

Some new integral inequalities for n-times differentiable log-convex functions

Imdat Iscan¹, Huriye Kadakal², Mahir Kadakal¹

¹Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Turkey

²Institute of Science, Ordu University, Ordu, Turkey

Received: 2 January 2017, Accepted: 4 March 2017

Published online: 3 April 2017.

Abstract: In this work, by using an integral identity together with both the Hölder and the Power-Mean integral inequality we establish several new inequalities for n -time differentiable log-convex functions.

Keywords: Convex function, Log-Convex function, Hölder Integral inequality and Power-Mean Integral inequality.

1 Introduction

In this paper, we establish some new inequalities for functions whose n th derivatives in absolute value are log-convex functions. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. For some inequalities, generalizations and applications concerning convexity see [6,9,10,11]. Recently, in the literature there are so many papers about n -times differentiable functions on several kinds of convexities. In references [3,4,5,8,12,13,17,19], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of log-convex functions see for instance the recent papers [1,2,7,14,15,16,18,20] and the references within these papers.

Definition 1. A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Definition 2. A positive function f is called log-convex on a real interval $I = [a, b]$, if for all $x, y \in [a, b]$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq [f(x)]^t + [f(y)]^{1-t}.$$

If f is a positive log-concave function, then the inequality is reversed. Equivalently, a function f is log-convex on I if f is positive and $\log f$ is convex on I . Also, if $f > 0$ and f' exists on I , then f is log-convex if and only if $ff'' - (f')^2 \geq 0$.

Let $0 < a < b$, throughout this paper we will use

$$A(a, b) = \frac{a+b}{2}, G(a, b) = \sqrt{ab}, L(a, b) = \frac{b-a}{\ln b - \ln a}, L_p(a, b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, a \neq b, p \in \mathbb{R}, p \neq -1, 0$$

for the arithmetic, geometric, logarithmic, generalized logarithmic mean for $a, b > 0$ respectively.

2 Main results

We will use the following Lemma [13] for we obtain the main results.

Lemma 1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable mapping on I° for $n \in \mathbb{N}$ and $f^{(n)} \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$, we have the identity

$$\sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx = \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x)dx.$$

where an empty sum is understood to be nil.

Theorem 1. For $\forall n \in \mathbb{N}$; let $f : I \subseteq [0, \infty) \rightarrow (0, \infty)$ be n -times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ for $q > 1$ is log-convex on $[a, b]$, then the following inequality holds:

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \leq \frac{1}{n!} (b-a) L_{np}^n(a, b) L^{\frac{1}{q}} \left(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q \right)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $|f^{(n)}|^q$ for $q > 1$ is log-convex on $[a, b]$, using Lemma 1, the Hölder integral inequality and

$$|f^{(n)}(x)|^q = \left| f^{(n)} \left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a \right) \right|^q \leq \left[|f^{(n)}(b)|^q \right]^{\frac{x-a}{b-a}} \left[|f^{(n)}(a)|^q \right]^{\frac{b-x}{b-a}}$$

we have

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \leq \frac{1}{n!} \int_a^b x^n f^{(n)}(x)dx \\ & \leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b \left[|f^{(n)}(b)|^q \right]^{\frac{x-a}{b-a}} \left[|f^{(n)}(a)|^q \right]^{\frac{b-x}{b-a}} dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} |f^{(n)}(a)| \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b \left[\frac{|f^{(n)}(b)|^q}{|f^{(n)}(a)|^q} \right]^{\frac{x-a}{b-a}} dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} |f^{(n)}(a)| \left[\frac{b^{np+1} - a^{np+1}}{(np+1)} \right]^{\frac{1}{p}} \times \left\{ \frac{b-a}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} \left(\frac{|f^{(n)}(b)|^q}{|f^{(n)}(a)|^q} - 1 \right) \right\}^{\frac{1}{q}} \\ & = \frac{1}{n!} \left[\frac{b^{np+1} - a^{np+1}}{(np+1)(b-a)} \right]^{\frac{1}{p}} \left\{ \frac{|f^{(n)}(b)|^q - |f^{(n)}(a)|^q}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} \right\}^{\frac{1}{q}} = \frac{1}{n!} (b-a) L_{np}^n(a, b) L^{\frac{1}{q}} \left(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q \right) \end{aligned}$$

This completes the proof of theorem.

