (4) and (8), we can obtain first and second curvatures of curve φ

$$k_{1\varphi} = \frac{k_1^4}{k_1^4} = 1$$
$$k_{2\varphi} = 0.$$

Since $\|T \wedge N \wedge \varphi'''\| = 0$, $B_{1\varphi}$, $B_{2\varphi}$ and $k_{3\varphi}$ are not defined.

3.3 First binormal indicatrices of spacelike partially null curve in E_2^4

Let $\alpha = \alpha(s)$ be spacelike partially null unit speed curve in E_2^4. By means of Frenet equations, let us calculate following differentiations respect to s.

$$\begin{aligned}
&\lambda(s) = B_1(s)\\
&\lambda' = k_3 B_1\\
&\lambda'' = (k_3' + k_3^2) B_1\\
&\lambda''' = (k_3'' + 3k_3k_3' + k_3^3) B_1
\end{aligned} \tag{9}$$

From the equations (1), (3) and (9), first binormal indicatrices are undefined.

3.4 Second binormal indicatrices of spacelike partially null curve in E_2^4

Let $\alpha = \alpha(s)$ be spacelike partially null unit speed curve in E_2^4. By means of Frenet equations, let us calculate following differentiations respect to s.

$$\begin{aligned}
&\psi(s) = B_2(s)\\
&\psi' = k_2 N - k_3 B_2\\
&\psi'' = k_1k_2 T + (k_2' + k_2k_3) N + k_1^2 B_1 + (k_3^2 - k_3') B_2\\
&\psi''' = a_1(s) T + a_2(s) N + a_3(s) B_1 + a_4(s) B_2
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&a_1(s) = k_1'k_2 + 2k_1k_2' - k_1k_2k_3\\
&a_2(s) = k_1^2k_2 + (k_3' - k_2k_3)' + k_2(k_3^2 - k_3')\\
&a_3(s) = 3k_2k_2'\\
&a_4(s) = (k_3^2 - k_3')' - k_3(k_3^2 - k_3')
\end{aligned} \tag{11}$$

and fourth differential of ψ

$$\begin{aligned}
\psi^{(IV)}(s) = {} & (a_1' + k_1a_2) T + (k_1a_1 + a_2' + k_2a_4) N\\
& + (k_2a_2 + a_3' + k_3a_3) B_1 + (a_4' - k_3a_4) B_2
\end{aligned} \tag{12}$$

From the equations (1), (3), (10) and (11), we have

$$T_\psi = \frac{k_2 N - k_3 B_2}{k_2}$$

$$N_\psi = \frac{k_2b_1 T + k_2b_2 N + k_2b_3 B_1 + b_4 B_2}{k_2\sqrt{\left|k_1^2k_2^2 - [2(k_2' + k_2k_3)]^2 + (k_2k_3^2 - (k_2k_3)')\right|}}$$

where $b_1(s) = k_1k_2$, $b_2(s) = 2(k_2^{'} + k_2k_3) + k_1^2k_3$, $b_3(s) = k_2^2$, $b_4(s) = k_2k_3^2 - (k_2k_3)^{'}$ to obtain B_2 we need $T \wedge N \wedge \alpha^{''''}$ if we product tree vectors as cross product, we get

$$T \wedge N \wedge \alpha^{''''} = c_1(s)T + c_2(s)N + c_3(s)B_1 + c_4(s)B_2$$

where

$$\begin{aligned} c_1(s) &= k_3b_3a_2 - (k_3b_2 + k_2b_4)a_3 + k_2b_3a_4 \\ c_2(s) &= k_3b_3a_1 - k_2b_1a_3 \\ c_3(s) &= (k_2b_4 + k_3b_2)a_{31} - k_3b_1a_2 - k_2b_1a_4 \\ c_4(s) &= k_3b_3a_1 - k_2b_1a_3 \end{aligned} \tag{13}$$

equations (1), (3), (10), (11), (12) and (13), we have $B_{2\psi}$ and $B_{1\psi}$

$$\begin{aligned} B_{2\psi} = &\frac{c_1}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}T + \frac{c_2}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}N \\ &-\frac{c_3}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}B_1 - \frac{c_4}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}B_2 \end{aligned}$$

and

$$\begin{aligned} B_{1\psi} = &\frac{1}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}\{[-k_3b_2c_3 - k_2b_3c_3 - k_2b_4c_3 + k_3b_3c_2]T \\ &+[k_3b_3c_1 - k_3b_1c_3]N \\ &-[k_2b_4c_1 - k_3b_1c_2 - k_2b_2c_4 + k_3b_2c_1]B_1 \\ &-[k_2b_3c_1 - k_2b_1c_3]B_2 \end{aligned}$$

