New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.151

A new two step iterative scheme for a finite family of nonself I-asymptotically nonexpansive mappings in Banach space

Birol Gunduz

Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, Turkey.

Received: 17 November 2016, Accepted: 10 January 2017
Published online: 15 April 2017.

Abstract: Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E and let $T_i : K \to E$ be N I_i-asymptotically nonexpansive nonself mappings and I_i be N asymptotically nonexpansive nonself mappings. It is proved that a new two step iterative algorithm converges weakly to a $q \in F$ in a real uniformly convex Banach space such that its dual has the Kadec-Klee property and strongly under condition (B) in a real uniformly convex Banach space. It presents some new results in this paper.

Keywords: Nonself I-asymptotically nonexpansive mapping, Fréchet differentiable norm, Opial property, Kadec-Klee property, B condition, fixed point.

1 Introduction

Let K be a nonempty subset of a real normed linear space E. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. Throughout this paper, we always assume that $F(T) \neq \emptyset$. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - T^n y\| \leq (1+k_n)\|x-y\|$ for all $x,y \in K$ and $n \geq 1$.

The class of asymptotically nonexpansive self-mappings was introduced by Goebel and Kirk [1] in 1972, who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on K, then T has a fixed point.

A mapping $T : K \to K$ is called uniformly L-Lipschitzian if there exists constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x-y\|$ for all $x,y \in K$ and $n \geq 1$. Also T is called asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x \in K$, the following inequality holds:

$$\|T^n x - p\| \leq k_n \|x-p\|,\ \forall p \in F(T),\ n \geq 1.$$

From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive. Every asymptotically nonexpansive mapping with a fixed point is asymptotically quasi-nonexpansive, but the converse may be not true.

In 2007, Agarwal et al. [10] introduced a new iteration process.

* Corresponding author e-mail: birolgndz@gmail.com

For K a convex subset of a linear space E and T a mapping of K into itself, the iterative sequence $\{x_n\}$ is generated from $x_1 \in K$, and is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases}, \quad n \geq 1, \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

They showed that their process is independent of Mann and Ishikawa iteration processes and converges faster than both of these (See [10, Proposition 3.1]).

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to K$ is said to be a retraction if $P^2 = P$. It follows that, if a map P is a retraction, then $Py = y$ for all y in the range of P.

In 2003, as the generalization of asymptotically nonexpansive self-mappings Chidume et al. [11] introduced the concept of nonself asymptotically nonexpansive mappings. The nonself asymptotically nonexpansive mapping is defined as follows.

Definition 1. [11] *Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E into K.*

(i) *A nonself mapping $T : K \to E$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [0,\infty)$ with $k_n \to 0$ as $n \to \infty$ such that*

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq (1+k_n)\|x-y\|, \quad \forall x,y \in K, \ n \geq 1.$$

(ii) *A nonself mapping $T : K \to E$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that*

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq L\|x-y\|, \quad \forall x,y \in K, \ n \geq 1.$$

(iii) *A nonself mapping $T : K \to E$ is called asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \in [0,\infty)$ with $k_n \to 0$ as $n \to \infty$ such that for all $x \in K$,*

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}p\right\| \leq (1+k_n)\|x-p\|, \quad \forall p \in F, \ n \geq 1.$$

If T is a self-mapping, then P becomes the identity mapping.

By using the following iterative algorithm:

$$x_1 \in K, \quad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad \forall n \geq 1. \tag{2}$$

Chidume et al. [11] established demiclosed principle, strong and weak convergence theorems for nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces.

In 2009, Thianwan [13,14] considered a new iterative scheme for two nonself asymptotically nonexpansive mappings as follows.

$$\begin{cases} x_{n+1} = P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$.

Recently, Temir [2,3] introduced the following definitions and studied Ishikawa iteration processes for a finite family of nonself I-asymptotically nonexpansive mappings.

Definition 2. *Let $T,I:K\to K$ be two mappings. T is said to be I-asymptotically nonexpansive [2,3] if there exists a sequence $\{u_n\}\subset[0,\infty)$ with $\lim_{n\to\infty}u_n=0$ such that*

$$\|T^nx-T^ny\|\le(1+u_n)\|I^nx-I^ny\| \tag{4}$$

for all $x,y\in K$ and $n\ge1$. T is said to be I-uniformly Lipschitz if there exists $\Gamma>0$ such that

$$\|T^nx-T^ny\|\le\Gamma\|I^nx-I^ny\|,\ x,y\in K \text{ and } n\ge1. \tag{5}$$

Incorporating the ideas of Chidume et al. [11] and Temir [2,3], Yang and Xie [4] introduced the following generalized definition recently.

Definition 3. [4] *Let K be a nonempty subset of a real normed space E and $P:E\to K$ be a nonexpansive retraction of E onto K. Let $T,I:K\to E$ be two mappings. T is called nonself I-asymptotically nonexpansive if there exists sequence $\{v_n\}\subset[0,\infty)$ with $\lim_{n\to\infty}v_n=0$, such that*

$$\|T(PT)^{n-1}x-T(PT)^{n-1}y\|\le(1+v_n)\|I(PI)^{n-1}x-I(PI)^{n-1}y\|$$

for all $x,y\in K$ and $n\ge1$. T is said to be Γ-uniformly Lipschitzian if there exists $\Gamma>0$ such that for all $x,y\in K$ and all positive integer n

$$\|T(PT)^{n-1}x-T(PT)^{n-1}y\|\le\Gamma\|I(PI)^{n-1}x-I(PI)^{n-1}y\|.$$

Yang and Xie [4] also introduced an iteration scheme for a finite family of nonself I-asymptotically nonexpansive mappings as follows.