Corollary 1. Under the conditions Theorem 1 for $n = 1$ we have the following inequality:

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq L_p(a,b)L^{\frac{1}{q}} \left(|f'(b)|^q, |f'(a)|^q \right).$$

Proposition 1. Let $a, b \in (0, \infty)$ with $a < b, p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and we have

$$L_{1-\frac{1}{q}}^{1-\frac{1}{q}}(a,b) \leq L_p(a,b) \left[\frac{L(a,b)}{G^2(a,b)} \right]^{\frac{1}{q}}$$

Proof. Under the assumption of the Proposition, let $f(x) = \frac{q}{q-1}x^{1-\frac{1}{q}}, x \in (0, \infty)$. Then

$$|f'(x)| = x^{-\frac{1}{q}}$$

is log-convex on $(0, \infty)$ and the result follows directly from Corollary 1.

Theorem 2. For $\forall n \in \mathbb{N}$; let $f : I \subseteq [0, \infty) \rightarrow (0, \infty)$ be n -times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f^{(n)}|^q \in L[a, b]$ and $|f^{(n)}|^q$ for $q \geq 1$ is log-convex on $[a, b]$, then the following inequality holds:

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \leq \frac{1}{n!} |f^{(n)}(a)| (b-a)^{1-\frac{1}{q}} L_n^{n(1-\frac{1}{q})}(a,b) M^{\frac{1}{q}}(a,b,n,q)$$

where $M(a,b,n,q) = \int_a^b x^n \left[\frac{|f^{(n)}(b)|^q}{|f^{(n)}(a)|^q} \right]^{\frac{x-a}{b-a}} dx$.

Proof. From Lemma 1 and Power-mean integral inequality, we obtain

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \\ & \leq \frac{1}{n!} \left(\int_a^b x^n dx \right)^{1-\frac{1}{q}} \left(\int_a^b x^n |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \leq \frac{1}{n!} \left(\int_a^b x^n dx \right)^{1-\frac{1}{q}} \left(\int_a^b x^n \left[|f^{(n)}(b)|^q \right]^{\frac{x-a}{b-a}} \left[|f^{(n)}(a)|^q \right]^{\frac{b-x}{b-a}} dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} |f^{(n)}(a)| \left(\int_a^b x^n dx \right)^{1-\frac{1}{q}} \left(\int_a^b x^n \left[\frac{|f^{(n)}(b)|^q}{|f^{(n)}(a)|^q} \right]^{\frac{x-a}{b-a}} dx \right)^{\frac{1}{q}} = \frac{1}{n!} |f^{(n)}(a)| \left[\frac{b^{n+1} - a^{n+1}}{b-a} \right]^{1-\frac{1}{q}} M^{\frac{1}{q}}(a,b,n,q) \\ & = \frac{1}{n!} |f^{(n)}(a)| (b-a)^{1-\frac{1}{q}} \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{1-\frac{1}{q}} M^{\frac{1}{q}}(a,b,n,q) = \frac{1}{n!} |f^{(n)}(a)| (b-a)^{1-\frac{1}{q}} L_n^{n(1-\frac{1}{q})}(a,b) M^{\frac{1}{q}}(a,b,n,q). \end{aligned}$$

This completes the proof of theorem.

Corollary 2. Under the conditions Theorem 2 for $n = 1$ we have the following inequality.

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq A^{1-\frac{1}{q}}(a,b) \left\{ \frac{b |f'(b)|^q - a |f'(a)|^q}{\ln |f'(b)|^q - \ln |f'(a)|^q} - \frac{(b-a)L(|f'(b)|^q, |f'(a)|^q)}{\ln |f'(b)|^q - \ln |f'(a)|^q} \right\}^{\frac{1}{q}}.$$

Proposition 2. Let $a, b \in (0, \infty)$ with $a < b$, $q \geq 1$ and, we have

$$L_{1-\frac{1}{q}}^{1-\frac{1}{q}}(a, b) \leq A^{1-\frac{1}{q}}(a, b)G^{-\frac{2}{q}}(a, b)L^{\frac{2}{q}}(a, b)$$

Proof. The result follows directly from Corollary 2 for the function $f(x) = \frac{q}{q-1}x^{1-\frac{1}{q}}$, $x \in (0, \infty)$.