From the equations (1), (3) (10), (11), (12) and (13), we get curvatures of ψ as follow

$$k_{1\psi} = \frac{\sqrt{|k_1^2k_2^2 - [2(k_2^{'} + k_2k_3)]^2 + k_2(k_2k_3^2 - (k_2k_3)^{'})]^2|}}{k_2^2}$$

$$k_{2\psi} = \frac{\sqrt{|k_1^2k_2^2 - [2(k_2^{'} + k_2k_3)]^2 + k_2(k_2k_3^2 - (k_2k_3)^{'})]^2|}}{k_2\sqrt{|k_1^2k_2^2 - [2(k_2^{'} + k_2k_3)]^2 + (k_2k_3^2 - (k_2k_3)^{'})|}}$$

$$\begin{aligned} k_{3\psi} = &\frac{1}{k_2}\{c_1(a_1^{'} + k_1a_2) - c_2(k_1a_1 + a_2^{'} + k_2a_4) \\ &-c_3(a_4^{'} - k_3a_4) - c_4(k_2a_2 + a_3^{'} + k_3a_3)\} \end{aligned}$$

4 Spherical indicatrices of timelike partially null curve in E_2^4

In this section, we examine timelike curve with normal is spacelike, first and second binormal are null vectors.

Theorem 4. *Let $\alpha = \alpha(s)$ be timelike partially null unit speed curve with normal is spacelike, first and second binormal are null vectors and $k_1(s)$, $k_2(s)$ and $k_3(s)$ first, second and third curvature of $\alpha(s)$ respectively in E_2^4. If the curve $\alpha(s)$ is pseudo hyper spherical curve then*

$$k_1^{''} = \frac{k_1^{'}}{k_1k_2}[(k_1k_2)^{'} + k_2(k_1^{'} + k_1k_3)]$$

where $k_1 \neq 0$, $k_2 \neq 0$ and $k_3 \neq 0$.

Proof. Equation of pseudo hyper sphere is defined by

$$(\overrightarrow{C}-\overrightarrow{\alpha}(s))\cdot(\overrightarrow{C}-\overrightarrow{\alpha}(s))=r^2$$

where r is radius and C is centre of pseudo hyper sphere. If the curve $\alpha(s)$ is pseudo hyper spherical curve then

$$f^{\prime}(s)=f^{\prime\prime}(s)=f^{\prime\prime\prime}(s)=f^{(\mathrm{IV})}(s)=0$$

where $f(s)=(\overrightarrow{\alpha}(s)-\overrightarrow{C})^2-r^2$ then we can write

$$\begin{array}{ll} \text{if } f^{\prime}(s)=0, & \text{then } (\overrightarrow{\alpha}-\overrightarrow{C})\cdot\overrightarrow{T}=0 \\ \text{if } f^{\prime\prime}(s)=0, & \text{then } (\overrightarrow{\alpha}-\overrightarrow{C})\cdot k_1\overrightarrow{N}+1=0 \\ \text{if } f^{\prime\prime\prime}(s)=0, & \text{then } (\overrightarrow{\alpha}-\overrightarrow{C})\cdot[k_1^{\prime}\overrightarrow{N}+k_1k_2\overrightarrow{B}_1]=0 \\ \text{if } f^{(\mathrm{IV})}(s)=0, & \text{then } (\overrightarrow{\alpha}-\overrightarrow{C})\cdot[k_1^{\prime\prime}\overrightarrow{N}+\{(k_1k_2)^{\prime}+k_1^{\prime}k_2+k_1k_2k_3\}\overrightarrow{B}_1]=0. \end{array} \tag{14}$$

Using equations (14) and (2) we get

$$\begin{aligned} (\overrightarrow{\alpha}-\overrightarrow{C})\cdot\overrightarrow{N}&=-\tfrac{1}{k_1}, \\ (\overrightarrow{\alpha}-\overrightarrow{C})\cdot\overrightarrow{B}_1&=\tfrac{k_1^{\prime}}{k_1^2k_2}, \end{aligned} \tag{15}$$

substituting equations (14) into equation $(15)_4$ we obtain

$$k_1^{\prime\prime}=\frac{k_1^{\prime}}{k_1k_2}[(k_1k_2)^{\prime}+k_2(k_1^{\prime}+k_1k_3)].$$