Let $\{T_i\}_{i=1}^N$ be a finite family of I_i-asymptotically nonexpansive nonself-mappings and $\{I_i\}_{i=1}^N$be a finite family of asymptotically nonexpansive nonself-mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0,1]$. Then the sequence $\{x_n\}_{n\ge1}$ is generated as follows:

$$\begin{cases} y_n=P\left((1-\beta_n)x_n+\beta_nT_i(PT_i)^{n-1}x_n\right) \\ x_{n+1}=P\left((1-\alpha_n)y_n+\alpha_nI_i(PI_i)^{n-1}y_n\right) \end{cases},\ \forall n\ge1, \tag{6}$$

where $n=(k-1)N+i$, $i=i(n)\in I_0:=\{1,2,\dots,N\}$, $k=k(n)\ge1$ is a positive integer and $k(n)\to\infty$ as $n\to\infty$.

In the light of the above facts, a new two-step iterative scheme for a finite family of nonself I-asymptotically nonexpansive mappings is introduced and studied in this paper. Our iterative scheme reads as follows.

Let K be a nonempty subset of a Banach space E. Let $T_i:K\to E$ be N nonself I_i-asymptotically nonexpansive mappings and $\{I_i\}_{i=1}^N$ be N nonself asymptotically nonexpansive mappings. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0,1]$. Then the sequence $\{x_n\}$ is generated as follows.

$$\begin{cases} y_n=P\left((1-\beta_n)x_n+\beta_nT_i(PT_i)^{n-1}x_n\right) \\ x_{n+1}=P\left((1-\alpha_n)I_i(PI_i)^{n-1}y_n+\alpha_nT_i(PT_i)^{n-1}y_n\right) \end{cases},\ \forall n\ge1, \tag{7}$$

where $n=(k-1)N+i,\ i=i(n)\in J:=\{1,2,\dots,N\}$ is a positive integer and $k(n)\to\infty$ as $n\to\infty$. Thus, (7) can be expressed in the following form:

$$\begin{cases} y_n = P\left((1-\beta_n)x_n+\beta_n T_i(PT_i)^{k(n)-1}x_n\right) \\ x_{n+1} = P\left((1-\alpha_n)I_i(PI_i)^{k(n)-1}y_n+\alpha_n T_i(PT_i)^{k(n)-1}y_n\right) \end{cases},\ n\geq 1.$$

Remark. Our process (7) generalizes Chidume-Ofoedu-Zegeye process (2), Thianwan process (3) and, Yang and Xie process (6). Our process (7) is able to compute common fixed points like (3), (6) and Temir process (see [2]) but at a better rate.

Some fixed point theorems using an iteration process for asymptotically nonexpansive mappings in different spaces was proved by different authors (see [20,24,26,25,27]).

2 Preliminaries

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of E is the function $\delta_E(\varepsilon):(0,2]\to[0,1]$ defined by

$$\delta_E(\varepsilon)=\inf\left\{1-\left\|\frac{1}{2}(x+y)\right\|:\|x\|=1,\ \|y\|=1,\ \varepsilon=\|x-y\|\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon)>0$ for all $\varepsilon\in(0,2]$.

A Banach space E is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in E, $x_n\rightharpoonup x$ implies that

$$\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|$$

for all $y\in E$ with $y\neq x$, where $x_n\rightharpoonup x$ means that $\{x_n\}$ converges weakly to x.

A Banach space E is said to have a Fréchet differentiable norm [15] if for all $x\in S_E=\{x\in E:\|x\|=1\}$

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists and is attained uniformly in $y\in S_E$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n\rightharpoonup x^*\in D(T)$ and $Tx_n\to p$ then $Tx^*=p$.

A mapping $T:K\to K$ is said to be semicompact if, for any bounded sequence $\{x_n\}$ in K such that $\|x_n-Tx_n\|\to 0$ as $n\to\infty$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some x^* in K.

Lemma 1. [5] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1}\leq(1+\delta_n)a_n+b_n,\ n\geq 1.$$

If $\sum_{n=1}^{\infty}b_n<\infty$ and $\sum_{n=1}^{\infty}\delta_n<\infty$, then $\lim_{n\to\infty}a_n$ exists.

Lemma 2. [8] *Suppose that E is a uniformly convex Banach space and* $0 < p \le t_n \le q < 1$ *for all* $n \ge 1$. *Also, suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \ \limsup_{n\to\infty} \|y_n\| \le r \text{ and } \lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$$

hold for some $r \ge 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 3. [11] *Let E be a real uniformly convex Banach space, K a nonempty closed subset of E, and let* $T : K \to E$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\} \subset [0,\infty)$ *and* $k_n \to 0$ *as* $n \to \infty$, *then* $(\check{I} - T)$ *is demiclosed at zero, where* $\check{I}$ *is an identity mapping.*

A Banach space E is said to have the Kadec–Klee property if, for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $\|x_n \to x\| \to 0$. Every locally uniformly convex space has the Kadec-Klee property. In particular, L_p spaces, $1 < p < \infty$ have this property.