Corollary 3. Under the conditions Theorem 2 for $q = 1$ we have the following inequality:

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \leq \frac{1}{n!} |f^{(n)}(a)| M(a, b, n, 1)$$

Theorem 3. For $\forall n \in \mathbb{N}$; let $f : I \subseteq [0, \infty) \rightarrow (0, \infty)$ be n-times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f^{(n)}|^q \in L[a, b]$ and $|f^{(n)}|^q$ for $q > 1$ is log-convex on $[a, b]$, then the following inequality holds.

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \\ & \leq \frac{1}{n!} (b-a) L_{p(n-1)+1}^{n-1+1/p}(a, b) \times \left\{ \frac{b |f^{(n)}(b)|^q - a |f^{(n)}(a)|^q}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} - \frac{(b-a)L(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} \right\}^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $|f^{(n)}|^q$ for $q > 1$ is log-convex on $[a, b]$, using Lemma 1 and the Hölder integral inequality, we have the following inequality:

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \leq \frac{1}{n!} \int_a^b x^{n-\frac{1}{q}} x^{\frac{1}{q}} |f^{(n)}(x)| dx \\ & \leq \frac{1}{n!} \left[\int_a^b (x^{n-\frac{1}{q}})^p dx \right]^{\frac{1}{p}} \left[\int_a^b (x^{\frac{1}{q}})^q |f^{(n)}(x)|^q dx \right]^{\frac{1}{q}} \leq \frac{1}{n!} \left[\int_a^b x^{p\frac{qn-1}{q}} dx \right]^{\frac{1}{p}} \left[\int_a^b x [|f^{(n)}(b)|^q]^{\frac{x-a}{b-a}} [|f^{(n)}(a)|^q]^{\frac{b-x}{b-a}} dx \right]^{\frac{1}{q}} \\ & = \frac{1}{n!} |f^{(n)}(a)| \left[\int_a^b x^{p(n-1)+1} dx \right]^{\frac{1}{p}} \left[\int_a^b x \left[\frac{|f^{(n)}(b)|^q}{|f^{(n)}(a)|^q} \right]^{\frac{x-a}{b-a}} dx \right]^{\frac{1}{q}} = \frac{1}{n!} (b-a)^{\frac{1}{p}} \left[\frac{b^{p(n-1)+2} - a^{p(n-1)+2}}{(p(n-1)+2)(b-a)} \right]^{\frac{1}{p}} \\ & \times \left\{ \frac{(b-a) [b |f^{(n)}(b)|^q - a |f^{(n)}(a)|^q]}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} - \frac{(b-a)^2 L(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} \right\}^{\frac{1}{q}} = \frac{1}{n!} (b-a) \left[\frac{b^{p(n-1)+2} - a^{p(n-1)+2}}{(p(n-1)+2)(b-a)} \right]^{\frac{1}{p}} \\ & \times \left\{ \frac{b [b |f^{(n)}(b)|^q - a |f^{(n)}(a)|^q]}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} - \frac{(b-a)L(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} \right\}^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a) L_{p(n-1)+1}^{n-1+1/p}(a, b) \times \left\{ \frac{b [b |f^{(n)}(b)|^q - a |f^{(n)}(a)|^q]}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} - \frac{(b-a)L(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)}{\ln |f^{(n)}(b)|^q - \ln |f^{(n)}(a)|^q} \right\}^{\frac{1}{q}}. \end{aligned}$$

Corollary 4. Under the conditions Theorem 3 for $n = 1$ we have the following inequality.

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq A^{\frac{1}{p}}(a,b) \left\{ \frac{b[b|f'(b)|^q - a|f'(a)|^q]}{\ln|f'(b)|^q - \ln|f'(a)|^q} - \frac{(b-a)L(|f'(b)|^q, |f'(a)|^q)}{\ln|f'(b)|^q - \ln|f'(a)|^q} \right\}^{\frac{1}{q}}$$

Proposition 3. Let $a, b \in (0, \infty)$ with $a < b, p, q > 1$ $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$L_{1-\frac{1}{q}}^{1-\frac{1}{q}}(a,b) \leq A^{\frac{1}{p}}(a,b) G^{-\frac{2}{q}}(a,b) L^{\frac{2}{q}}(a,b)$$

Proof. The result follows directly from Corollary 4 for the function $f(x) = \frac{q}{q-1}x^{1-\frac{1}{q}}, x \in (0, \infty)$.