Let be $\overrightarrow{C}=\theta_1(s)T+\theta_2(s)N+\theta_3(s)B_1+\theta_4(s)B_2$ is vectorial equation of center of pseudo hyper sphere. Using equations (14) and (15), we obtain

$$\overrightarrow{C}=\overrightarrow{\alpha}-\frac{1}{k_1}\overrightarrow{N}-\frac{k_1^{\prime}}{k_1^2k_2}\overrightarrow{B}_2.$$

4.1 Tangent indicatrices of timelike partially null curve in E_2^4

Let $\alpha=\alpha(s)$ be timelike partially null unit speed curve in E_2^4. By means of Frenet equations, let us calculate following differentiations respect to s.

$$\begin{aligned} &\tfrac{d\alpha(s)}{ds}=T(s)=\beta(s) \\ &\beta^{\prime}=k_1N \\ &\beta^{\prime\prime}=k_1^2T+k_1^{\prime}N+k_1k_2B_1 \\ &\beta^{\prime\prime\prime}=3k_1k_1^{\prime}T+(k_1^3+k_1^{\prime\prime})N+(2k_1^{\prime}k_2+k_1k_2^{\prime}+k_1k_2k_3)B_1 \end{aligned} \tag{16}$$

From the equations (2), (3) and (16) we have tangent and normal indicatrices

$$T_\beta=\tfrac{k_1N}{k_1}=N$$

$$N_\beta=\tfrac{k_1^2T+k_2B_1}{k_1}$$

Using, the equations (2), (4) and (16), we can obtain first and second curvatures of curve β

$$k_{1\beta} = 1$$
$$k_{2\beta} = 0.$$

Since $\|T \wedge N \wedge \beta^{'''}\| = 0$, $B_{1\beta}$, $B_{2\beta}$ and $k_{3\beta}$ are not defined.

4.2 Normal indicatrices of timelike partially null curve in E_2^4

Let $\alpha = \alpha(s)$ be timelike partially null unit speed curve in E_2^4. By means of Frenet equations, let us calculate following differentiations respect to s.

$$\begin{aligned}
&\varphi(s) = N(s)\\
&\varphi^{'} = k_1 T + k_2 B_1\\
&\varphi^{''} = k_1^{'} T + k_1^2 N + (k_2^{'} + k_2 k_3) B_1\\
&\varphi^{'''} = (k_1^3 + k_1^{''}) T + 3k_1 k_1^{'} N + (k_1^2 k_2 + k_2^{''} + (k_2 k_3)^{'} + k_2^{'} k_3 + k_2 k_3^2) B_1
\end{aligned} \tag{17}$$

From the equations (2), (3) and (17), we have

$$T_\varphi = \frac{k_1 T + k_2 B_1}{k_1}$$

$$N_\varphi = \frac{2k_1^2 k_1^{'} T + k_1^4 N + k_1[(k_1 k_2) + k_1 k_2 k_3] B_1}{k_1^2 \sqrt{\left|k_1^4 - (2k_1^{'})^2\right|}}$$

Using, the equations (2), (4) and (17), we can obtain first and second curvatures of curve φ

$$k_{1\varphi} = \frac{\sqrt{\left|k_1^4 - (2k_1^{'})^2\right|}}{k_1^2}$$

$$k_{2\varphi} = 0.$$

Since $\|T \wedge N \wedge \varphi^{'''}\| = 0$, $B_{1\varphi}$, $B_{2\varphi}$ and $k_{3\varphi}$ are not defined.

4.3 First binormal indicatrices of timelike partially null curve in E_2^4

Let $\alpha = \alpha(s)$ be spacelike partially null unit speed curve in E_2^4. By means of Frenet equations, let us calculate following differentiations respect to s.

$$\begin{aligned}
&\lambda(s) = B_1(s)\\
&\lambda^{'} = k_3 B_1\\
&\lambda^{''} = (k_3^{'} + k_3^2) B_1\\
&\lambda^{'''} = (k_3^{''} + 3k_3 k_3^{'} + k_3^3) B_1
\end{aligned} \tag{18}$$

From the equations (2), (3) and (18), first binormal indicatrices are undefined.