Let $\omega_w \{x_n\} = \left\{x : \exists x_{n_j} \rightharpoonup x\right\}$ denote the weak limit set of $\{x_n\}$.

Lemma 4. [12] *Let E be a real reflexive Banach space such that its dual E has Kadec-Klee property. Let* $\{x_n\}$ *be a bounded sequence in E and* $q_1, q_2 \in \omega_w \{x_n\}$. *Suppose* $\lim_{n\to\infty} \|\alpha x_n + (1-\alpha) q_1 - q_2\| = 0$ *exists for all* $\alpha \in [0,1]$. *Then* $q_1 = q_2$.

The mapping $T : K \to K$ with $F(T) \ne \emptyset$ is said to satisfy condition (A) [7] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \ge f(d(x, F(T)))$ for all $n \ge 1$. Senter and Dotson [7] pointed out that every continuous and semi-compact mapping must satisfy Condition (A).

Let $\{T_i : i \in I_0\}$ and $\{I_i : i \in I_0\}$ be two family of nonself mappings with nonempty fixed points set F. These families are said to satisfy condition (B) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$\text{either } \max_{i\in I_0} \|x - T_i x\| \ge f(d(x,F)) \text{ or } \max_{i\in I_0} \|x - I_i x\| \ge f(d(x,F)).$$

3 Main results

Lemma 5. *Let E be a real Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let* $T_i : K \to E$ $(i \in J)$ *be nonself* I_i*-asymptotically nonexpansive mappings with sequences* $\{u_{in}\} \subset [0,\infty)$ *and* $I_i : K \to E$ $(i \in J)$ *be asymptotically nonexpansive nonself-mappings with* $\{v_{in}\} \subset [0,\infty)$ *such that* $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \ne \emptyset$. *For an arbitrary* $x_0 \in K$, *let* $\{x_n\}$ *be the sequence generated by* (7) *satisfying the following conditions.*

(1) $\sum_{n=1}^{\infty} h_n < \infty$, *where* $h_n = \max\{u_{in} : i \in J\} \vee \max\{v_{in} : i \in J\}$;

(2) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $\tau_1 \le (1-\alpha_n), (1-\beta_n) \le \tau_2$, $\forall n \ge 1$.

Then $\lim_{n\to\infty} \|x_n - p\|$ *exists for each* $p \in F$.

Proof. We suppose that $q \in F$. From (7) we have

$$\begin{aligned}
\|y_n - q\| &= \left\| P\left((1-\beta_n)x_n + \beta_n T_i (PT_i)^{n-1} x_n\right) - Pq \right\| \\
&\le \left\| (1-\beta_n)(x_n - q) + \beta_n \left(T_i (PT_i)^{n-1} x_n - q\right) \right\| \\
&\le (1-\beta_n)\|x_n - q\| + \beta_n \left\| T_i (PT_i)^{n-1} x_n - q \right\| \\
&\le (1-\beta_n)\|x_n - q\| + \beta_n (1+h_n) \left\| I_i (PI_i)^{n-1} x_n - q \right\| \\
&\le (1-\beta_n)\|x_n - q\| + \beta_n (1+h_n)^2 \|x_n - q\| \\
&\le \left(1 + \beta_n \left(2h_n + h_n^2\right)\right)\|x_n - q\| \le (1+h_n)^2 \|x_n - q\|.
\end{aligned} \tag{8}$$

Using (7) and (8), we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \left\| P\left((1-\alpha_n) I_i (PI_i)^{n-1} y_n + \alpha_n T_i (PT_i)^{n-1} y_n\right) - Pq \right\| \\
&\le \left\| (1-\alpha_n)\left(I_i (PI_i)^{n-1} y_n - q\right) + \alpha_n \left(T_i (PT_i)^{n-1} y_n - q\right) \right\| \\
&\le (1-\alpha_n)\left\| I_i (PI_i)^{n-1} y_n - q \right\| + \alpha_n \left\| T_i (PT_i)^{n-1} y_n - q \right\| \\
&\le (1-\alpha_n)(1+h_n)\|y_n - q\| + \alpha_n (1+h_n)^2 \|y_n - q\| \\
&\le (1 + h_n + \alpha_n h_n (1+h_n))\|y_n - q\| \\
&\le (1+h_n)^2 \|y_n - q\|.
\end{aligned} \tag{9}$$

Substituting (8) into (9), we get that

$$\|x_{n+1} - p\| \le (1+h_n)^2 (1+h_n)^2 \|x_n - q\| \le (1+\delta_n)\|x_n - q\|, \tag{10}$$

where $\delta_n = h_n \left(4 + 5h_n + 4h_n^2 + h_n^3\right)$. Since $\sum_{n=1}^{\infty} h_n < \infty$, we obtain $\sum_{n=1}^{\infty} \delta_n < \infty$. By Lemma 1 and (10), we get $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof.