3 Conclusions

In this paper, by using an integral identity we obtain some new type inequalities for n -time differentiable log-convex functions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

- [1] M. Alomari and M. Darus, "On The Hadamard's Inequality for Log-Convex Functions on the Coordinates", Hindawi Publishing Corporation Journal of Inequalities and Applications, Volume 2009, Article ID 283147, 13 pages, doi:10.1155/2009/283147.
- [2] M. A. Ardiç and M. Emin Özdemir, "Inequalities for log-convex functions via three times differentiability", arXiv:1405.7480v1 [math.CA] 29 May 2014.
- [3] S.-P. Bai, S.-H. Wang and F. Qi, "Some Hermite-Hadamard type inequalities for n -time differentiable (α, m) -convex functions", Jour. of Ineq. and Appl., 2012, 2012:267.
- [4] P. Cerone, S.S. Dragomir and J. Roumeliotis, "Some Ostrowski type inequalities for n -time differentiable mappings and applications", Demonstratio Math., 32 (4) (1999), 697–712.
- [5] P. Cerone, S.S. Dragomir, J. Roumeliotis and J. Sunde, "A new generalization of the trapezoid formula for n -time differentiable mappings and applications", Demonstratio Math., 33 (4) (2000), 719–736.
- [6] S.S. Dragomir and C.E.M. Pearce, "Selected Topics on Hermite-Hadamard Inequalities and Applications", RGMIA Monographs, Victoria University, 2000, online: http://www.staxo.vu.edu.au/RGMIA/monographs/hermite_hadamard.html.
- [7] S.S. Dragomir, "New Jensen's type inequalities for differentiable log-convex functions of selfadjoint operators in Hilbert spaces", Sarajevo Journal of Mathematics, Vol.7 (19) (2011), 67-80.
- [8] D.-Y. Hwang, "Some Inequalities for n -time Differentiable Mappings and Applications", Kyung. Math. Jour., 43 (2003), 335–343.
- [9] İ. İşcan, "Ostrowski type inequalities for p -convex functions", New Trends in Mathematical Sciences, 4 (3) (2016), 140-150.
- [10] İ. İşcan and S. Turhan, "Generalized Hermite-Hadamard-Fejér type inequalities for GA-convex functions via Fractional integral", Moroccan J. Pure and Appl. Anal.(MJPA), Volume 2(1) (2016), 34-46.

- [11] İ. İscan, “Hermite-Hadamard type inequalities for harmonically convex functions”, Hacettepe Journal of Mathematics and Statistics, Volume 43 (6) (2014), 935–942.
- [12] W.-D. Jiang, D.-W. Niu, Y. Hua and F. Qi, “Generalizations of Hermite-Hadamard inequality to n-time differentiable function which are s -convex in the second sense”, Analysis (Munich), 32 (2012), 209–220.
- [13] S. Maden, H. Kadakal, M. Kadakal and İ. İscan, “Some new integral inequalities for n-times differentiable convex and concave functions”, <https://www.researchgate.net/publication/312529563>, (Submitted).
- [14] M. Mansour, M. A. Obaid, “A Generalization of Some Inequalities for the log-Convex Functions”, International Mathematical Forum, 5, 2010, no. 65, 3243 – 3249.
- [15] C. P. Niculescu, “The Hermite–Hadamard inequality for log-convex functions”, Nonlinear Analysis 75 (2012) 662–669.
- [16] J. Park, “Some Hermite-Hadamard-like Type Inequalities for Logarithmically Convex Functions”, Int. Journal of Math. Analysis, Vol. 7, 2013, no. 45, 2217-2233.
- [17] S.H. Wang, B.-Y. Xi and F. Qi, “Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex”, Analysis (Munich), 32 (2012), 247–262.
- [18] G. S. Yang, K. L. Tseng and H. T. Wang, “A note on integral inequalities of Hadamard type for log-convex and log-concave functions”, Taiwanese Journal of Mathematics, Vol. 16, No. 2, pp. 479-496, April 2012.
- [19] Ç. Yıldız, “New inequalities of the Hermite-Hadamard type for n-time differentiable functions which are quasiconvex”, Journal of Mathematical Inequalities, 10, 3(2016), 703-711.
- [20] X. Zhang, W. Jiang, “Some properties of log-convex function and applications for the exponential function”, Computers and Mathematics with Applications 63 (2012) 1111–1116.