4.4 Second binormal indicatrices of timelike partially null curve in E_2^4

Let $\alpha=\alpha(s)$ be timelike partially null unit speed curve in E_2^4. By means of Frenet equations, let us calculate following differentiations respect to s.

$$\begin{aligned}
&\psi(s)=B_2(s)\\
&\psi' = k_2N-k_3B_2\\
&\psi'' = k_1k_2T+(k_2'+k_2k_3)N+k_1^2B_1+(k_3^2-k_3')B_2\\
&\psi''' = a_1(s)T+a_2(s)N+a_3(s)B_1+a_4(s)B_2
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
&a_1(s)=k_1k_2k_3-k_1'k_2-2k_1k_2'\\
&a_2(s)=(k_3'-k_2k_3)'-k_1^2k_2-k_2(k_3^2-k_3')\\
&a_3(s)=-3k_2k_2'\\
&a_4(s)=(k_3^2-k_3')'-k_3(k_3^2-k_3')
\end{aligned} \tag{20}$$

and fourth differential of ψ

$$\begin{aligned}
\psi^{(\mathrm{IV})}(s) = {} & (a_1'+k_1a_2)T+(k_1a_1+a_2'+k_2a_4)N\\
& +(k_2a_2+a_3'+k_3a_3)B_1+(a_4'-k_3a_4)B_2
\end{aligned} \tag{21}$$

From the equations (2), (3), (19) and (20), we have

$$T_\psi = \tfrac{-k_2N-k_3B_2}{k_2}$$

$$N_\psi = \frac{b_1T+k_2b_2N+b_3B_1+b_4B_2}{k_2\sqrt{\left|-k_1^2k_2^2+2k_2^2k_3^2-k_2(k_2'k_3-k_2k_3')\right|}}$$

where $b_1(s)=-k_1k_2^3$, $b_2(s)=k_2^3k_3$, $b_3(s)=-k_2^4$, $b_4(s)=-k_2^2k_3^2+k_2(k_2k_3'-k_2'k_3)$. To obtain $B_{2\psi}$ we need $T\wedge N\wedge\alpha'''$ if we product tree vectors as cross product, we get

$$T\wedge N\wedge\alpha''' = c_1(s)T+c_2(s)N+c_3(s)B_1+c_4(s)B_2$$

where

$$\begin{aligned}
&c_1(s)=-k_3b_3a_2+(-k_3b_2+k_2b_4)a_3-k_2b_3a_4\\
&c_2(s)=+k_3b_3a_1-k_3b_1a_3\\
&c_3(s)=(k_2b_4-k_3b_2)a_1+k_3b_1a_2-k_2b_1a_4\\
&c_4(s)=k_3b_3a_1-k_2b_1a_3,
\end{aligned} \tag{22}$$

equations (2), (3), (19), (20), (21) and (22), we have $B_{2\psi}$ and $B_{1\psi}$

$$\begin{aligned}
B_{2\psi} = {} & -\tfrac{c_1}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}T-\tfrac{c_2}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}N\\
& +\tfrac{c_3}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}B_1+\tfrac{c_4}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}B_2
\end{aligned}$$

and

$$\begin{aligned} B_{1\psi} = & \frac{1}{\sqrt{|c_1^2\text{-}c_2^2+c_4c_3|}}\{[k_3b_2c_3+k_2b_3c_3-k_2b_4c_3+k_3b_3c_2]T \\ & +[k_3b_3c_1+k_3b_1c_3]N \\ & +[k_2b_4c_1-k_3b_1c_2+k_2b_2c_4-k_3b_2c_1]B_1 \\ & +[k_2b_3c_1+k_2b_1c_3]B_2 \end{aligned}$$

From the equations (2), (4) (19), (20), (21) and (22), we have curvatures of ψ as follow:

$$k_{1\psi} = \frac{\sqrt{\left|-k_1^2k_2^2+2k_2^2k_3^2-k_2(k_2^{\prime}k_3-k_2k_3^{\prime})\right|}}{k_2^2}$$

$$k_{2\psi} = \frac{\sqrt{\left|c_1^2-c_2^2+c_4c_3\right|}}{k_2\sqrt{\left|-k_1^2k_2^2+[2(k_2^{\prime}+k_2k_3)]^2+k_2(k_2k_3^2-(k_2k_3)^{\prime})\right|}}$$

$$\begin{aligned} k_{3\psi} = & \frac{1}{k_2}\{c_1(a_1^{\prime}+k_1a_2)-c_2(k_1a_1+a_2^{\prime}-k_2a_4) \\ & +c_3(a_4^{\prime}-k_3a_4)+c_4(k_2a_2+a_3^{\prime}+k_3a_3)\} \end{aligned}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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