Lemma 6. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T_i : K \to E$ $(i \in J)$ be nonself I_i-asymptotically nonexpansive mappings with sequences $\{u_{in}\} \subset [0,\infty)$ and $I_i : K \to E$ $(i \in J)$ be asymptotically nonexpansive nonself-mappings with $\{v_{in}\} \subset [0,\infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. For an arbitrary $x_0 \in K$, let $\{x_n\}$ be the sequence generated by (7) satisfying the following conditions.*

(1) $\sum_{n=1}^{\infty} h_n < \infty$, *where* $h_n = \max\{u_{in} : i \in J\} \vee \max\{v_{in} : i \in J\}$;

(2) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $\tau_1 \le (1-\alpha_n), (1-\beta_n) \le \tau_2$, $\forall n \ge 1$.

Then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - I_i x_n\| = 0$ *for all* $i \in J$.

Proof. In view of Lemma 5, we obtain that the limit of the sequence $\{\|x_n - q\|\}$ exits. Next, we assume that $\lim_{n\to\infty} \|x_n - q\| = d$, for some $d > 0$. It follows from (8) and (9) that

$$\lim_{n\to\infty} \|y_n - q\| = d. \tag{11}$$

Note that using I_i-asymptotically nonexpansivity of T_i $(i \in J)$, we get

$$\left\| T_i (PT_i)^{n-1} x_n - q \right\| \le (1+h_n)\left\| I_i (PI_i)^{n-1} x_n - q \right\| \le (1+h_n)^2 \|x_n - q\|.$$

It follows that

$$\limsup_{n\to\infty}\left\|T_i\left(PT_i\right)^{n-1}x_n-q\right\|\le d. \tag{12}$$

Moreover, from (8), we have

$$\|y_n-q\|\le\left\|(1-\beta_n)(x_n-q)+\beta_n\left(T_i\left(PT_i\right)^{n-1}x_n-q\right)\right\|\le(1+h_n)^2\|x_n-q\|$$

and (11) implies

$$\lim_{n\to\infty}\left\|(1-\beta_n)(x_n-q)+\beta_n\left(T_i\left(PT_i\right)^{n-1}x_n-q\right)\right\|=d. \tag{13}$$

Combining $\lim_{n\to\infty}\|x_n-q\|=d$, (12) with (13) and from Lemma 6, we obtain that

$$\lim_{n\to\infty}\left\|x_n-T_i\left(PT_i\right)^{n-1}x_n\right\|=0. \tag{14}$$

In similar way, from (9), we can prove that

$$\lim_{n\to\infty}\left\|(1-\alpha_n)\left(I_i\left(PI_i\right)^{n-1}y_n-q\right)+\alpha_n\left(T_i\left(PT_i\right)^{n-1}y_n-q\right)\right\|=d. \tag{15}$$

Thus $\left\|I_i\left(PI_i\right)^{n-1}y_n-q\right\|\le(1+h_n)\|y_n-q\|$ for all $n\ge1$ implies that

$$\limsup_{n\to\infty}\left\|I_i\left(PI_i\right)^{n-1}y_n-q\right\|\le d. \tag{16}$$

Since

$$\limsup_{n\to\infty}\left\|T_i\left(PT_i\right)^{n-1}y_n-q\right\|\le\limsup_{n\to\infty}(1+h_n)\left\|I_i\left(PI_i\right)^{n-1}y_n-q\right\|\le\limsup_{n\to\infty}(1+h_n)^2\|y_n-q\|$$

and $\lim_{n\to\infty}\|y_n-q\|=d$, we have

$$\limsup_{n\to\infty}\left\|T_i\left(PT_i\right)^{n-1}y_n-q\right\|\le d. \tag{17}$$

Now, using (15), (16), (17), and Lemma 6, we obtain

$$\lim_{n\to\infty}\left\|I_i\left(PI_i\right)^{n-1}y_n-T_i\left(PT_i\right)^{n-1}y_n\right\|=0 \tag{18}$$

By $y_n=P\left((1-\beta_n)x_n+\beta_nT_i\left(PT_i\right)^{n-1}x_n\right)$ and (14), since P is nonexpansive mapping, we have

$$\begin{aligned}\|y_n-x_n\|&=\left\|P\left((1-\beta_n)x_n+\beta_nT_i\left(PT_i\right)^{n-1}x_n\right)-Px_n\right\|\\&\le\left\|(1-\beta_n)x_n+\beta_nT_i\left(PT_i\right)^{n-1}x_n-x_n\right\|\le\left\|x_n-T_i\left(PT_i\right)^{n-1}x_n\right\|\\&\to0\ (\text{as } n\to\infty).\end{aligned} \tag{19}$$

Also

$$\begin{aligned}\left\|I_i\left(PI_i\right)^{n-1}y_n-x_n\right\|&\le\left\|I_i\left(PI_i\right)^{n-1}y_n-T_i\left(PT_i\right)^{n-1}y_n\right\|+\left\|T_i\left(PT_i\right)^{n-1}y_n-T_i\left(PT_i\right)^{n-1}x_n\right\|+\left\|T_i\left(PT_i\right)^{n-1}x_n-x_n\right\|\\&\le\left\|I_i\left(PI_i\right)^{n-1}y_n-T_i\left(PT_i\right)^{n-1}y_n\right\|+(1+h_n)^2\|y_n-x_n\|+\left\|T_i\left(PT_i\right)^{n-1}x_n-x_n\right\|\end{aligned}$$

implies by (14), (18), and (19) that

$$\lim_{n\to\infty}\left\|I_i(PI_i)^{n-1}y_n-x_n\right\|=0. \tag{20}$$

From (18), and (20), we have

$$\begin{aligned}\|x_{n+1}-x_n\|&=\left\|P\left((1-\alpha_n)I_i(PI_i)^{n-1}y_n+\alpha_nT_i(PT_i)^{n-1}y_n\right)-x_n\right\|\\&\leq\left\|I_i(PI_i)^{n-1}y_n-x_n\right\|+\alpha_n\left\|I_i(PI_i)^{n-1}y_n-T_i(PT_i)^{n-1}y_n\right\|\\&\to 0\ (\text{as } n\to\infty).\end{aligned}\tag{21}$$

It follows from (19) and (21) that

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0 \tag{22}$$

Using (19) and (20), we have

$$\begin{aligned}\left\|I_i(PI_i)^{n-1}x_n-x_n\right\|&\leq\left\|I_i(PI_i)^{n-1}x_n-I_i(PI_i)^{n-1}y_n\right\|+\left\|I_i(PI_i)^{n-1}y_n-x_n\right\|\\&\leq(1+h_n)\|x_n-y_n\|+\left\|I_i(PI_i)^{n-1}y_n-x_n\right\|\\&\to 0\ (\text{as } n\to\infty).\end{aligned}\tag{23}$$

Now

$$\left\|I_i(PI_i)^{n-1}y_n-x_{n+1}\right\|\leq\left\|I_i(PI_i)^{n-1}y_n-x_n\right\|+\|x_n-x_{n+1}\| \tag{24}$$

gives by (20) and (21) that

$$\lim_{n\to\infty}\left\|I_i(PI_i)^{n-1}y_n-x_{n+1}\right\|=0. \tag{25}$$

Next, we also have from (14), (22) and (25)

$$\begin{aligned}\|x_n-T_ix_n\|&\leq\left\|x_n-T_i(PT_i)^{n-1}x_n\right\|+\left\|T_i(PT_i)^{n-1}x_n-T_i(PT_i)^{n-1}y_{n-1}\right\|+\left\|T_i(PT_i)^{n-1}y_{n-1}-T_ix_n\right\|\\&\leq\left\|x_n-T_i(PT_i)^{n-1}x_n\right\|+(1+h_n)^2\|x_n-y_{n-1}\|+\Gamma\left\|I_i(PI_i)^{n-1}y_{n-1}-x_n\right\|\\&\leq\left\|x_n-T_i(PT_i)^{n-1}x_n\right\|+(1+h_n)^2\|x_n-y_{n-1}\|+\Gamma I\left\|I_i(PI_i)^{n-2}y_{n-1}-x_n\right\|\\&\to 0\ (n\to\infty).\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=0. \tag{26}$$

From (19), (20), and (21), we have

$$\begin{aligned}\left\|x_{n+1}-I_i(PI_i)^{n-1}x_n\right\|&\leq\|x_{n+1}-x_n\|+\left\|x_n-I_i(PI_i)^{n-1}y_n\right\|+\left\|I_i(PI_i)^{n-1}y_n-I_i(PI_i)^{n-1}x_n\right\|\\&\leq\|x_{n+1}-x_n\|+\left\|x_n-I_i(PI_i)^{n-1}y_n\right\|+(1+h_n)\|x_n-y_n\|\end{aligned}$$

and so

$$\lim_{n\to\infty}\left\|x_{n+1}-I_i(PI_i)^{n-1}x_n\right\|=0 \tag{27}$$

By means of asymptotically nonexpansivity of I_i $(i \in J)$ we get

$$\begin{aligned}\|x_n - I_i x_n\| &\leq \left\|x_n - I_i (PI_i)^{n-1} x_n\right\| + \left\|I_i (PI_i)^{n-1} x_n - I_i (PI_i)^{n-1} x_{n-1}\right\| + \left\|I_i (PI_i)^{n-1} x_{n-1} - I_i x_n\right\| \\ &\leq \left\|x_n - I_i (PI_i)^{n-1} x_n\right\| + (1+h_n)\|x_n - x_{n-1}\| + L\left\|I_i (PI_i)^{n-2} x_{n-1} - x_n\right\|.\end{aligned}$$

It follows from (21), (23) and (27) that

$$\lim_{n\to\infty} \|x_n - I_i x_n\| = 0 \tag{28}$$

Lemma 7. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T_i : K \to E$ $(i \in J)$ be nonself I_i-asymptotically nonexpansive mappings with sequences $\{u_{in}\} \subset [0,\infty)$ and $I_i : K \to E$ $(i \in J)$ be asymptotically nonexpansive nonself-mappings with $\{v_{in}\} \subset [0,\infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. For an arbitrary $x_0 \in K$, let $\{x_n\}$ be the sequence generated by (7) satisfying the following conditions.*

(1) $\sum_{n=1}^{\infty} h_n < \infty$, *where* $h_n = \max\{u_{in} : i \in J\} \vee \max\{v_{in} : i \in J\}$;

(2) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $\tau_1 \leq (1-\alpha_n), (1-\beta_n) \leq \tau_2$, $\forall n \geq 1$.

Then $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ *exists for all* $t \in [0,1]$ *and* $p_1, p_2 \in F$.

Proof. Let $a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$, then $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ exists. It follows from Lemma 5 that $\lim_{n\to\infty} a_n(1) = \|x_n - p_2\|$ exists. It now remains to prove the lemma for $t \in (0,1)$. Let $U_n, W_n : K \to E$ be defined as follows:

$$\begin{cases} U_n x = P\left((1-\beta_n)x + \beta_n T_i (PT_i)^{n-1} x\right) \\ W_n x = P\left((1-\alpha_n) I_i (PI_i)^{n-1} U_n x + \alpha_n T_i (PT_i)^{n-1} U_n x\right) \end{cases}, \; n \geq 1,$$

for all $x \in K$. Then for all $x, y \in K$, we have

$$\begin{aligned}\|U_n x - U_n y\| &\leq \left\|P\left((1-\beta_n)x + \beta_n T_i (PT_i)^{n-1} x\right) - P\left((1-\beta_n)y + \beta_n T_i (PT_i)^{n-1} y\right)\right\| \\ &\leq \left\|(1-\beta_n)(x-y) + \beta_n\left(T_i (PT_i)^{n-1} x - T_i (PT_i)^{n-1} y\right)\right\| \\ &\leq (1-\beta_n)\|x-y\| + \beta_n (1+h_n)\left\|I_i (PI_i)^{n-1} x - I_i (PI_i)^{n-1} y\right\| \\ &\leq (1-\beta_n)\|x-y\| + \beta_n (1+h_n)^2 \|x-y\| \\ &\leq (1+l_n)^2 \|x-y\|.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\|W_n x - W_n y\| &\leq \left\|P\left((1-\alpha_n) I_i (PI_i)^{n-1} U_n x + \alpha_n T_i (PT_i)^{n-1} U_n x\right) - P\left((1-\alpha_n) I_i (PI_i)^{n-1} U_n y + \alpha_n T_i (PT_i)^{n-1} U_n y\right)\right\| \\ &\leq \left\|(1-\alpha_n)\left(I_i (PI_i)^{n-1} U_n x - I_i (PI_i)^{n-1} U_n y\right) + \alpha_n\left(T_i (PT_i)^{n-1} U_n x - T_i (PT_i)^{n-1} U_n y\right)\right\| \\ &\leq (1-\alpha_n)\left\|I_i (PI_i)^{n-1} U_n x - I_i (PI_i)^{n-1} U_n y\right\| + \alpha_n\left\|T_i (PT_i)^{n-1} U_n x - T_i (PT_i)^{n-1} U_n y\right\| \\ &\leq (1-\alpha_n)(1+h_n)\|U_n x - U_n y\| + \alpha_n (1+h_n)(1+h_n)\|U_n x - U_n y\| \\ &\leq (1+h_n + \alpha_n h_n (1+h_n))\|U_n x - U_n y\| \\ &\leq (1+h_n)^2 \|U_n x - U_n y\| \\ &\leq (1+h_n)^4 \|x-y\|.\end{aligned}$$

This implies that $W_n : K \to K$ is Lipschitz with the Lipschitz constant $(1+h_n)^4$ and $x_{n+1} = W_n x_n$. Setting $H_n = \prod_{j=n}^{\infty}(1+h_j)^4$ for $n \geq 1$, then $H_n \to 1$ as $n \to \infty$. Putting

$$S_{n,m} = W_{n+m-1}W_{n+m-2}\cdots W_n,\ n,m \geq 1,$$

then $S_{n,m} : K \to K$ is Lipschitz with the Lipschitz constant H_n, $S_{n,m}x_n = x_{n+m}$ and $S_{n,m}p = p$ for each $p \in F$.
Then the rest of the proof follows as in the proof of Lemma 3 of [9]. This completes the proof.

Theorem 1. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec–Klee property and K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T_i : K \to E$ $(i \in J)$ be nonself I_i-asymptotically nonexpansive mappings with sequences $\{u_{in}\} \subset [0,\infty)$ and $I_i : K \to E$ $(i \in J)$ be asymptotically nonexpansive nonself-mappings with $\{v_{in}\} \subset [0,\infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. For an arbitrary $x_0 \in K$, let $\{x_n\}$ be the sequence generated by (7) satisfying the following conditions.*

(1) $\sum_{n=1}^{\infty} h_n < \infty$, *where* $h_n = \max\{u_{in} : i \in J\} \vee \max\{v_{in} : i \in J\}$;
(2) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $\tau_1 \leq (1-\alpha_n), (1-\beta_n) \leq \tau_2$, $\forall n \geq 1$.
Then $\{x_n\}$ converges weakly to some $q \in F$.

Proof. It follows from Lemma 5 that $\{x_n\}$ is bounded. Since E is a uniformly convex Banach space, $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}$ such that converges weakly to p. Since $\{x_n\} \subset K$ and K is weakly closed, then $p \in K$. Moreover, $\lim_{n\to\infty} \left\| x_n - T_i (PT_i)^{n-1} x_n \right\| = 0$ and $\lim_{n\to\infty} \left\| x_n - I_i (PI_i)^{n-1} x_n \right\| = 0$ for all $i \in J$ by Lemma 6 and so $p \in F$ by Lemma 3.
Now, we show that $\{x_n\}$ converges weakly to p. Suppose that $\{x_{n_k}\}$ is another subsequence of $\{x_n\}$ which converges weakly to some $q \in K$. By the same method as above, we have $q \in F$ and $p,q \in \omega_w\{x_n\}$. By Lemma 7,

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$ and so $p = q$ by Lemma 4. Therefore, the sequence $\{x_n\}$ converges weakly to p. This completes the proof.

Remark. In [16], it is point out that there exist uniformly convex Banach spaces which have neither a Fréchet differentiable norm nor the Opial property but their duals do have the Kadec–Klee property. And the duals of reflexive Banach spaces with Fréchet differentiable norms or the Opial property have the Kadec–Klee property.

Theorem 2. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T_i : K \to E$ $(i \in J)$ be nonself I_i-asymptotically nonexpansive mappings with sequences $\{u_{in}\} \subset [0,\infty)$ and $I_i : K \to E$ $(i \in J)$ be asymptotically nonexpansive nonself-mappings with $\{v_{in}\} \subset [0,\infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. For an arbitrary $x_0 \in K$, let $\{x_n\}$ be the sequence generated by (7) satisfying the following conditions.*

(1) $\sum_{n=1}^{\infty} h_n < \infty$, *where* $h_n = \max\{u_{in} : i \in J\} \vee \max\{v_{in} : i \in J\}$;
(2) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $\tau_1 \leq (1-\alpha_n), (1-\beta_n) \leq \tau_2$, $\forall n \geq 1$.
Then $\{x_n\}$ converges strongly to some $q \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity is obvious. Let us proof the sufficiency part of theorem. For any given $p \in F$, we have (see (10))

$$\|x_{n+1} - p\| \leq (1+\delta_n)\|x_n - q\|. \tag{29}$$

Taking the infimum over all $p \in F$ in the inequalities (29), we get

$$d(x_{n+1}, F) \leq (1 + \delta_n) d(x_n, F) \tag{30}$$

Now applying Lemma 1 to (30) we obtain the existence of the limit $\lim_{n\to\infty} d(x_n, F)$. By hypothesis, we see

$$\lim_{n\to\infty} d(x_n, F) = \lim_{n\to\infty} \inf d(x_n, F) = 0.$$

It will be proved that $\{x_n\}$ is a Cauchy sequence. In fact, as $1 + t \leq \exp(t)$ for all $t > 0$, from (29), we obtain

$$\|x_{n+1} - p\| \leq \exp(\delta_n) \|x_n - q\| \tag{31}$$

Thus, for any positive integers m, n, iterating (31) and noting $\sum_{n=1}^{\infty} \delta_n < \infty$, we get

$$\begin{aligned}
\|x_{n+m} - p\| &\leq \exp\{\delta_{n+m-1}\} \|x_{n+m-1} - p\| \\
&\leq \exp\{\delta_{n+m-1}\} [\exp\{\delta_{n+m-2}\} \|x_{n+m-2} - p\|] \\
&= \exp\{\delta_{n+m-1} + \delta_{n+m-2}\} \|x_{n+m-2} - p\| \leq \cdots \\
&\leq \exp\left\{\sum_{j=n}^{n+m-1} \delta_j\right\} \|x_n - p\| \leq M \|x_n - p\|,
\end{aligned}$$

where $M = \sum_{j=1}^{\infty} \delta_j < \infty$. Therefore,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq (1 + M) \|x_n - p\|, \tag{32}$$

for all $p \in F$. Taking the infimum over $p \in F$ in (32) we obtain

$$\|x_{n+m} - x_n\| \leq (1 + M) \lim_{n\to\infty} d(x_n, F). \tag{33}$$

Due to $\lim_{n\to\infty} d(x_n, F) = 0$, given $\varepsilon > 0$ there exists an integer $N_0 > 0$ such that for all $n > N_0$ we have $d(x_n, F) < \frac{\varepsilon}{1+M}$. Consequently, for all integers $n > N_0$ and $m \geq 1$, from (33) we get $\|x_{n+m} - x_n\| < \varepsilon$, which means that $\{x_n\}$ is a Cauchy sequence in E. Since the space E is complete, $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = q$. Then, $\lim_{n\to\infty} d(x_n, F) = 0$ implies that $\lim_{n\to\infty} d(q, F) = 0$. F is closed, thus $q \in F$. This completes the proof.

Applying Theorem 2, we obtain a strong convergence theorem using the iterative sequence (7) under the condition (B) as follows.

Theorem 3. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T_i : K \to E$ $(i \in J)$ be nonself I_i-asymptotically nonexpansive mappings with sequences $\{u_{in}\} \subset [0, \infty)$ and $I_i : K \to E$ $(i \in J)$ be asymptotically nonexpansive nonself-mappings with $\{v_{in}\} \subset [0, \infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. For an arbitrary $x_0 \in K$, let $\{x_n\}$ be the sequence generated by (7) satisfying the following conditions.*

(1) $\sum_{n=1}^{\infty} h_n < \infty$, *where* $h_n = \max\{u_{in} : i \in J\} \vee \max\{v_{in} : i \in J\}$;
(2) *there exist constants* $\tau_1, \tau_2 \in (0, 1)$ *such that* $\tau_1 \leq (1 - \alpha_n), (1 - \beta_n) \leq \tau_2$, $\forall n \geq 1$.

If $\{T_1, T_2, \ldots, T_N\}$ and $\{I_1, I_2, \ldots, I_N\}$ satisfy Condition (B) then $\{x_n\}$ converges strongly to some $q \in F$.

Proof. In Lemma 6, we proved that

$$\lim_{n\to\infty}\left\|x_n - T_i(PT_i)^{n-1}x_n\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|x_n - I_i(PI_i)^{n-1}x_n\right\| = 0$$

for all $i \in J$. Thus from the condition (B), we get $\lim_{n\to\infty} f(d(x_n,F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n,F) = 0$.
Now all the conditions of Theorem 2 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of F.

Remark. (i) Our results can be viewed generalization of result of Akbulut et al. [6].

(ii) If the error terms are added in (7) and assumed to be bounded, then the results of this paper still hold.

(iii) Our results can be viewed extension of the result given in [17, 18, 19, 20, 21, 22, 23].

References

[1] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc., 35 (1972), 171-174.

[2] S. Temir, On the convergence theorems of implicit iteration process for a finite family of I-asymptotically nonexpansive mappings, J. Comput. Appl. Math. 225 (2009) 398–405.

[3] S. Temir, O. Gul, Convergence theorem for I-asymptotically quasi-nonexpansive mapping in Hilbert space, J. Math. Anal. Appl. 329 (2007) 759–765.

[4] L. Yang, X. Xie, Weak and strong convergence theorems for a finite family of I-asymptotically nonexpansive mappings, Appl. Math. Comput. 216 (2010) 1057–1064.

[5] K.K. Tan and H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, Journal of Mathematical Analysis and Applications, vol. 178, no. 2, pp. 301–308, 1993.

[6] S. Akbulut, S. H. Khan, M. Ozdemir, An iteration process for common fixed points of two nonself asymptotically nonexpansive mappings, An. Şt. Univ. Ovidius Constanta, Vol. 20(1), 2012, 15-30.

[7] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc., 44(1974), 375-380.

[8] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991), 153-159.

[9] M.O. Osilike, A. Udomene, Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder–Petryshyn type, J. Math. Anal. Appl. 256 (2001) 431–445.

[10] R.P. Agarwal, Donal O'Regan and D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J.Nonliear Convex. Anal.8(1)(2007), 61–79.

[11] C.E. Chidume, E.U. Ofoedu, H. Zegeye, Strong and weak convergence theorems for asymptotically nonexpansive mappings, J. Math. Anal. Appl., 280 (2003), 364-374.

[12] J.G. Falset, W. Kaczor, T. Kuczumow, S. Reich, Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. 43 (2001) 377–401.

[13] S. Thianwan, Common fixed point of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space, J. Comput. Appl. Math. 224 (2009) 688-695.

[14] S. Thianwan, New Iterations with Errors for Approximating Common Fixed Points for two Generalized Asymptotically Quasi-Nonexpansive Nonself-Mappings, Mathematical Notes, 2011, Vol. 89, No. 3, pp. 397-407.

[15] M.O. Osilike, A. Udomene, Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder–Petryshyn type, J. Math. Anal. Appl. 256 (2001) 431-445.

[16] W. Kaczor, Weak convergence of almost orbits of asymptotically nonexpansive commutative semigroups, J. Math. Anal. Appl. 272 (2002) 565–574.

[17] B. Gunduz and S. Akbulut, Convergence theorems of a new three-step iteration for nonself asymptotically nonexpansive mappings, Thai J. Math. 13 (2015), no. 2, 465-480.

[18] B. Gunduz and S. Akbulut, On weak and strong convergence theorems for a finite family of nonself I-asymptotically nonexpansive mappings, Math. Morav. 19 (2015), no. 2, 49-64.

[19] B. Gunduz, S.H. Khan, and S. Akbulut, On convergence of an implicit iterative algorithm for non self asymptotically nonexpansive mappings, Hacet. J. Math. Stat. 43 (2014), no. 3, 399-411.

[20] B. Gunduz and S. Akbulut, A one-step implicit iterative process for a finite family of I-nonexpansive mappings in Kohlenbach hyperbolic spaces, Math Sci, 2016, 10, 55-61.

[21] B. Gunduz and S. Akbulut, On the convergence of an iteration process for totally asymptotically I-nonexpansive mappings, J. Nonlinear Anal. Optim., Vol.7, No.1, (2016), 17-30.

[22] I. Yildirim and F. Gu, A new iterative process for approximating common fixed points of nonself I-asymptotically quasi-nonexpansive mappings, Appl. Math. J. Chinese Univ. Ser. B 27 (2012), no. 4, 489–502.

[23] S. H. Khan and H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, Nonlinear Anal.TMA, 61(8) 2005, 1295-1301.

[24] B. Gunduz, A new multistep iteration for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces, J. Nonlinear Sci. Appl. 9 (2016), 1365-1372.

[25] B. Gunduz, Convergence of a new multistep iteration in convex cone metric spaces, Commun. Korean Math. Soc. 32(1) (2017), 39-46.

[26] A. Sahin and M. Basarir, On the strong convergence of a modified S-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space, Fixed Point Theory Appl. 2013, Article ID 12, 2013.

[27] A. Sahin and M. Basarir, Convergence and data dependence results of an iteration process in a hyperbolic space, Filomat, 30:3 (2016), 569–582.